

Name: \_\_\_\_\_

Quiz Score: \_\_\_\_/10

Answer each question completely in the area below. Show all work and explain your reasoning. If the work is at all ambiguous, it is considered incorrect. No phones, calculators, or notes are allowed. Anyone found violating these rules will be asked to leave immediately. Point values are in the square to the left of the question. If there are any other issues, please ask the instructor.

1. An empty bucket of water that weighs 4 lb and a rope of negligible weight are being pulled up a building of height 100 ft at a constant speed. As the bucket is being pulled up, it starts raining and the bucket is filled at a constant rate of 0.2 lb per foot the bucket is pulled up. Find the total work done by pulling the bucket to the top of the building.

- 2 (a) Write down the weight of the bucket as a function of  $y$ , the distance measured from the bottom of the building.

**Solution:**

If the bucket is at the bottom of the building  $y = 0$ , it's clear the weight is only 4 lb. For every  $y$  feet it is pulled up, it gains 0.2 lb. In equation form, this tells us exactly, the force, or weight of the bucket is:

$$F(y) = 4 + 0.2y.$$

- 3 (b) Find the total work done by pulling the bucket to the top of the building.

**Solution:** Note, the force to pull up the bucket does **not** remain constant, so we cannot use  $\text{work} = \text{force} \cdot \text{distance}$ . The whole motivation behind the integral solution we've discussed in class is: if we only consider a very short distance, the force is approximately constant, so we can compute  $\text{work} = \text{force} \cdot \text{distance}$ .

Thus, consider the bucket at position  $y$  and pulling it up a tiny distance,  $\Delta y$ . We know the force to pull the bucket is  $F(y)$  from part (a), and we're considering pulling it distance  $\Delta y$ , meaning the work done by pulling a tiny amount, which I'll denote  $w(y)$  is:

$$w(y) = F(y)\Delta y = (4 + 0.2y)\Delta y.$$

How do we add these tiny amounts up? We treat  $\Delta y$  small, calling it  $dy$  and we integrate, starting with the bucket being at the bottom  $y = 0$  and bringing it to the top  $y = 100$ :

$$\int_0^{100} F(y) dy = \int_0^{100} (4 + 0.2y) dy \tag{1}$$

$$= \left[ 4y + \frac{0.2y^2}{2} \right]_{y=0}^{y=100} \tag{2}$$

$$= 4(100) + 0.1(100)^2 \tag{3}$$

$$= 1400. \tag{4}$$

2. Suppose your friends are impressed by your ability to balance a pencil on your finger, and demand you teach them how. Being a good calculus and physics student, you tell them they simply must simply find the center of mass of the pencil. Suppose that the pencil is 8 inches long and that we can treat it as a thin rod with a non-constant density of  $\rho(x) = \frac{1}{2}\sqrt{x}$  g/in since the eraser adds extra weight.



- 2 (a) Compute the total mass of the pencil.

**Solution:** Similar in spirit to the first problem: if the pencil had a constant density, then we could say mass = density · length, but it does not, so we cannot use this. Instead, we can think of slicing the pencil into tiny slices, that *do* approximately have constant density. Thus, consider a tiny slice of length  $\Delta x$  at position  $x$ , which has mass, which I'll denote  $m(x)$ :

$$m(x) = \rho(x) \cdot \Delta x = \frac{1}{2}\sqrt{x}\Delta x.$$

How do we get the total mass of the pencil? We simply add the slices by considering  $\Delta x$  small, which we'll now call  $dx$  and integrating:

$$M = \int_0^8 \rho(x) dx \tag{5}$$

$$= \int_0^8 \frac{1}{2}\sqrt{x} dx \tag{6}$$

$$= \left[ \frac{1}{2} \frac{x^{3/2}}{3/2} \right]_{x=0}^{x=8} \tag{7}$$

$$\approx 7.542 \tag{8}$$

- 3 (b) Compute the center of mass of the pencil.

**Solution:** Again, consider slicing the pencil into tiny slices of length  $\Delta x$ . To compute the center of mass, we must sum mass · distance as we did with the point mass example in class. The mass of a slice, in this case, is the same as above:

$$m(x) = \rho(x) \cdot \Delta x = \frac{1}{2}\sqrt{x}\Delta x.$$

What is the distance away from the  $y$ -axis, or  $x = 0$ ? It's simply just  $x$ . Thus, in this case, our mass · distance of a single slice, which is typically called a *moment* or *centroid*, which I'll call  $m_x(x)$  is:

$$m_x(x) = m(x)x = \frac{1}{2}\sqrt{x}\Delta x \cdot x = \frac{1}{2}x^{3/2} \Delta x.$$

Again, we simply add these up via an integral to find that:

$$M_x = \int_0^8 x\rho(x) dx. \quad (9)$$

$$= \int_0^8 \frac{1}{2}x^{3/2} dx \quad (10)$$

$$= \left[ \frac{1}{2} \frac{x^{5/2}}{5/2} \right]_{x=0}^{x=8} \quad (11)$$

$$\approx 36.2 \quad (12)$$

From this, we know the center of mass can be obtained by dividing the moment by the total mass:

$$\bar{x} = \frac{M_x}{M} = \frac{36.2}{7.52} \approx 4.8.$$