

Name: _____

Quiz Score: _____/10

Answer each question completely in the area below. Show all work and explain your reasoning. If the work is at all ambiguous, it is considered incorrect. No phones, calculators, or notes are allowed. Anyone found violating these rules will be asked to leave immediately. Point values are in the square to the left of the question. If there are any other issues, please ask the instructor.

- 4 1. **Using the integral test**, show that the following series **diverges**:

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n}.$$

Make sure to state all of the conditions to apply the test and explain why this particular case satisfies them. Note that the series does not start at $n = 1$.

Solution: Here, we must apply the integral test, considering $f(x) = 1/(x \ln x)$. It's clear that the first requirement, $f(n) = a_n$ is satisfied. We need now check that $f(x)$ is continuous, positive, and decreasing.

- (1) Continuous: Notice that this function only has discontinuities when $x \ln x = 0$ or when $x = 0$ or when $\ln x = 0 \implies x = 1$. Since we're starting at $n = 2$, which corresponds to $n = 2$, this function is continuous for values we care about.
- (2) Positive: If $x \geq 2$, we see that $x \geq 2 > 0$ and $\ln 2 \geq \ln 2 > 0$. Thus, this function is always positive.
- (3) Decreasing: This is the hardest to see or prove. It's intuitively clear since the denominator gets larger, meaning we divide by a larger number, meaning our value must decrease, but we can show this more formally by considering the derivative of $f(x)$, by the chain rule:

$$f'(x) = -\frac{\ln x + 1}{(x \ln x)^2}.$$

Immediately notice that the denominator is always positive since it's something squared. Next, notice that $\ln x > -1$, thus the numerator is always positive too, meaning the whole derivative is negative for all x we care about. Thus, $f'(x) < 0$ and therefore $f(x)$ is decreasing.

We've satisfied all conditions of the integral test, meaning we can now apply it, which states that we examine the behavior of the following indefinite integral:

$$\begin{aligned} \int_2^{\infty} \frac{dx}{x \ln x} &= \lim_{t \rightarrow \infty} \int_2^t \frac{dx}{x \ln x} \\ &= \lim_{t \rightarrow \infty} \int_{\ln 2}^{\ln t} \frac{du}{u} && (u = \ln x, du = 1/x) \\ &= \lim_{t \rightarrow \infty} \ln(\ln x) \Big|_{x=2}^{x=t} \\ &= \infty. \end{aligned}$$

Thus, this integral diverges, and the integral test says that the series divergence or convergence matches that of the integral, so our series diverges as well.

- 5 2. (a) **Using the alternating series test**, show that series converges:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} \sqrt{n}}{n+4}.$$

Again, make sure to state and verify all of the conditions of the test. Make sure to show work to justify your claims.

Solution: The alternating series test says that if we have a series of the following form:

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n \quad (b_n > 0),$$

then, we can check conditions on b_n that determine if the series converges. In this case, it's clear from the form listed above, that $b_n = \sqrt{n}/(n+4)$. The alternating series test then, specifically requires two conditions for convergence:

- (1) $b_{n+1} \leq b_n$ or decreasing terms. To see this, we'll use the same trick. Consider $f(x) = \sqrt{x}/(x+4)$, whose derivative is:

$$f'(x) = \frac{4-x}{2\sqrt{x}(x+4)^2}.$$

Notice, if $x \geq 4$, then $f'(x)$ is negative. This is classic case where it's maybe not decreasing initially, but it ultimately does and that's all we care about.

- (2) $\lim_{n \rightarrow \infty} b_n = 0$. To see this, consider $\lim_{n \rightarrow \infty} \frac{\sqrt{n}}{n+4}$. We notice the 4 is negligible and therefore we only need to consider the different powers of n : $\lim_{n \rightarrow \infty} \frac{\sqrt{n}}{n} = 0$. Thus, this condition is satisfied.

The alternating series test says that if the above conditions are satisfied then indeed, the series converges.

- 1 (b) On this quiz, you've used both the integral test and the alternating series test to prove *convergence* of series. Does either test allow you to prove *divergence*? Why or why not?

Solution: This question made way more sense before the typo in the first problem. Ultimately, I was hoping that you would write that the integral test can be used to show convergence or divergence, whereas the alternating series test can only be used to show convergence.