

Math 1320 – Midterm 2: April 6, 2015

As with the previous, this outline is not meant to be an end-all exhaustive study replacement, but rather a structured baseline for your studying. That is, this is a study *companion*.

The topics covered on the exam are roughly described below. Identify which topics are your own weaknesses by practicing problems without your notes.

In class on Friday before the exam, I'll review, so please come with questions. Matthew will also review on Thursday. I'll make an effort to be in my office as much as possible during the week prior to the exam. I'm particularly likely to be in my office if you see nothing on my schedule, found here: <http://www.math.utah.edu/~miles/#contact>.

This exam will likely be more calculation oriented than the last, but don't assume that will mean it is easier. I do not expect you to memorize the names of surfaces, but be able to identify them if I give you an equation by considering slices or any other technique. This also means I can give you surfaces in cylindrical or spherical coordinates as well (*hint*). Otherwise, I'll provide practice problems that are a good resource to practice the calculations expected on the exam.

Chapter 9: Vectors and the Geometry of Space

9.1: 3 Dimensions

- Cartesian, or rectangular coordinates in 3 dimensions: (x, y, z)
- In 2D, $y = 2$ is a line, in 3D, $y = 2$ is a plane
- Know other common examples, for instance $x^2 + y^2 = 1$ is a cylinder
- Distance between two points $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$.
- Sphere with center (h, k, l) and radius r : $(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2$.

9.2: Vectors

- New fundamental quantity: vectors have direction and magnitude (think of arrows)
- Two vectors are equal if they have the same direction, magnitude
- 0 vector has 0 magnitude and is the only vector with no direction
- Displacement vector: describing moving from A to B , denoted \overrightarrow{AB} .
- We can add vectors by placing them tip (of first) to tail of second and draw resulting total arrow.
- Scalar multiplication: rescales vector magnitude and/or flips if negative.
- We think of vectors as their components: $\mathbf{v} = \langle a, b, c \rangle$.
- From this, displacement vector is easy: $\overrightarrow{AB} = \langle b_1 - a_1, b_2 - a_2, b_3 - a_3 \rangle$.
- Magnitude of vector, denoted $|\mathbf{a}|$ or $\|\mathbf{a}\|$: distance formula.
- Unit vector = length 1.
- Standard coordinate vectors: $\mathbf{i} = \langle 1, 0, 0 \rangle$, $\mathbf{j} = \langle 0, 1, 0 \rangle$ and $\mathbf{k} = \langle 0, 0, 1 \rangle$.
- We can write any vector in $\mathbf{i}, \mathbf{j}, \mathbf{k}$ form (convince yourself of why this is true)
- Be familiar with tension example: forces in different directions

9.3: Dot product

- Motivated definition of dot by notion of work.
- If \mathbf{F} is some force exerted an angle away θ from the displacement \mathbf{D} then we found that $W = |\mathbf{F}||\mathbf{D}|\cos\theta$

- We take this as definition: $\mathbf{a} \cdot \mathbf{b} \stackrel{\text{def}}{=} |\mathbf{a}| |\mathbf{b}| \cos \theta$, where $\theta \in [0, \pi]$.
- Sometimes called the scalar product because you put in two vectors and get a scalar out.
- If $\mathbf{a} \cdot \mathbf{b} = 0$, they are orthogonal, also the converse is true. Why?
- Understand dot product differences when θ is acute, obtuse, right.
- Know how to obtain θ for vectors.
- Component formula: $\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$.
- Projecting b onto a : $\text{proj}_a \mathbf{b} = \text{comp}_a \mathbf{b} \frac{\mathbf{a}}{|\mathbf{a}|}$, where $\text{comp}_a \mathbf{b} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|}$. Know the picture in different cases and where these formulae come from

9.4: Cross product

- Similar to the dot product, we motivate this by physics: imagine a wrench of some direction described by \mathbf{r} and exerting a force on it in another direction \mathbf{F} . We get a *torque* out
- torque: $\tau = |\mathbf{r}| |\mathbf{F}| \sin \theta$, where \mathbf{n} is a vector orthogonal to the other two.
- $\mathbf{a} \times \mathbf{b} \stackrel{\text{def}}{=} |\mathbf{a}| |\mathbf{b}| \sin \theta \mathbf{n}$, again $\theta \in [0, 2\pi]$ and \mathbf{n} is unit orthogonal to both vectors, determined by the right hand rule.
- Sometimes called the vector product because you put in two vectors and get a vector out.
- Angle θ always measured from \mathbf{a} to \mathbf{b} . Drawing tails attached helps figure this out often
- If $\mathbf{a} \times \mathbf{b} = 0$, they are parallel, converse is also true.
- Cross product does NOT commute, but $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$.
- Know how to use determinants to obtain the cross product. General result (probably don't want to memorize): $\mathbf{a} \times \mathbf{b} = \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle$.
- $|\mathbf{a} \times \mathbf{b}|$ gives you the area of the parallelogram described by the two vectors
- Useful formula: $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$.
- Another useful formula: $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c}) \mathbf{b} - (\mathbf{a} \cdot \mathbf{b}) \mathbf{c}$.

9.5: Equations of lines, planes

- Vector equation of a line is $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$, where \mathbf{r}_0 is a vector pointing to some point, and \mathbf{v} is a vector pointing in the direction of the line. Any point on the line can be described this way for some t .
- Alternatively, if $\mathbf{v} = \langle a, b, c \rangle$, we can say $\langle x, y, z \rangle = \langle x_0 + at, y_0 + bt, z_0 + ct \rangle$.
- Parametric equations: $x = x_0 + at, \dots$ and so on. Just take each component of above formula.
- We can eliminate t to obtain the symmetric equations: $\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$.
- If we want a line segment from \mathbf{r}_0 to \mathbf{r}_1 , we can parameterize by $\mathbf{r} = (1-t)\mathbf{r}_0 + t\mathbf{r}_1$, where $0 \leq t \leq 1$.
- Two lines are skew if they do not intersect and are not parallel
- To define a plane, we need two things: any point on the plane $P_0 = (x_0, y_0, z_0)$ and the normal vector \mathbf{n} .
- Notice that \mathbf{n} is orthogonal to any displacement vector in the plane, so we have $\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0$, which is the vector equation of a plane.
- More usefully, if $\mathbf{n} = \langle a, b, c \rangle$, we have $\langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$, which is often called the scalar equation of a plane.
- Angle between two planes: angle between normals.
- Parallel planes: same normal.

- Distance D from a point P_1 to a plane is $D = |\text{comp}_n \mathbf{b}| = \frac{|\mathbf{n} \cdot \mathbf{b}|}{n}$, where $\mathbf{b} = \langle x_1 - x_0, y_1 - y_0, z_1 - z_0 \rangle$. Know geometrically why this is true.
- Distance between two planes: pick a point on one plane, repeat above.
- Distance between two lines: can be thought of as distance between two planes. Find the normal and pick points.

9.6: Functions and surfaces

- We can think of $f(x, y) = z$, where we put in two variables and get a height z out.
- Know how to find domain/range of these functions. Pretty much the same as previous.
- Be familiar with basic shapes. For instance, $z = ax + by + c$ or any linear function of x, y, z is a plane.
- Key idea for identifying shapes: take one coordinate to be constant, say $x = k$. This is like slicing the picture. Piece together from this.
- Be able to identify shapes given equations or the reverse.

9.7: Cylindrical and spherical Coordinates

- Cylindrical coordinates: exactly the same as polar. $x = r \cos \theta, y = r \sin \theta, z = z$.
- We also have $r^2 = x^2 + y^2, \tan \theta = y/x$.
- Be able to convert points and functions between rectangular/cylindrical.
- $z = r$ is a cone, $r = 2$ is a cylinder. Know basic shapes and be able to recognize them.
- Spherical coordinates: new idea. θ is angle from x axis, ϕ is angle from z axis, ρ is distance from origin.
- Actual formulae: $x = \rho \sin \phi \cos \theta, y = \rho \sin \phi \sin \theta, z = \rho \cos \phi$. We get this by recalling that $r = \rho \sin \phi$ and computing two triangles.
- Know basic shapes: $\rho = c$ is a sphere, $\theta = c$, a plane, $\phi = c$, a cone.
- Be able to recognize basic shapes and match them.
- Be able to convert between points and equations of spherical and convert to rectangular or the opposite.

Chapter 10: Vector Functions

10.1: Vector functions, Space curves

- We now think of vector functions. Input: some variable, typically t output: a vector. This is new.
- Typically: $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$.
- Limit of r : just take limit of each component.
- Continuity of r : $\lim_{t \rightarrow a} \mathbf{r}(t) = \mathbf{r}(a)$.
- We can think of $\mathbf{r}t$ as an "arm" pointing to different points as t changes. We can alternatively (and often more usefully) think of the space curve described by $x = f(t), y = g(t), z = h(t)$, which describes the path of the tip of the arm. Understand the difference between these two.
- Helix: $\mathbf{r} = \langle \cos t, \sin t, t \rangle$.
- Know how to find intersection of two surfaces: space curve.

10.2: Derivatives and the integrals of vector functions

- Derivative of vector function: just component-wise derivative
- Geometrically, derivative of vector function gives us a vector tangent at the point

- We can normalize to length 1, called unit tangent vector: $\mathbf{T}(t) = \mathbf{r}'(t)/|\mathbf{r}'(t)|$.
- Important theorem: if $|\mathbf{r}| = c$ then $\mathbf{r}' \perp \mathbf{r}$, that is $\mathbf{r}' \cdot \mathbf{r} = 0$. Know the proof. Is the converse true? Try to prove it.
- Integrals of vector functions: just integral of each component.

10.3: Arc length and curvature

- Arc length formula the same as before: $L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2} dt$, but notice that $|\mathbf{r}'(t)| = \sqrt{f'^2 + g'^2 + h'^2}$
- Nicer formula: $L = \int_a^b |\mathbf{r}'(t)| dt$.
- We can consider $s(t)$, the arc length function from some point $t = a$ as $s(t) = \int_a^t |\mathbf{r}'(t)| dt$, which means that $ds/dt = |\mathbf{r}'(t)|$.
- Issue: same curves have different parameterizations. How do describe the curve itself, independent of the actual parameterization? Parameterize by arc length, s .
- As curve gets curvier, T changes rapidly, and when curve is flatter, T changes slowly, we can give this quantity a name, curvature: $\kappa = |d\mathbf{T}/ds|$.
- Not very useful formula, but we can recognize that $\kappa = |\mathbf{T}'(t)|/|\mathbf{r}'(t)|$. Understand the proof of this.
- Harder to prove, but worth studying the proof: $\kappa = |\mathbf{r}' \times \mathbf{r}''|/|\mathbf{r}'|^3$. Most useful formula if I give you \mathbf{r} and ask you to compute κ .
- Special case: if $y = f(x)$, then $\kappa = |f''|/[1 + f'^2]^{3/2}$.
- \mathbf{T} is the tangent vector, so what is tangent to the tangent? The unit normal vector, $\mathbf{N} = \mathbf{T}'/|\mathbf{T}'|$. Also, there is a third vector orthogonal to both called the binormal $\mathbf{B} = \mathbf{T} \times \mathbf{N}$. Be able to calculate these.

10.4: Velocity and acceleration

- Often useful to think of $\mathbf{v}(t)$, the velocity of an object as a vector function
- Now we have machinery to compute position $\mathbf{r}(t) = \mathbf{v}'(t)$ and acceleration $\mathbf{a}(t) = \int \mathbf{v}(t) dt$.
- Other relationships: $\mathbf{v}(t) = \mathbf{v}(t_0) + \int_{t_0}^t \mathbf{a}(u) du$ and $\mathbf{r}(t) = \mathbf{r}(t_0) + \int_{t_0}^t \mathbf{v}(u) du$.
- $\mathbf{F} = m\mathbf{a}$ still holds. Now we get a force vector.
- Projectile motion: acceleration due to gravity: $-mg\mathbf{j}$. We can use vector relationships to find that $\mathbf{r}(t) = -\frac{1}{2}gt^2\mathbf{j} + t\mathbf{v}_0 + \mathbf{r}_0$. Be able to interpret extensions/variations of this.
- Useful to realize that acceleration is $\mathbf{a} = |\mathbf{v}'|\mathbf{T} + \kappa|\mathbf{v}|^2\mathbf{N}$, that is acceleration has a tangential and normal component (think slamming on your breaks vs sharply turning).