

Midterm Exam I

Math 1320 - Engineering Calculus II

February 27, 2015

Answer each question completely in the area below. Show all work and explain your reasoning. If the work is at all ambiguous, it is considered incorrect. No phones, calculators, or notes are allowed. Anyone found violating these rules will be asked to leave immediately. Point values are in the square to the left of the question. **If there are any other issues, please ask the instructor.**

By signing below, you are acknowledging that you have read and agree to the above paragraph, as well as agree to abide University Honor Code:

Name: _____
Signature: _____
uID: _____

Solutions

| Question | Points | Score |
|----------|--------|-------|
| 1 | 15 | |
| 2 | 10 | |
| 3 | 5 | |
| 4 | 15 | |
| 5 | 10 | |
| 6 | 10 | |
| 7 | 15 | |
| 8 | 20 | |
| Total: | 100 | |

Note: There are 8 questions total on the exam, as well as a challenging bonus question at the end of the exam that is worth trying if and only if you have extra time.

- 10 1. (a) Find the volume of the solid generated by revolving the region bounded by $y = x^3$, $y = 0$, $x = 2$ around the line $x = 3$ **using the method of shells**.

Solution: Here, you were instructed to use the method of shells. The axis of rotation is a vertical line, and the method of shells says that our slices are parallel to the axis of rotation, meaning we also have vertical slices. Thus, we are integrating dx .

To compute the method of shells, we need two components: the radius of the shell and the height. Here, if we consider what the radius from $x = 3$, the axis of rotation, to some arbitrary point x is, it's simply $r = 3 - x$. The height is then just the height created by the function, $h = x^3$, and our x ranges from 0 to 2. Thus, we have

$$V = \int_0^2 2\pi r h dx = 2\pi \int_0^2 (3-x)x^3 dx = 2\pi \int_0^2 [3x^3 - x^4] = 56\pi/5.$$

- 5 (b) Set up (but **do not evaluate**) the integral for computing the same volume **using the method of washers**.

Solution: The washer method specifies that we take slices *perpendicular* to the axis of rotation, meaning we must consider horizontal slices, and therefore integrate dy .

To compute the washer method, we need two components: the inner radius and the outer radius. The inner radius is just determined by the distance from our axis of rotation to the right side of the object we're integrating, so $r_{in} = 3 - 2 = 1$. The outer radius is determined by the distance from the axis of rotation to the left side, which is determined by the function $y = x^3$. Notice, we want the x value an expression of y , so $x = \sqrt[3]{y}$ and therefore $r_{out} = (3 - \sqrt[3]{y})$. Lastly, we just need the range of y we're integrating over, and it clearly starts at $y = 0$ but the end is when $x = 2$ or $y = 8$, thus:

$$V = \int_0^8 \pi [r_{out}^2 - r_{in}^2] dy = \int_0^8 \pi [(3 - \sqrt[3]{y})^2 - 1] dy.$$

If you evaluated this, you would get the same result as part (a) but this final step was not necessary.

- 8 2. (a) Coulomb's Law states that the electrical force F , between two charged objects is inversely proportional to the square of the distance between the two objects:

$$F(r) = \frac{kq_1q_2}{r^2},$$

where k is a constant, q_1, q_2 are the charges of the objects, and r is the distance the objects are apart.

Calculate the work required to move the charged objects from a distance of 5m apart to a distance of 1m apart. Your answer should be in terms of k, q_1, q_2 .

Solution: This question was directly from the review. We've established many times that work is the integral of force. The reason for this, is that if force F and distance d are constant, then we know that the work $W = Fd$. Thus, even if the force (or distance) is not constant, we can slice our problem into tiny slices who effectively change so little that we can consider the force (and distance) constant within the slice.

In this problem, we can consider tiny slices of length Δr occurring at r , meaning the force at this slice is $F(r)$. Thus, the work done to move the particles closer together a tiny distance Δr when they're currently r apart is $W_{\Delta r} = F(r)\Delta r$. If we consider the limit of $\Delta r \rightarrow 0$ and adding up infinitely many of these, we get our classic integral:

$$W = \int_5^1 F(r) dr = \int_5^1 \frac{kq_1q_2}{r^2} dr = \left[-\frac{kq_1q_2}{r} \right]_{r=5}^{r=1} = -\frac{4}{5}kq_1q_2.$$

If you're curious, notice that you get a different sign (energy put in or taken out) depending if the particles attract (different charges) or repel (same charge).

- 2 (b) Why can't we use $W = Fd$, that is, work = force \times distance, in this case? How does this relate to your answer to part (a)?

Solution: See the answer above. Effectively, this is only true when both force and distance are constant. The best answer would have been restating the result from lab, that actually $W = F_{\text{ave}}d$.

- 5 3. Verify that $y(x) = 4e^{3x} \sin x$ is a solution to the following initial value problem:

$$\frac{dy}{dx} = 3y + 4e^{3x} \cos x, \quad y(0) = 0.$$

Solution: To verify that this is indeed a solution to the differential equation, we simply have to plug it into the differential equation and check that the left hand side is equal to the right. In this case, we just have $y'(x) = dy/dx$ on the left hand side, so we just have to check that the derivative of the supposed solution is equal to $3y + 4e^{3x} \cos x$, so notice, by the product rule:

$$\frac{dy}{dx} = 3 \underbrace{4e^{3x} \sin x}_{=y(x)} + 4e^{3x} \cos x.$$

Thus, we indeed recover the right hand side of the differential equation, meaning this is a solution. We also need to check that this satisfies the initial condition, which, since $\sin 0 = 0$, we see that $y(0) = 0$ and therefore we have a solution to the initial value problem.

- 10 4. (a) Solve the following differential equation (that is, get a form $y(x) = \dots$):

$$\frac{dy}{dx} = xy - 2x - y + 2.$$

Solution: Notice that this is nearly identical to the problem on the review sheet. We can factor:

$$\frac{dy}{dx} = (x - 1)(y - 2).$$

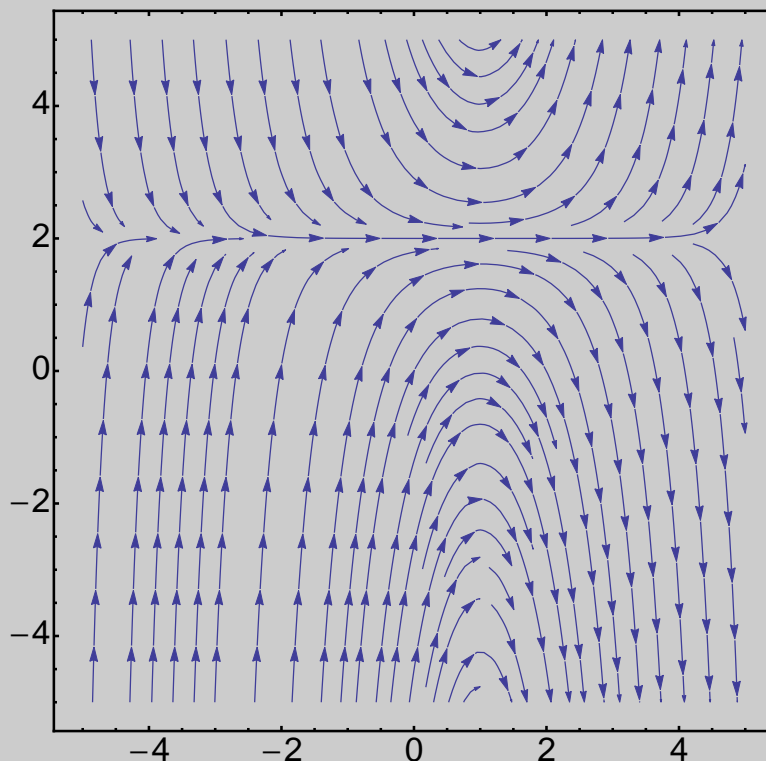
From here, it's easy to see how to separate and integrate:

$$\begin{aligned} \int \frac{dy}{y - 2} &= \int (x - 1) dx \\ \ln |y - 2| &= x^2 - x + K \\ |y - 2| &= e^{x^2 - x + K} \\ y(x) &= Ce^{x^2 - x} + 2. \end{aligned}$$

Notice, we got rid of the absolute value issue by absorbing the \pm into our constant C as we have done many times.

- 5 (b) Draw a slope field for the differential equation from part (a).

Solution: While this isn't completely like a slope field we have done before, it's easy to figure out. First, notice where derivative is zero: $x = 1$, $y = 2$. We know the slope must be flat here. This basically produces 4 different regions where we can check whether the slope is positive or negative and draw accordingly. The result should look something like the following:



- 10 5. **Choose one the following series** and prove whether it diverges or converges. In either case, make sure to state the requirements of the test you're using and why they are satisfied. Also, make sure to clearly indicate which series you are answering for.

$$(1) \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}.$$

$$(2) \sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{n^2 + 1}.$$

Solution: The purpose of this problem was to force you to use a test with some advanced criteria. While there may have been other ways to do this, I think the most suggestive tests were integral and alternating series (respectively).

- (1) This looks very, very similar to the integral test problem we had on a quiz, particularly because the integral of this function is relatively straightforward.

For the integral test, we consider $f(x) = 1/(x \ln^2 x)$ since $f(n) = a_n$. We need to check several conditions:

- (i) $f(x)$ is positive: since we're only considering $x \geq 2$, we see that the \ln is squared and therefore always positive, meaning this function is always positive for $x \geq 2$.
- (ii) $f(x)$ is continuous: we see that the only discontinuities are when $x = 0$, but we're only considering $x \geq 2$, so this is not a worry.
- (iii) $f(x)$ is decreasing: this is definitely the hardest condition to verify, but we can simply compute the derivative:

$$f'(x) = -\frac{\ln(x) + 2}{x^2 \ln^3(x)} \leq 0, \quad \text{when } x \geq 2,$$

thus the function is decreasing.

We've therefore verified all of the conditions of the integral test meaning we can actually examine the behavior of the integral, note that this is the same u -substitution: $u = \ln x$, which yields

$$\begin{aligned} \int_2^{\infty} \frac{dx}{x \ln^2 x} &= \lim_{t \rightarrow \infty} \int_2^t \frac{dx}{x \ln^2 x} \\ &= \lim_{t \rightarrow \infty} \left[-\frac{1}{\ln x} \right]_{x=2}^{x=t} \\ &= \frac{1}{\ln 2}. \end{aligned}$$

Thus, this integral converges. The integral test (once all the conditions are verified), says that the convergence (or divergence) of the series matches that of the integral. In other words, since the integral converges, so does the series.

- (2) This is a classic case of an application of the alternating series test, which states that if we have a series of the form:

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n,$$

then we can examine the behavior of b_n to prove this series converges. Here, we see that $b_n = n/(n^2 + 1)$. The two conditions are:

- (i) $b_{n+1} \leq b_n$, or the series is decreasing. Again, we'll consider $f(x) = x/(x^2 + 1)$ and take a derivative:

$$f'(x) = \frac{2 - x^2}{(x^2 + 1)^2} < 0 \quad \text{when } x > 2.$$

Notice, this function isn't decreasing immediately, but it is eventually ($x > 2$), which is all we care about.

- (ii) $b_n \rightarrow 0$ as $n \rightarrow \infty$. Notice that the power of n in the bottom is larger than that in the top. Preferrably, you could divide everything by n^2 to see that you get:

$$\lim_{n \rightarrow \infty} \frac{n}{n^2 + 2} = \lim_{n \rightarrow \infty} \frac{\frac{1}{n}}{\frac{2}{n^2} + 1} = 0,$$

but any argument that talks about the relative powers sufficed.

Since the series satisfies all of the conditions of the alternating series test, it must converge.

- 10 6. Determine for which values of p the following series converges:

$$\sum_{n=1}^{\infty} \frac{n^p}{2 + n^3}.$$

Hint: what other series does this look like for large n ?

Solution: Notice, as $n \rightarrow \infty$, the 2 becomes insignificant and this looks like n^{p-3} . While you could use the limit comparison test, this is actually even simpler since you can compare directly:

$$\frac{n^p}{2 + n^3} < \frac{n^p}{n^3} \text{ for large } n.$$

Thus, we can actually just use the direct comparison test, which states that if $b_n \leq a_n$ and $\sum a_n$ converges, then $\sum b_n$ converges. Here, our $\sum b_n$ is our original series and $a_n = n^{p-3}$. We can rewrite this in a more suggestive form:

$$\frac{n^p}{n^3} = \frac{1}{n^{3-p}}.$$

Notice, this is exactly a p series, meaning this converges when the exponent is > 1 or $3 - p > 1$, which implies that $p < 2$.

- 10 7. (a) Find the Taylor series for the following function around $x = 2$:

$$g(x) = (x - 2)^2 e^x.$$

Solution: This problem is easiest if we consider the Taylor series for e^x around $a = 2$ separately. By definition, our Taylor series around $x = a$ is

$$f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \frac{f'''(a)}{3!}(x - a)^3 + \dots$$

Notice here, every derivative of $f(x) = e^x$ is just e^x . That is, $f^{(n)} = e^x$ regardless of n and therefore $f^{(n)}(2) = e^2$. Thus, this series becomes:

$$e^2 + e^2(x - 2) + \frac{e^2}{2}(x - 2)^2 + \frac{e^2}{3!}(x - 2)^3 + \dots = \sum_{n=0}^{\infty} \frac{e^2}{n!}(x - 2)^n.$$

Now, to consider the series for $(x - 2)^2 e^2$ around $a = 2$, then notice we can just multiply the above series by $(x - 2)^2$, yielding:

$$e^2(x - 2)^2 + e^2(x - 2)^3 + \frac{e^2}{2}(x - 2)^4 + \frac{e^2}{3!}(x - 2)^5 + \dots = \sum_{n=0}^{\infty} \frac{e^2}{n!}(x - 2)^{n+2}.$$

- 5 (b) What is the radius of convergence for the series from part (a)?

Solution: We can consider rewriting the series above as the following for convenience:

$$\sum_{n=0}^{\infty} \frac{e^2}{n!}(x - 2)^{n+2} = \sum_{n=2}^{\infty} \underbrace{\frac{e^2}{(n - 2)!}}_{=a_n}(x - 2)^n.$$

Now, we simply apply the ratio test, which states that we examine the behavior of:

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{e^2(x - 2)^{n+1}}{(n + 1 - 2)!} \cdot \frac{(n - 2)!}{e^2(x - 2)^n} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{(n - 2)!}{(n + 1 - 2)!}(x - 2) \right| \\ &= \lim_{n \rightarrow \infty} \underbrace{\left| \frac{1}{n - 1} \right|}_{\rightarrow 0} |x - 2| \end{aligned}$$

Notice, the above limit goes to 0 regardless of x , and therefore is definitely < 1 to appease the ratio test. Thus, this converges for all x , or $R = \infty$.

- 10 8. (a) Derive the Maclaurin series for the following function:

$$f(x) = \ln(1 + x).$$

Hint: It may be useful to consider the function $\frac{1}{1+x}$ and how it is related to $f(x)$.

Solution: This is very similar to many examples we did in class, in lab, and on the homework. Notice that we can think of $1/(1+x)$ as a geometric series:

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = 1 - x + x^2 - x^3 + x^4 + \dots = \sum_{n=1}^{\infty} (-x)^{n-1}, \quad |x| < 1.$$

We now integrate the above, to yield:

$$\int \frac{1}{1+x} = \ln(1+x)$$

which tells us that our desired series is simply

$$\int (1 - x + x^2 - x^3 + x^4 + \dots) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}.$$

- 5 (b) Using the result from part (a), find the Maclaurin series of the following function:

$$f(x) = \ln(1 - x^2).$$

Solution: Notice that if we consider $\ln(1+y)$, we know the series from part (a), but we can consider $y = -x^2$, which gives us exactly the Taylor series we hope to find. Thus, just plug in $-x^2$ everywhere you see an x in the series above to yield:

$$(-x^2) - \frac{(-x^2)^2}{2} + \frac{(-x^2)^3}{3} - \frac{(-x^2)^4}{4} + \dots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(-x^2)^n}{n},$$

if you want, you could clean this up to be

$$-x^2 - \frac{x^4}{2} - \frac{x^6}{3} - \frac{x^8}{4} - \dots = -\sum_{n=1}^{\infty} \frac{x^{2k}}{k}.$$

- 5 (c) What is the radius and interval of convergence of each of the series from part (a) and (b)? Why?

Solution: Notice that our original interval of convergence from part (a) was $|x| < 1$ (not including endpoints because a geometric series does not converge when $r = 1$), and integrating to get part (b) does not change this. Thus, the radius and interval are the same.

For part c, we just consider now $|-x^2| < 1$, which in fact, produces the same interval and radius as well.

9. **(Extra credit)** Taylor series are tremendously useful in numerical analysis, the study (and practice) of numerically approximating mathematical problems.

One particular use is that they provide a very natural way to compute derivatives called *finite differences*. By expanding f in a Taylor series around x and then evaluating at the appropriate values, argue what the following quantities represent by examining the resulting leading term:

- (a) $\frac{1}{2h} [f(x+h) - f(x-h)]$.
 (b) $\frac{1}{h^2} [f(x+h) + f(x-h) - 2f(x)]$.

Solution: Consider $T(z)$, the Taylor expansion of the above around some point x . I chose z just to avoid confusion with x . We typically treat x as a variable, but here we're treating it as a single point. This is confusing at first, but actually leads to a prettier result. We can then take our general Taylor series:

$$T(z) = f(x) + f'(x)(z-x) + \frac{f''(x)}{2}(z-x)^2 + \frac{f'''(x)}{6}(z-x)^3 + \dots$$

Now, consider evaluating at the quantities in each of the problems. That is, we consider $T(x+h)$. Notice, that this just means that $z-x = h$. We can think of this as the size of the step we are taking away. Now, plugging this in:

$$\begin{aligned} \frac{1}{2h} [f(x+h) - f(x-h)] &= \frac{1}{2h} [T(x+h) - T(x-h)] \\ &= \frac{1}{2h} \left[\left(\cancel{f(x)} + f'(x)h + \frac{f''(x)}{2}h^2 + \dots \right) \right. \\ &\quad \left. - \left(\cancel{f(x)} - f'(x)(-h) + \frac{f''(x)}{2}(-h)^2 + \dots \right) \right] \\ &= \frac{1}{2h} 2f'(x)h + \dots \\ &= f'(x) + \dots \end{aligned}$$

Several terms cancel and we examine the leading term, which simply resulted in $f'(x)$. As with all of our Taylor series analysis, we're assuming the remainder terms are negligible and therefore, this gives us an approximation of the derivative of $f'(x)$.

Similarly, for the second expression:

$$\begin{aligned} \frac{1}{h^2} [f(x+h) + f(x-h) - 2f(x)] &= \frac{1}{h^2} [T(x+h) + T(x-h) - 2T(x)] \\ &= \frac{1}{h^2} \left[\left(\cancel{f(x)} + f'(x)h + \frac{f''(x)}{2}h^2 + \frac{f'''(x)}{6}h^3 + \dots \right) \right. \\ &\quad \left. + \left(\cancel{f(x)} + f'(x)(-h) + \frac{f''(x)}{2}(-h)^2 + \frac{f'''(x)}{6}(-h)^3 + \dots \right) \right. \\ &\quad \left. - \left(\cancel{f(x)} + f'(x)(0) + \frac{f''(x)}{2}(0)^2 + \frac{f'''(x)}{6}(0)^3 + \dots \right) \right] \\ &= \frac{1}{h^2} f''(x)h^2 + \dots \\ &= f''(x) + \dots \end{aligned}$$

Again, after cancelling we see that the leading order term is simply the second derivative $f''(x)$.

Notice how useful this is for two particular reasons:

- (1) Often, we have functions that are simply too difficult to take the derivative of by hand. This says that to approximate the derivative at x , we simply have to evaluate f at two different points, a much easier computation.
- (2) In physics and engineering, it's often the case that we experimentally measure a set of data but do not know the underlying function that produced it. Therefore, we may have x_1, \dots, x_n , some sample data with spacing h and now we can approximate the derivative of our data by using the formula we just derived. That is: we don't even need to know what $f(x)$ is, we just need $f(x)$ evaluated at a few points. This is extremely powerful.

Also note that the terms included in the \dots actually can provide us with an estimate of the error of these approximations.