

**Final Exam**  
**Math 1320 - Engineering Calculus II**  
**April 30, 2015**

Answer each question completely in the area below. Show all work and explain your reasoning. If the work is at all ambiguous, it is considered incorrect. No phones, calculators, or notes are allowed. Anyone found violating these rules will be asked to leave immediately. Point values are in the square to the left of the question. **If there are any other issues, please ask the instructor.**

By signing below, you are acknowledging that you have read and agree to the above paragraph, as well as agree to abide University Honor Code:

Name: \_\_\_\_\_  
Signature: \_\_\_\_\_  
uID: \_\_\_\_\_

**Solutions**

Question	Points	Score
1	25	
2	10	
3	20	
4	15	
5	35	
6	25	
7	20	
8	15	
9	15	
10	15	
11	15	
Total:	210	

**Note:** There are 11 total questions on the exam with 210 possible points total, but the exam will be graded out of 200.

1. Suppose this morning, you put a Pop-Tart<sup>®</sup> in the toaster and let it heat up. When the toaster pops, you grab the Pop-Tart<sup>®</sup> and suddenly realize how hot it is, throwing it on the counter-top, allowing it to cool off in the surrounding air of 25°C. Newton's Law of Cooling states that the temperature of the Pop-Tart<sup>®</sup>, at time  $t$ , which we'll call  $P(t)$  is determined by the following:

$$\frac{dP}{dt} = k(P - 25).$$



Figure 1: A Pop-Tart<sup>®</sup> in a toaster. Not drawn to actual size.

- 5 (a) By considering whether  $dP/dt$  should be positive or negative, argue whether  $k < 0$  or  $k > 0$  for this physical situation.
- 15 (b) Suppose the Pop-Tart<sup>®</sup> was 100° C (very hot!) when you took it out of the toaster. Solve for the temperature of the Pop-Tart<sup>®</sup> as a function of time. That is, solve the initial value problem for  $P(t)$ .
- 5 (c) What is  $\lim_{t \rightarrow \infty} P(t)$ ? Why does this make sense? That is, what is the physical interpretation of this calculation and its result?

### Solution:

- (a) Since the Pop-tart starts out hotter than the surrounding temperature, that is:  $P > 25$ , the  $(P - 25)$  term in the differential equation is positive, but we need the temperature of the Pop-tart to be going down (cooling), so overall, we need  $dP/dt < 0$ , meaning  $k$  must be negative:  $k < 0$ .

- (b) We, in total have the following initial value problem:

$$\frac{dP}{dt} = k(P - 25), \quad P(0) = 100.$$

The only technique we know for solving this is to separate and integrate, which we've done many times on problems of this form. Doing so:

$$\begin{aligned} \frac{dP}{dt} &= k(P - 25) \\ \frac{dP}{P - 25} &= kdT \\ \int \frac{dP}{P - 25} &= \int kdT \\ \ln |P - 25| &= kt + C. \end{aligned}$$

Note,  $P > 25$ , so we can remove the absolute value sign and exponentiate, yielding:

$$P(t) - 25 = De^{kt} \implies P(t) = 25 + De^{kt}.$$

What is  $D$ , the integration constant? We need to use the initial condition  $P(0) = 100$ :

$$P(0) = 100 = 25 + D \implies D = 75.$$

Thus, our total solution is:

$$P(t) = 25 + 75e^{kt}.$$

- (c) By part (b), since  $k < 0$ , the exponential term decays and  $P(t) \rightarrow 25$  as  $t \rightarrow \infty$ . This also makes physical sense because the Pop-tart will eventually cool to the surrounding temperature if left out.

- 10 2. In a cylindrical tube, the work done to expand a gas using **constant pressure**  $\rho$  is  $W = \rho V$ , where  $V$  is the change in volume. That is,  $V = V_2 - V_1$ , where  $V_1$  is the starting volume and  $V_2$  is the ending volume.

If the pressure applied is non-constant, that is, we now have  $\rho(v)$ , where  $v$  is the current volume of the gas, **derive an integral expression for the work done** to expand the gas from a volume of  $V_1$  to  $V_2$ .

*Hint:* recall how we derived the integral expression for  $W = FD$ . We split  $D$  into intervals of width  $\Delta x$  and used a local approximation. Do something similar for intervals of width  $\Delta v$ .

**Solution:** If we consider partitioning the interval  $[V_1, V_2]$  into  $n$  tiny subintervals of width  $\Delta v = (V_2 - V_1)/n$ , where each starts at  $v_i$ . Then, we can make the approximation that the pressure  $\rho(v)$ , is close enough to constant in each subinterval, so the work done in the  $i$ th interval is:  $W_i = \rho(v_i)\Delta v$ . We now just need to sum the total work done:

$$W \approx \sum_{i=1}^n W_i = \sum_{i=1}^n \rho(v_i)\Delta v.$$

If we take the limit as  $n \rightarrow \infty$ , we finally get the integral, as the above is just a Riemann sum.

$$W = \lim_{n \rightarrow \infty} \sum_{i=1}^n \rho(v_i)\Delta v = \int_{V_1}^{V_2} \rho(v) dv.$$

Note, this is exactly analogous to the  $W = Fd$  case.

- 10 3. (a) Suppose  $\sum_{n=1}^{\infty} a_n$  diverges and  $\sum_{n=1}^{\infty} b_n$  diverges. Does  $\sum_{n=1}^{\infty} (a_n + b_n)$  necessarily diverge? If so, prove it. If not, provide a counterexample and explain.
- 10 (b) Find the values of  $\alpha$  for which the series converges. Then, find the sum of the series for those values of  $\alpha$ .

$$\sum_{n=1}^{\infty} \frac{(\alpha + 1)^n}{4^n}.$$

**Solution:** Note this question was taken **directly** from lab.

- (a) No, the sum of two divergent series is not necessarily divergent. For example, let  $\sum_{n=1}^{\infty} a_n = 1/n$ , which is a divergent harmonic series, and let  $\sum_{n=1}^{\infty} b_n = -1/n$ , another divergent harmonic series. Their sum is equal to the series in which every term is zero, however, so it converges.
- (b) This series can be thought of as a geometric series with ratio  $r = (\alpha + 1)/4$ . The series converges if and only if  $|r| < 1$ , which happens if and only if  $-5 < \alpha < 3$ . In this case, the sum of the series is

$$\frac{a}{1-r} = \frac{(\alpha + 1)/4}{1 - (\alpha + 1)/4} = \frac{\alpha + 1}{3 - \alpha}, \quad \alpha \in (-5, 3).$$

- 15 4. Determine the interval of convergence for the following series:

$$\sum_{n=1}^{\infty} \frac{(3x-2)^n}{n3^n}.$$

*Hint:* do not forget to check the endpoints.

**Solution:** We first use the ratio test to determine the interval of convergence. Denote  $a_n$  as the  $n$ th term in the series:

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(3x-2)^{n+1}}{(n+1)3^{n+1}} \cdot \frac{n3^n}{(3x-2)^n} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{(3x-2)}{3} \cdot \frac{n}{n+1} \right| \\ &= \frac{|3x-2|}{3} \lim_{n \rightarrow \infty} \left| \frac{n}{n+1} \right| \\ &= \frac{|3x-2|}{3}. \end{aligned}$$

Note, we need this quantity to be less than 1, so we have:

$$\begin{aligned} \frac{|3x-2|}{3} &< 1 \\ -1 &< \frac{3x-2}{3} < 1 \\ -3 &< 3x-2 < 3 \\ -1 &< 3x < 5 \\ -\frac{1}{3} &< x < \frac{5}{3}. \end{aligned}$$

Now, we must check the endpoints. First, consider  $x = -1/3$ , then our series turns into:

$$\sum_{n=1}^{\infty} \frac{[3(-1/3) - 2]^n}{n3^n} = \sum_{n=1}^{\infty} \frac{(-3)^n}{n3^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}.$$

Note, we know, by the Alternating Series Test, that this converges.

Now, consider  $x = 5/3$ , leaving our series at:

$$\sum_{n=1}^{\infty} \frac{[3(5/3) - 2]^n}{n3^n} = \sum_{n=1}^{\infty} \frac{(3)^n}{n3^n} = \sum_{n=1}^{\infty} \frac{1^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n}.$$

This is the Harmonic series, which diverges. Thus, our interval of convergence is  $x \in [-1/3, 5/3)$ .

5. Consider the following function and interval:

$$f(x) = \sin(x), \quad x \in [-\pi, \pi].$$

- 15 (a) Derive, from the definition, the third-order (cubic) Taylor series approximation,  $T_3(x)$ , of  $f(x)$ , around the point  $a = 0$ . Note, this is also the Maclaurin series.
- 15 (b) Using Taylor's theorem, find a bound for the maximum error  $|T_3(x) - f(x)|$  between the approximation and  $f(x)$  on the interval  $[-\pi, \pi]$ .
- 5 (c) Suppose that for a practical application, we require that the error not exceed on significant digit (1/10) on this interval. Based on your bound in the previous problem, will the third order  $p_3(x)$  approximation ensure the required accuracy?

**Solution:**

$$f'(x) = \cos(x), \quad f''(x) = -\sin(x), \quad f'''(x) = -\cos(x)$$

$$\begin{aligned} T_3(x) &= 0 + \frac{\cos(0)}{1!}x + 0 - \frac{\cos(0)}{3!}x^3 \\ &= x - \frac{x^3}{6} \end{aligned}$$

$$f^{(4)}(x) = \sin(x)$$

$$\max_{x \in [-\pi, \pi]} |\sin(x)| = M = 1$$

$$|T_3(x) - f(x)| \leq \frac{M\pi^4}{4!} = \frac{\pi^4}{24}$$

Because  $\frac{\pi^4}{24} > \frac{1}{10}$ , the  $p_3(x)$  approximation will not guarantee the accuracy requirement.

6. Consider the following vector valued function:

$$\mathbf{r}(t) = \langle \cos 3t, \sin 3t, 2t \rangle.$$

- 10 (a) Parameterize  $\mathbf{r}(t)$  with respect to arc length  $s(t)$  from  $t = 0$ . That is, write  $\mathbf{r}(s(t))$ .
- 10 (b) Without computing  $\mathbf{N}(t)$ , the unit normal vector, or  $\mathbf{T}(t)$ , the unit tangent vector, what is  $\mathbf{T} \cdot \mathbf{N}$ ?
- 5 (c) Write down an expression for (but do not compute) the curvature  $\kappa(t)$ . What does this geometrically represent?

**Solution:**

- (a) Recall that  $s(t) = \int_0^t \|\mathbf{r}'(x)\| dx$ , so we need to compute  $\mathbf{r}'(t) = \langle -3 \sin 3t, 3 \cos 3t, 2 \rangle$ . We can take the magnitude of this:

$$\|\mathbf{r}'(t)\| = \sqrt{(-3 \sin 3t)^2 + (3 \cos 3t)^2 + (2)^2} = \sqrt{9 + 4} = \sqrt{13}.$$

Thus, we now can compute the arc-length, which is just the integral of a constant:

$$s(t) = \int_0^t \|\mathbf{r}'(x)\| dx = \int_0^t \sqrt{13} dx = \sqrt{13}t.$$

Since we have  $s = \sqrt{13}t$ , this implies that  $t = s/\sqrt{13}$ , which we can plug into our original function:

$$\mathbf{r}'(s) = \left\langle 3 \cos \frac{s}{\sqrt{13}}, 3 \sin \frac{s}{\sqrt{13}}, \frac{2s}{\sqrt{13}} \right\rangle.$$

- (b) By the definition of these two vectors, they must be orthogonal, and therefore  $\mathbf{T} \cdot \mathbf{N} = 0$ .
- (c) Many different formulae were acceptable, including:

$$\kappa(t) = \frac{\|\mathbf{r}' \times \mathbf{r}''\|}{\|\mathbf{r}'\|^3}, \quad \kappa(t) = \left| \frac{dT}{ds} \right|, \quad \kappa(t) = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|}.$$

In a hand-wavy sense, curvature measures how quickly the tangent line is changing, or how “curvy” a curve is. Notice we want to measure this as a function of arc length because we don’t want the parameterization speed to affect this measure of curviness.

- 10 7. (a) Find the equation for a plane through the point  $P_0 = (1, 2, 3)$  that contains the directions described by the following two vectors:

$$\mathbf{v}_1 = \langle 2, 2, 2 \rangle, \quad \mathbf{v}_2 = \langle -1, 3, 2 \rangle.$$

- 10 (b) Find the distance from the plane from part (a) to the point  $P_1 = (5, 0, 0)$ .

**Solution:**

- (a) We need the normal vector to the plane, which is orthogonal to both  $\mathbf{v}_1$  and  $\mathbf{v}_2$  and therefore obtained by  $\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2$ :

$$\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & 2 \\ -1 & 3 & 2 \end{vmatrix} = \langle -2, -6, 8 \rangle.$$

We know that the equation of a plane is then:

$$\mathbf{n} \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0,$$

and plugging in our actual values yields:

$$-2(x - 1) - 6(y - 2) - 8(z - 3) = 0.$$

- (b) Denote  $\mathbf{b}$  to be the vector from an arbitrary point on the plane to our point  $P_1 = (5, 0, 0)$ . We know then, that the distance from  $P_1$  to the plane is just the magnitude of the projection of  $\mathbf{b}$  onto the normal vector.

For convenience, rewrite the plane equation above as  $-2x - 6y - 8z + 28 = 0$ , which we'll call  $ax + by + cz + d = 0$ , then:

$$D = \|\text{proj}_{\mathbf{n}} \mathbf{b}\| = \text{comp}_{\mathbf{n}} \mathbf{b} = \frac{|\mathbf{n} \cdot \mathbf{b}|}{\|\mathbf{n}\|} = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}} = \frac{|-2(5) + 28|}{\sqrt{2^2 + 6^2 + 8^2}}.$$



- 15 8. In class, we proved the following statement:

$\|\mathbf{r}(t)\| = c$ , for all  $t$ , where  $c$  is a constant, then  $\mathbf{r}(t) \perp \mathbf{r}'(t)$ , that is, they are orthogonal.

On the previous exam, you were asked to prove the converse:

If  $\mathbf{r}(t) \perp \mathbf{r}'(t)$ , for all  $t$ , then  $\|\mathbf{r}(t)\| = c$ , a constant.

**Choose one of these statements** and prove it.

**Solution:**

- (a) We could prove both of these for  $\mathbf{r}$  in 2D or 3D, but it's just as easy in general vector form. This proof was done in class and is also in the book. We start with supposing that  $\|\mathbf{r}\| = c$  and square both sides, yielding  $\|\mathbf{r}\|^2 = c^2$ . We now make the observation that the left hand side is simply the dot product. That is,  $\|\mathbf{r}\|^2 = \mathbf{r} \cdot \mathbf{r}$ . Now, we take a derivative:

$$\frac{d}{dt} \{\mathbf{r}(t) \cdot \mathbf{r}(t)\} = \frac{d}{dt} \{c^2\} = 0.$$

Thus, we have  $\frac{d}{dt} \{\mathbf{r} \cdot \mathbf{r}\} = 0$ , since the derivative of a constant is 0, but we can now use the product rule for dot products:

$$0 = \frac{d}{dt} \{\mathbf{r} \cdot \mathbf{r}\} = \mathbf{r}(t) \cdot \mathbf{r}'(t) + \mathbf{r}'(t) \cdot \mathbf{r}(t) = 2\mathbf{r}(t) \cdot \mathbf{r}'(t).$$

Notice, this implies that  $\mathbf{r} \cdot \mathbf{r}' = 0$ , which is exactly that they are perpendicular and we're done the proof.

- (b) We start with the assumption that  $\mathbf{r} \perp \mathbf{r}'$ , which means that their dot product is 0:

$$\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0,$$

and we now can integrate. Noting, by the chain rule that  $\int f(t)f'(t) dt = f(t)^2/2$ . If you don't believe this, consider letting  $u = f(t)$ , then  $du = f'(t)dt$ . Integrating both sides, noting that the integral of 0 is a constant, we get:

$$\mathbf{r}(t) \cdot \mathbf{r}'(t) dt = \mathbf{r}(t) \cdot \mathbf{r}(t) = c,$$

but note, we can manipulate the left hand side:

$$\mathbf{r}(t) \cdot \mathbf{r}(t) = \|\mathbf{r}(t)\|^2,$$

so we have

$$\|\mathbf{r}(t)\|^2 = c \implies \|\mathbf{r}(t)\| = k,$$

where  $k = \sqrt{c}$ . Thus, we're done the proof, as we've shown the magnitude of  $\mathbf{r}$  is constant for all  $t$ .

9. Consider the following function:

$$f(x, y) = x^2 + \frac{y^2}{2^2}.$$

- 10 (a) Write a vector equation for the **normal line** at the point (1,2,2).
- 5 (b) Draw the level curves for  $f(x, y)$ .

**Solution:**

- (a) We know that the gradient at a point provides the normal direction, we first rewrite our equation as  $F(x, y, z) = x^2 + y^2/4 - z = 0$  and take the gradient  $\nabla F = \langle 2x, y/2, -1 \rangle$ . If we evaluate at our point, we find  $\nabla F(1, 2, 2) = \langle 2, 1, -1 \rangle$ . Thus, this is our normal direction.

The vector equation of a line is  $\mathbf{r} = \mathbf{r}_0 + \mathbf{v}t$ , where  $\mathbf{r}_0$  is our starting point, in this case  $\mathbf{r}_0 = \langle 1, 2, 2 \rangle$ , and  $\mathbf{v}$  is the direction of motion, which we just computed to be  $\mathbf{v} = \langle 2, 1, -1 \rangle$ . Thus, our line is:

$$\mathbf{r}(t) = \langle 1, 2, 2 \rangle + \langle 2, 1, -1 \rangle t.$$

- (b) The level curves are obtained by setting  $f(x, y) = k$  for various constants  $k$ . In this case, we see that we get  $x^2 + y^2/2^2 = k$ , which is just an ellipse with varying radius, that is "fatter" in the  $y$ -direction.

10. Consider the following function:

$$f(x, y) = e^{xy}.$$

- 5 (a) Write down equations for  $x$  and  $y$  describing the coordinate change from  $x, y$  to  $r, \theta$ , where  $r, \theta$  correspond to cylindrical coordinates.
- 10 (b) **Using the chain rule**, compute  $\partial f(r, \theta)/\partial r$  and  $\partial f(r, \theta)/\partial \theta$  by considering  $x(r, \theta)$  and  $y(r, \theta)$ .

**Solution:**

(a) The change of coordinates is just:  $x = r \cos \theta, y = r \sin \theta$ .

(b) The chain rule gives us:

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial r} = r \sin \theta e^{(r \cos \theta)(r \sin \theta)} \cos \theta + r \cos \theta e^{(r \cos \theta)(r \sin \theta)} \sin \theta,$$

since  $\partial f/\partial x = ye^{xy} = r \sin \theta e^{(r \cos \theta)(r \sin \theta)}$  and similar for  $\partial f/\partial y$ . Also,  $\partial x/\partial r = \cos \theta$ , and  $\partial y/\partial r = \sin \theta$ . You did not need to simplify.

We get a very similar expression for  $\theta$ :

$$\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial \theta} = r \sin \theta e^{(r \cos \theta)(r \sin \theta)} (-r) + r \cos \theta e^{(r \cos \theta)(r \sin \theta)} r,$$

where the only difference is that we now have  $\partial x/\partial \theta = -r$  and  $\partial y/\partial \theta = r$  also.

- 15 11. Let  $f(x, y) = -x^2 + y^2$ . Find  $x$ - $y$  points that produce the maximum value of  $f$  subject to the constraint that  $x$  and  $y$  pairs also satisfy  $g(x, y) = x^2 + y = 0$ .

**Solution:** Use Lagrange multipliers:

$$\nabla f = \begin{bmatrix} -2x \\ 2y \end{bmatrix}$$

$$\nabla g = \begin{bmatrix} 2x \\ 1 \end{bmatrix}$$

$$\nabla f = \lambda \nabla g$$

$$2 \begin{bmatrix} -x \\ y \end{bmatrix} = \lambda \begin{bmatrix} 2x \\ 1 \end{bmatrix}$$

$$-2x = \lambda 2x \implies \lambda = -1$$

$$2y = -1 \implies y = -1/2, \quad (\text{from constraint equation})$$

$$x = \frac{1}{\sqrt{2}}$$