4. $z = f(x, y) = xe^{xy} \Rightarrow f_x(x, y) = xye^{xy} + e^{xy}$, $f_y(x, y) = x^2e^{xy}$, so $f_x(2, 0) = 1$, $f_y(2, 0) = 4$, and an equation of the tangent plane is $z - 2 = f_x(2, 0)(x - 2) + f_y(2, 0)(y - 0) \Rightarrow z - 2 = 1(x - 2) + 4(y - 0)$ or $z = x + 4y$.

12. $f(x,y)=x^3y^4$. The partial derivatives are $f_x(x,y)=3x^2y^4$ and $f_y(x,y)=4x^3y^3$, so $f_x(1,1)=3$ and $f_y(1,1)=4$. Both f_x and f_y are continuous functions, so f is differentiable at $(1,1)$ by Theorem 8. The linearization of f at $(1,1)$ is given by $L(x,y)=f(1,1)+f_x(1,1)(x-1)+f_y(1,1)(y-1)=1+3(x-1)+4(y-1)=3x+4y-6$.

30. $dx = \Delta x = -0.04$, $dy = \Delta y = 0.05$, $z = x^2 - xy + 3y^2$, $z_x = 2x - y$, $z_y = 6y - x$. Thus when $x = 3$ and $y = -1$,

$dz = (7)(-0.04) + (-9)(0.05) = -0.73$ while $\Delta z = (2.96)^2 - (2.96)(-0.95) + 3(-0.95)^2 - (9 + 3 + 3) = -0.7189$.

4. $z = \tan^{-1}(y/x), x = e^t, y = 1 - e^{-t} \quad \Rightarrow$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = \frac{1}{1+(y/x)^2}(-yx^{-2}) \cdot e^t + \frac{1}{1+(y/x)^2}(1/x) \cdot (-e^{-t})(-1)$$

$$= -\frac{y}{x^2+y^2} \cdot e^t + \frac{1}{x+y^2/x} \cdot e^{-t} = \frac{xe^{-t} - ye^t}{x^2+y^2}$$

10. $z = e^{x+2y}$, $x = s/t$, $y = t/s$ $\Rightarrow$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = (e^{x+2y})(1/t) + (2e^{x+2y})(-ts^{-2}) = e^{x+2y}\left(\frac{1}{t} - \frac{2t}{s^2}\right)$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = (e^{x+2y})(-st^{-2}) + (2e^{x+2y})(1/s) = e^{x+2y}\left(\frac{2}{s} - \frac{s}{t^2}\right)$$

14. By the Chain Rule (3), $\frac{\partial W}{\partial s} = \frac{\partial W}{\partial u}\frac{\partial u}{\partial s} + \frac{\partial W}{\partial v}\frac{\partial v}{\partial s}$. Then

$$W_s(1,0) = F_u(u(1,0), v(1,0))\, u_s(1,0) + F_v(u(1,0), v(1,0))\, v_s(1,0) = F_u(2,3)u_s(1,0) + F_v(2,3)v_s(1,0)$$
$$= (-1)(-2) + (10)(5) = 52$$

Similarly, $\frac{\partial W}{\partial t} = \frac{\partial W}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial W}{\partial v}\frac{\partial v}{\partial t} \quad \Rightarrow$

$$W_t(1,0) = F_u(u(1,0), v(1,0))\, u_t(1,0) + F_v(u(1,0), v(1,0))\, v_t(1,0) = F_u(2,3)u_t(1,0) + F_v(2,3)v_t(1,0)$$
$$= (-1)(6) + (10)(4) = 34$$

8. $f(x,y) = y^2/x$

(a) $\nabla f(x,y) = \dfrac{\partial f}{\partial x}\,\mathbf{i} + \dfrac{\partial f}{\partial y}\,\mathbf{j} = y^2(-x^{-2})\mathbf{i} + (2y/x)\,\mathbf{j} = -\dfrac{y^2}{x^2}\mathbf{i} + \dfrac{2y}{x}\mathbf{j}$

(b) $\nabla f(1,2) = -4\,\mathbf{i} + 4\,\mathbf{j}$

(c) By Equation 9, $D_{\mathbf{u}}\, f(1,2) = \nabla f(1,2) \cdot \mathbf{u} = (-4\,\mathbf{i} + 4\,\mathbf{j}) \cdot \frac{1}{3}\left(2\,\mathbf{i} + \sqrt{5}\,\mathbf{j}\right) = \frac{1}{3}\left(-8 + 4\sqrt{5}\right) = \frac{4}{3}\left(\sqrt{5} - 2\right).$

14. $g(r,s) = \tan^{-1}(rs) \;\Rightarrow\; \nabla g(r,s) = \left(\dfrac{1}{1+(rs)^2}\cdot s\right)\mathbf{i} + \left(\dfrac{1}{1+(rs)^2}\cdot r\right)\mathbf{j} = \dfrac{s}{1+r^2s^2}\mathbf{i} + \dfrac{r}{1+r^2s^2}\mathbf{j}$,

$\nabla g(1,2) = \frac{2}{5}\,\mathbf{i} + \frac{1}{5}\,\mathbf{j}$, and a unit vector in the direction of $\mathbf{v}$ is $\mathbf{u} = \frac{1}{\sqrt{5^2+10^2}}(5\,\mathbf{i} + 10\,\mathbf{j}) = \frac{1}{5\sqrt{5}}(5\,\mathbf{i} + 10\,\mathbf{j}) = \frac{1}{\sqrt{5}}\mathbf{i} + \frac{2}{\sqrt{5}}\mathbf{j}$,

so $D_{\mathbf{u}}\, g(1,2) = \nabla g(1,2)\cdot\mathbf{u} = \left(\frac{2}{5}\,\mathbf{i} + \frac{1}{5}\,\mathbf{j}\right)\cdot\left(\frac{1}{\sqrt{5}}\,\mathbf{i} + \frac{2}{\sqrt{5}}\,\mathbf{j}\right) = \frac{2}{5\sqrt{5}} + \frac{2}{5\sqrt{5}} = \frac{4}{5\sqrt{5}}$ or $\frac{4\sqrt{5}}{25}$.

22. $f(p,q) = qe^{-p} + pe^{-q} \quad \Rightarrow \quad \nabla f(p,q) = \langle -qe^{-p} + e^{-q}, e^{-p} - pe^{-q} \rangle$.

$\nabla f(0,0) = \langle 1, 1 \rangle$ is the direction of maximum rate of change and the maximum rate is $|\nabla f(0,0)| = \sqrt{2}$.

38. (a) From Equation 9 we have $D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = \langle f_x, f_y \rangle \cdot \langle a, b \rangle = f_x a + f_y b$ and from Exercise 37 we have

$$D_{\mathbf{u}}^2 f = D_{\mathbf{u}}\left[D_{\mathbf{u}} f\right] = \nabla\left[D_{\mathbf{u}} f\right] \cdot \mathbf{u} = \langle f_{xx}a + f_{yx}b, f_{xy}a + f_{yy}b \rangle \cdot \langle a, b \rangle = f_{xx}a^2 + f_{yx}ab + f_{xy}ab + f_{yy}b^2.$$

But $f_{yx} = f_{xy}$ by Clairaut's Theorem, so $D_{\mathbf{u}}^2 f = f_{xx}a^2 + 2f_{xy}ab + f_{yy}b^2$.

(b) $f(x, y) = xe^{2y} \quad \Rightarrow \quad f_x = e^{2y}, f_y = 2xe^{2y}, f_{xx} = 0, f_{xy} = 2e^{2y}, f_{yy} = 4xe^{2y}$ and a

unit vector in the direction of $\mathbf{v}$ is $\mathbf{u} = \frac{1}{\sqrt{4^2+6^2}}\langle 4, 6 \rangle = \left\langle \frac{2}{\sqrt{13}}, \frac{3}{\sqrt{13}} \right\rangle = \langle a, b \rangle$. Then

$$D_{\mathbf{u}}^2 f = f_{xx}a^2 + 2f_{xy}ab + f_{yy}b^2 = 0 \cdot \left(\frac{2}{\sqrt{13}}\right)^2 + 2 \cdot 2e^{2y}\left(\frac{2}{\sqrt{13}}\right)\left(\frac{3}{\sqrt{13}}\right) + 4xe^{2y}\left(\frac{3}{\sqrt{13}}\right)^2 = \frac{24}{13}e^{2y} + \frac{36}{13}xe^{2y}.$$

40. Let $F(x,y,z) = x^2 - z^2 - y$. Then $y = x^2 - z^2 \Leftrightarrow x^2 - z^2 - y = 0$ is a level surface of F. $F_x(x,y,z) = 2x \Rightarrow$ $F_x(4,7,3) = 8$, $F_y(x,y,z) = -1 \Rightarrow F_y(4,7,3) = -1$, and $F_z(x,y,z) = -2z \Rightarrow F_z(4,7,3) = -6$.

(a) An equation of the tangent plane at $(4,7,3)$ is $8(x-4) - 1(y-7) - 6(z-3) = 0$ or $8x - y - 6z = 7$.

(b) The normal line has symmetric equations $\dfrac{x-4}{8} = \dfrac{y-7}{-1} = \dfrac{z-3}{-6}$ and parametric equations $x = 4 + 8t$, $y = 7 - t$, $z = 3 - 6t$.