

Math 1320 – Final Exam: April 30, 2015, 8 AM

This document outlines the topics covered since the second midterm exam. Again, this is not an exhaustive list, but meant to give you a rough checklist of topics to know. As with the previous exams, identify which topics on the list are your weaknesses and do practice problems to improve in this area.

In class on Monday and Tuesday before the exam, I'll review, so please come with questions. The week before (and during the week of) the exam, I'll attempt to be in my office as much as possible. Particularly, if my schedule is free during a time, I'm very likely to be in my office. You can check my schedule on my website: <http://www.math.utah.edu/~miles/#contact>.

The exam itself will be cumulative, covering topics from all year, weighted approximately 1/3 to each of the two exams, and (slightly more than) 1/3 to new material. It will *not* be double the length of a midterm, but you will have double the amount of time: 2 hours. I would study all of the materials from the class: notes, quizzes, lab problems, old exam problems. Many problems from these sources are likely to make a reappearance on the final exam.

Chapter 10: Vector Functions

10.5: Parametric surfaces

- Most general of our surfaces. We now have things of the form: $\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}$, which defines the surface.
- For instance, $x = 2 \cos u, y = v, z = 2 \sin u$ defines a cylinder.
- Plane equation: $\mathbf{r}(u, v) = \mathbf{r}_0 + u\mathbf{a} + v\mathbf{b}$, where \mathbf{a}, \mathbf{b} are some vectors in the plane.
- Sphere equation: $x = a \sin \phi \cos \theta, y = a \sin \phi \sin \theta, z = a \cos \phi$, basically spherical.
- Surface of revolution: rotate $f(x)$ around x axis yields: $x = x, y = f(x) \cos \theta, z = f(x) \sin \theta$.
- Again, be able to recognize these and match equations/graphs.
- Some examples we've seen: torus, intersection of a plane/cylinder.

Chapter 11: Multivariate Functions

11.1: Functions of Several Variables

- Now looking at things of the form $z = f(x, y)$ or $F(x, y, z) = k$.
- Typically think of z as height.
- For $z = f(x, y)$, a level curve is when we set $z = k$ and draw in the x, y plane the corresponding curve. Think of slicing at a height k .
- Be able to identify functions by their level curves (or vice versa)

11.2: Limits, Continuity

- New idea: 2D limits. For limit to exist, must be same in all directions
- Consequence: easy to show limits do not exist, just need 2 different paths
- Common paths: $y = 0, x = 0, y = mx, y = x^2$. Always check these first.
- Hard to show limits exist. Two examples: we used inequality $x^2 \leq x^2 + y^2$ and we also converted to polar. Know these examples.
- Continuous definition: the same as 1D. Limit = evaluation.

11.3: Partial Derivatives

- Partial derivative: for $z = f(x, y)$, $\partial_x f(x, y)$ describes the rate of change of $f(x, y)$ as JUST x is changed. We can think of y as fixed.

- Computationally: to compute ∂_x , we treat y as a constant and take a normal derivative of x .
- Geometrically: $\partial_x f(x, y)$ gives us the slope of the tangent line in the x direction at a point.
- Chain rule, implicit differentiation, product rule, etc. all still apply. Chain rule is investigated further in 11.5.
- Clairaut's theorem: for f_{xy} and f_{yx} both continuous, then $f_{xy} = f_{yx}$. That is, for most "nice" functions, we can take partial derivatives in any order we want. Also means that $f_{xyy} = f_{yyx} = f_{yyx}$ for instance.
- We can now define a partial differential equation. Example: Laplace's equation: $\partial_{xx}u + \partial_{yy}u = 0$. Can verify solutions by plugging in just like for regular differential equations.

11.4: Tangent Planes, Linear Approximations, Differentials

- Partial derivatives f_x, f_y basically define two vectors that generate the tangent plane, giving us the formula: $z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$.
- Can think of the tangent plane as a linear approximation around (x_0, y_0, z_0) .
- Technical note: tangent plane only exists if f_x, f_y are continuous.
- Linearization $L(x, y)$ defined the same way: $L(x, y) = z_0 + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$
- Theorem: if f_x, f_y exist and are continuous near (a, b) , then $f(x, y)$ is differentiable near (a, b) .
- Differentials: roughly same idea as linear approximation, just different way of interpreting
- Rough idea of differentials: absolute change in z as you take $x \rightarrow x + dx$ vs. the change you get by approximating via the tangent line (or plane). Ultimately, $dz = f_x dx + f_y dy$. Good approximation.
- If we have a parametric surface $\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle$, then we just compute \mathbf{r}_u and \mathbf{r}_v , the partial derivatives, which give us two vectors that generate the normal vector for the tangent plane. That is, $\mathbf{n} = \mathbf{r}_u \times \mathbf{r}_v$. If this cross product is 0, the tangent plane does not exist.

11.5: Chain Rule

- Case 1: $z = f(x, y), x = x(t), y = y(t)$, then $\frac{dz}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}$.
- Case 2: $z = f(x, y), x = x(s, t), y = y(s, t)$, then $\frac{\partial z}{\partial t} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial t}$. Similar can be defined for s .
- Useful tool: draw tree. Top of tree = thing you're taking the derivative of, then draw lines to everything its a function of, and then draw lines to everything those are a function of. To get full chain rule: start at top and move down to variable you're taking the derivative with respect to.
- For example: $\frac{\partial z}{\partial t}$, you would start at z and end at t . There are 2 paths so we get 2 terms. Each line corresponds to multiplication of some derivative.
- Second partials often involve using chain rule more than once. Know how to do this.
- Implicit differentiation gives us that if we have $F(x, y) = 0$, then $\frac{dy}{dx} = -\frac{F_x}{F_y}$.
- Similarly, $F(x, y, z) = 0$ implies $\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}$ or $\frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$.

11.6: Directional Derivatives, Gradient

- ∂_x, ∂_y are rates of change in the direction of x, y , but we can consider any arbitrary direction \mathbf{u} , where $\mathbf{u} = \langle a, b \rangle$ is a unit vector.
- Gives us: directional derivative $D_{\mathbf{u}}f(x, y)$. Know limit definition, but easier definition: $D_{\mathbf{u}}f = f_x a + f_y b$.
- Define the gradient $\nabla f = \langle f_x, f_y \rangle$. That is, the gradient is a vector function that contains the partial derivatives at a point.
- Convenient form: $D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u}$.
- Big idea #1: at a point, the direction \mathbf{u} that maximizes the directional derivative $D_{\mathbf{u}}f$ is in the direction of ∇f (since it is a vector) and the maximum value is $|\nabla f|$. Know why.

- Big idea #2: consider some surface $F(x, y, z) = k$, and define $\delta x = \langle x - x_0, y - y_0, z - z_0 \rangle$. Then the tangent plane at a point is $\nabla F \cdot \delta x = 0$.
- In the case we have that $z = f(x, y)$, this reduce to our same tangent plane equation, but useful because more general. Therefore: generally, the gradient points outward (normal) to any surface.
- Thus, at any level curve, ∇f gives us the direction of fastest rate of change, and is also perpendicular to any level curve.

11.7: Maxima and Minima

- Local minima/maxima: largest or smallest value near some point. Global maxima/minima: largest/smallest in domain.
- Same idea as calc 1: if (a, b) is a local minima or maxima, then $f_x(a, b) = 0$ AND $f_y(a, b) = 0$.
- We call points where $f_x = 0$ and $f_y = 0$ critical points. Not necessarily min or max.
- Again, similarity to calc 1: need to check 2nd derivative to tell if min/max.
- Define $D = f_{xx}f_{yy} - f_{xy}^2$. (Think of as a determinant of a matrix)
- Test says that if: (1) $D > 0, f_{xx} > 0$ then local min, (2) $D > 0, f_{xx} < 0$, local max, (3) $D < 0$, neither (called a saddle point).
- Note: if $D = 0$, no conclusion can be made.
- Thus, general procedure: find all critical points and then test D for each.
- Distance from (x, y, z) to a point (x_0, y_0, z_0) : $d = \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}$. Often easier (and valid) to minimize d^2 .
- Closed set: includes boundary. Open set: does not.
- Extreme (absolute/global min or max): must occur for any closed bounded set.
- Recipe for finding global/min or max: find critical numbers, check value of f at critical numbers, check value of f on boundaries. Key idea: max/min MUST occur on boundary or at a critical number.

11.8: Lagrange Multipliers

- Recipe for constrained optimization. That is, we want to maximize $f(x, y)$ constrained to $g(x, y) = k$.
- Equivalently: we find the largest c such that the level curve $f(x, y) = c$ intersects the level curve $g(x, y) = k$. This occurs when the level curves have a common tangent line!
- Common tangent line = same normal direction = parallel gradient vectors!
- Key idea: $\nabla f = \lambda \nabla g$ for some λ . Thus, we just need to find λ . This is called a Lagrange multiplier.
- General recipe: find all x, y, z, λ such that this is true, compare f on values of x, y, z .
- Component equations: $f_x = \lambda g_x, f_y = \lambda g_y, f_z = \lambda g_z, g = k$.
- If we have two constraints: that is $g = k, h = c$, we are looking for $f(x, y, z)$ that lies in the intersection of these two level surfaces. By gradient properties, ∇f is orthogonal to this point, and so are $\nabla g, \nabla h$. Thus, we just need to find λ, μ such that: $\nabla f = \lambda \nabla g + \mu \nabla h$. Same idea.