

Name: \_\_\_\_\_

Quiz Score: \_\_\_\_/10

Answer each question completely in the area below. Show all work and explain your reasoning. If the work is at all ambiguous, it is considered incorrect. No phones, calculators, or notes are allowed. Anyone found violating these rules will be asked to leave immediately. Point values are in the square to the left of the question. If there are any other issues, please ask the instructor.

1. Determine whether the following sequences converge or diverge. If it converges, state the limit. If it diverges, state why.

2 (a)  $\left\{ \frac{5\sqrt{n} - n^{1/3} + 6}{4n^{1/3} - 3\sqrt{n} + n^{1/5}} \right\}$ .

**Solution:** Here, we need to decide the behavior of the terms of this sequence as  $n \rightarrow \infty$ , that is:

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{5\sqrt{n} - n^{1/3} + 6}{4n^{1/3} - 3\sqrt{n} + n^{1/5}}.$$

The result is immediate: we notice that  $\sqrt{n}$  is the largest growing term in both the numerator and denominator, meaning those terms dominate, and the resulting limit is the ratio of their coefficients:  $\lim_{n \rightarrow \infty} a_n = -5/3$ . If you need to see this a different way, consider dividing both the numerator and denominator by  $\sqrt{n}$ . Notice all terms go to 0 except for the  $\sqrt{n}$  terms, which approach the aforementioned limit. Since this sequence approaches a finite number, it converges.

2 (b)  $\{ne^{-n}\}$ .

**Solution:** Again, we must consider the limit of the terms:

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} ne^{-n}.$$

Notice, however, that we have an indeterminate form, as  $n \rightarrow \infty$  and  $e^{-n} \rightarrow 0$ , thus we have a  $0 \times \infty$  situation, which is indeterminate. Thus, we must apply L'Hôpital's rule:

$$\lim_{n \rightarrow \infty} ne^{-n} = \lim_{n \rightarrow \infty} \frac{n}{e^n} \stackrel{\text{L'Hôp.}}{=} \lim_{n \rightarrow \infty} \frac{1}{e^n} = 0.$$

Again, this sequence approaches a number, and therefore converges.

- 2 2. *True or false?* If  $\lim_{n \rightarrow \infty} a_n = 0$ , then the series  $\sum a_n$  converges. If true, explain why. If false, state a counterexample.

**Solution:** False. While it may seem true, it is actually the converse is true from our theorem in class: if a series converges, the terms must go to 0. The statement in this problem is not necessarily the case. For instance, consider our harmonic series:  $\sum_{n=1}^{\infty} 1/n \rightarrow \infty$ , but  $1/n \rightarrow 0$ .

- 4 3. Determine whether the following series converges or diverges. If it converges, state the sum. If it diverges, state why.

$$\sum_{n=2}^{\infty} \left(\frac{1}{\sqrt{2}}\right)^n.$$

**Solution:** The slight curveball in this problem is that the series does not start at  $n = 1$ , but this is fairly insignificant, so long as we observe that this is indeed a geometric series. Our typical geometric series is of the form:

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar^2 + ar^3 + \dots.$$

Here, we have:

$$\sum_{n=2}^{\infty} \left(\frac{1}{\sqrt{2}}\right)^n = \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 \frac{1}{\sqrt{2}} + \left(\frac{1}{\sqrt{2}}\right)^2 \left(\frac{1}{\sqrt{2}}\right)^2 + \dots.$$

Thus, it's hopefully clear that this and our geometric series are the same with  $a = (1/\sqrt{2})^2 = 1/2$  and  $r = 1/\sqrt{2} = \sqrt{2}/2$ .

Recall that a geometric series converges if  $|r| < 1$  and in this case, since  $\sqrt{2} > 1$ , we know that  $r = 1/\sqrt{2} < 1$ , and therefore our series converges. In the case of a convergent geometric series, we know that it must converge to:

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}, \quad |r| < 1.$$

Thus, if we plug in our values of  $a = 1/2$  and  $r = \sqrt{2}/2$ , we have:

$$\sum_{n=2}^{\infty} \left(\frac{1}{\sqrt{2}}\right)^n = \frac{\frac{1}{2}}{1 - \frac{\sqrt{2}}{2}} = \frac{1}{2 - \sqrt{2}}.$$

If we wanted to rationalize (multiply on the top and bottom by  $2 + \sqrt{2}$ ), this ultimately equals  $1 + 1/\sqrt{2} \approx 1.70$ .