

Answer the questions neatly and clearly on another sheet of paper (or typeset, preferably) and explain your answers. If you answer all of the questions correctly, you will receive 5% total extra credit on your midterm exam 1 score. Otherwise, the amount of extra credit will scale proportionally to the number of correct problems answered, that is, there will be no partial credit. If the work is at all ambiguous or unclear, it will be considered incorrect. I'll be happy to discuss any of the problems with you, but I will not provide you with answers. This will be due **March 23**.

1. In this problem, we'll investigate what our calculator does when you press the $\sqrt{\quad}$ button. How would *you* compute the square root of a number? If you answered "use a calculator", remember that someone had to tell (program) the calculator how to do so. Notice, if we started with 6 and repeatedly pressed the $\sqrt{\quad}$ button:

6
 1.56508...
 1.**251**03...
 1.**118**5...
 1.**057**59...
 1.**028**39...

It seems like the fractional part gets approximately halved every time we take another square root. Maybe this could give us a hint as to what our calculator is actually doing? Does the "halving" phenomenon occur for other numbers? If so, which, and why? Follow the steps below and you'll find an answer to this mystery!

Non-calculus method

- (a1) Let ε be a small, positive number. Our observation that "the fractional part is roughly halved at each stage" can be expressed mathematically as:

The square root of $1 + \varepsilon$ is approximated by $\sqrt{1 + \varepsilon} \approx \underline{\hspace{2cm}}$.

Fill in the blank.

- (a2) Give an informal proof/justification for this approximation by analyzing what you get when you square both sides.
- (a3) If $x = 1 + \varepsilon$, another way to describe this phenomenon is:

The square root of x is approximated by $\sqrt{x} \approx \underline{\hspace{2cm}}$.

Again, fill in the blank with an expression in terms of x . Can you justify this one as well?

Taylor series method

Taylor series provide us a much more elegant way to understand when and why this is true.

- (b1) Use the Taylor series of $f(x) = \sqrt{x}$ around $a = 1$ to give a more formal justification to your approximation from (a3). You may assume that $f(x)$ is equal to its Taylor series.
- (b2) Now, use Taylor's Inequality to quantify how good of an approximation (a1) is, assuming that x is within $1/100$ of 1. Also, how good of an approximation is that of (a3)?
- (b3) Based on these answers, how do you think your calculator computes square roots?

2. The Trouble with Tripples

You are the captain of the Starship Exitprize and Tripples have invaded the hold of your ship. You are concerned enough to ask Spook, the Vulgar, for an analysis. As usual, Spook takes a superior attitude. Vulgars are experts with infinite series, although they never developed calculus. Hoping to teach you something, he gives his report in a series of riddles, to which you must supply the answers. Supply answers to questions (a) – (d) below, and then go Spook one better by showing him how Earthlings would use calculus to solve the problem in parts (e) – (h).

Spook speaks

Let y_0 be the population of tripples at some initial time $t = 0$, Captain. Then let y_n be the population after n intervals of length Δt have elapsed, i.e. at $t = n\Delta t$. If you allow one interval to applse, the population will have increased during the next interval by

$$\Delta y_n = y_{n+1} - y_n.$$

Suppose Δy_n is proportional to both y_n and Δt :

$$\Delta y_n = k\Delta t y_n,$$

where k is the growth rate. Tripples populate at a very fast rate, meaning their death rate is negligible.

- Give an expression for y_1 , the total number of Tripples after one interval Δt , in terms of y_0 , the initial population at $t = 0$.
- Give expressions for y_2, y_3 and y_n , the total after 2, 3 or n intervals respectively, in terms of y_0 . Make your expression as compact as possible and factor out y_0 . Rewrite your expression for y_n in terms of the total elapsed time t by setting $\Delta t = t/n$.
- Now expand the expression for y_n you found in part (b) by using the Binomial Theorem. In order to get better resolution an accuracy, let the time scale of your analysis become finer and finer; that is, let $n \rightarrow \infty$ as $\Delta t \rightarrow 0$. Assume you can take this limit term by term; your result is an infinite series. Write this series.

This series is your answer for the population at time t , Captain. You may calculate the answer to any desired accuracy by including enough terms from the series.

Captain Kork's comeback

- How do you know you can calculate the answer to arbitrary accuracy? For example, how far off would I be if I just added up the first hundred terms? the first N terms? Give me an upper bound for the error after 100 terms, and the error after N terms, Spook, and then prove to me that the error goes to 0 as $N \rightarrow \infty$. *Hint: The error depends on the time t in question. One can compare to a geometric series here.*

Now wipe that superior smile off of Spook's face by showing him what can be done with Earthling calculus.

- Start with the same expression for the increment, $\Delta y = (k\Delta t)y$, as Spook used in part (a), and convert it into a differential equation by taking the limit as $\Delta t \rightarrow 0$ and using the definition of the derivative. Solve this equation by separation of variables to find the function $y(t)$, which gives the population at time t in terms of the initial population y_0 . It is your turn to smile. (Spook had to run to the ship's library to look up separation of variables, but you know it is Section 7.3.)

Spook's retort

Captain, I claim that your solution is no different from mine. You have simply renamed my infinite series as your function because Earthlings cannot handle the concept of infinity. Allow me to demonstrate, Captain. In a book in the ship's library, I came across something you Earthlings call the "derivative".

- (f) Differentiate my series term by term with respect to time and see if the answer isn't k times my series. Therefore, my series does satisfy your differential equation, since its derivative is k times itself. Now compare your function from part (e) with your answer to part (c). Your function and my series must be one in the same!

Captain Kork's finale

Not necessarily, Spook. The same differential equation may have many different solutions. Allow me to demonstrate. . .

At about this time, McCool and Scooter burst into your cabin. Scooter says, "You gentlemen can debate theory 'til you're blue in the face, but that does'na help us with our problem. I'm prepared to use radiation from the ship's engine just one time to exterminate 99% of the little beggars right now. Just give me the word, Captain!"

McCool replies, "I cannot condone such a waste of life, Captain. I've got a drug that will cut their growth rate k in half. I think you'll find this a more effective and less brutal method than the one suggested by our engineer."

- (g) Decide between Scooter's and McCool's strategies. Which is more effective in the short run? In the long run? Convince them of your answers.
- (h) In the future, will these two treatments ever result in the same number of tripples again? When?