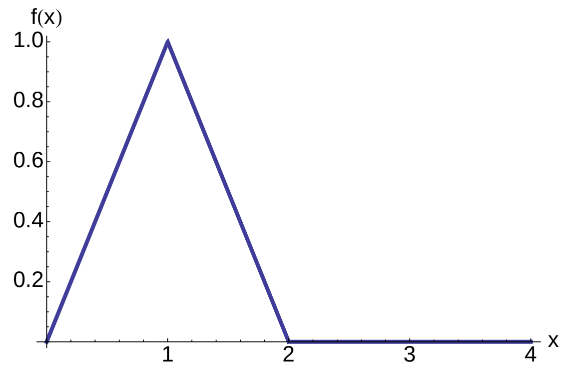
Name:________________________________ Quiz Score:_____/10

Answer each question completely in the area below. Show all work and explain your reasoning. If the work is at all ambiguous, it is considered incorrect. No phones, calculators, or notes are allowed. Anyone found violating these rules will be asked to leave immediately. Point values are in the square to the left of the question. If there are any other issues, please ask the instructor.

[7] 1. Using the definition, compute the Laplace transform of the following function, $f(x)$:



Solution: We know that the definition of the Laplace transform is:

$$F(s) = \mathcal{L}\{f(x)\} := \int_0^\infty e^{-sx} f(x)\, dx.$$

Here, we can see that $f(x)$ can be effectively broken down into three parts:

$$f(x) = \begin{cases} x & 0 \leq x < 1 \\ 2 - x & 1 \leq x < 2 \\ 0 & x \geq 2 \end{cases}.$$

Thus, this integral reduces to

$$\int_0^\infty e^{-sx} f(x) = \int_0^1 x e^{-sx}\, dx + \int_1^2 (2-x) e^{-sx}\, dx,$$

since everywhere else, the integral is 0. We have to compue this by parts, but it's not too bad:

$$\int_0^1 x e^{-sx}\, dx = -\left[\frac{x e^{-sx}}{s}\right]_{x=0}^{x=1} + \int_0^1 \frac{1}{s} e^{-sx}\, dx = \frac{1}{s^2}\left[1 - e^{-s}(s+1)\right].$$

Similarly, for the second term, we get

$$\int_1^2 (2-x) e^{-sx}\, dx = 2\int_1^2 e^{-sx} - \int_1^2 x e^{-sx} = \frac{e^{-2s}}{s^2}\left[e^{s}(s-1) + 1\right].$$

where we had already done the same integration by parts (making this a little easier).Thus, the total Laplace transform is just the sum of these two parts.

$$F(s) = \frac{1}{s^2}\left[1 - e^{-s}(s+1)\right] + \frac{e^{-2s}}{s^2}\left[e^{s}(s-1) + 1\right].$$

3 2. Find an explicit expression for $X(s)$, the Laplace transform of $x(t)$, but **do not** invert the transform to find $x(t)$:

$$x'' + 3x' + x = 0, \qquad x(0) = 1, \qquad x'(0) = -2.$$

Solution: Here, we use the linearity of the Laplace transform along with the properties we have below. To start, we Laplace transform both sides of the equation:

$$\begin{aligned} \mathcal{L}\left\{x'' + 3x' + x\right\} &= \mathcal{L}\{0\} \\ \mathcal{L}\left\{x''\right\} + 3\mathcal{L}\left\{x'\right\} + \mathcal{L}\left\{x\right\} &= 0. \end{aligned}$$

Note that we've used both the linearity of the transform and the fact that $\mathcal{L}\left\{0\right\} = 0$. Now, using the hints at the bottom individually:

$$\begin{aligned} \mathcal{L}\left\{x''\right\} &= s^2X(s) - sx(0) - x'(0) = s^2X(s) - s + 2, \\ \mathcal{L}\left\{x'\right\} &= sX(s) - x(0) = sX(s) - 1, \\ \mathcal{L}\left\{x\right\} &= X(s). \end{aligned}$$

Thus, we're left with

$$\left[s^2X(s) - s + 2\right] + 3\left[sX(s) - 1\right] + X(s) = 0.$$

Rearranging this a bit yields

$$\left[s^2 + 3s + 1\right]X(s) = s + 1, \qquad \Longrightarrow X(s) = \frac{s+1}{s^2 + 3s + 1}.$$

We could, in theory, invert this to find our original solution $x(t)$, but this was not required for the problem.

Useful facts:

$$\mathcal{L}\left\{f'(t)\right\} = sF(s) - f(0), \qquad \mathcal{L}\left\{f''(t)\right\} = s^2F(s) - sf(0) - f'(0).$$