

Name: _____

Quiz Score: ____/10

Answer each question completely in the area below. Show all work and explain your reasoning. If the work is at all ambiguous, it is considered incorrect. No phones, calculators, or notes are allowed. Anyone found violating these rules will be asked to leave immediately. Point values are in the square to the left of the question. If there are any other issues, please ask the instructor.

- 4 1. Consider the following subset W of \mathbb{R}^3 :

$$W = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 = 7x_3\}.$$

Prove or disprove whether W is a subspace of \mathbb{R}^3 .

Solution: The first issue is: should we try to prove or disprove this being a subspace? While your intuition should hopefully be building, we actually know for sure that this **is** a subspace. Why? What shape can we think of this object as? It's a line through the origin! Thus, we already know it's a subspace, but I wanted you to prove this directly, meaning we must show two properties.

1. The first property we must show is that if \mathbf{u}, \mathbf{v} are both in W then $\mathbf{w} = \mathbf{u} + \mathbf{v}$ is also in W . Note, I'm just giving the new vector you get a name, but it's still generic. I'm not saying what these particular vectors actually ARE. I'm just generically saying: consider adding two vectors and call the result \mathbf{w} .

If \mathbf{u}, \mathbf{v} are both in W , note that we can write them in the following way:

$$\mathbf{u} = (7u_3, u_2, u_3), \quad \mathbf{v} = (7v_3, v_2, v_3).$$

When we add them, we see

$$\begin{aligned} \mathbf{u} + \mathbf{v} &= (7u_3, u_2, u_3) + (7v_3, v_2, v_3) = (7u_3 + 7v_3, u_2 + v_2, u_3 + v_3) \\ &= (7\{u_3 + v_3\}, u_2 + v_2, u_3 + v_3) = (7w_3, w_2, w_3). \end{aligned}$$

Where again, I've just called $u_2 + v_2 = w_2$ and $u_3 + v_3 = w_3$ for convenience. Note that this looks EXACTLY like a vector in W . In fact, looking like this (that 7 times your third element is your first element) is what defines being in W . Thus, we can conclude $\mathbf{w} \in W$.

2. Similarly, our second condition is that if $\mathbf{u} \in W$, then $c\mathbf{u} \in W$ as well, where c can be ANY scalar. That is, if we take any vector in our set and re-scale it by any scalar, do we stay in our set? To see this, let

$$\mathbf{u} = (7u_3, u_2, u_3),$$

in which case this suggests

$$c\mathbf{u} = (7cu_3, cu_2, cu_3) = (7\tilde{u}_3, \tilde{u}_2, \tilde{u}_3),$$

where I am just giving the names $\tilde{u}_2 = cu_2$ and $\tilde{u}_3 = cu_3$. Note, the thing we've ended up with is again exactly the form of a vector in W , so we can conclude $c\mathbf{u} \in W$.

Thus, we've shown these properties to hold for ALL vectors in W , as I never specified what \mathbf{u} or \mathbf{v} were, so this is indeed a subspace.

- 1 2. (a) What is the definition of a **basis** of a vector space V ?

Solution: A basis, \mathcal{B} , is a collection of vectors in V , that have the following two properties:

1. The vectors in \mathcal{B} are linearly independent. That is, say

$$\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\},$$

for convenience. Note that a basis does **not** need to be finite, but I'm just supposing it is for the sake of explanation. The definition of linear independence is that if

$$c_1\mathbf{b}_1 + \dots + c_n\mathbf{b}_n = \mathbf{0},$$

then this necessarily implies that (all) $c_1 = \dots = c_n = 0$. That is, the only way to add these vectors to get $\mathbf{0}$ is by taking all the coefficients to be 0.

2. The second condition is that the vectors in \mathcal{B} span the vector space V . In symbols, $\text{span}\mathcal{B} = V$. What this means is that: given any vector $\mathbf{u} \in V$, there exists some a_1, \dots, a_n such that

$$\mathbf{u} = a_1\mathbf{b}_1 + \dots + a_n\mathbf{b}_n.$$

In other words, we can build anything in our vector space V as a linear combination of our basis elements.

- 4 (b) Consider the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Find a basis for the solution space of the homogeneous system $\mathbf{Ax} = \mathbf{0}$.

Solution: This is very similar to a number of examples we've done in class and in lab. Note that here we have two leading variables: x_1, x_3 and two free variables: x_2, x_4 . We parameterize the free variables: $x_2 = s$ and $x_4 = t$. We now remember that our system is the homogeneous system, so we solve for x_1, x_3 by setting their rows (equations) equal to 0. First, using the second row:

$$x_3 - x_4 = 0 \quad \implies \quad x_3 - t = 0 \quad \implies \quad x_3 = t.$$

Now, the first equation says that

$$x_1 + x_2 - 2x_4 = 0 \quad \implies \quad x_1 + s - 2t = 0 \quad \implies \quad x_1 = 2t - s.$$

Thus, any solution \mathbf{x} looks like:

$$\mathbf{x} = (2t - s, s, t, t) = s(-1, 1, 0, 0) + t(2, 0, 1, 1).$$

Thus, we see that any vector in our solution subspace can be written as a linear combination of the two vectors $(-1, 1, 0, 0)$ and $(2, 0, 1, 1)$. From this, we conclude our basis is

$$\mathcal{B} = \{(-1, 1, 0, 0), (2, 0, 1, 1)\}.$$

Note, we didn't explicitly check a lot of aspects of this. For one, we know from class that the set of vectors \mathbf{x} that satisfy the homogeneous equation is indeed a subspace of its original space. Thus, we know we're actually finding the basis of a vector space.

We also never explicitly checked that this set spans or is linearly independent, but it's immediate that we actually have a basis: we can construct any solution \mathbf{x} this way (thus it spans) and we couldn't have done it with any fewer vectors (thus they're linearly independent.)

- 1 (c) What is the dimension of the basis you found in part (b)?

Solution: We have two vectors, so $\dim \mathcal{B} = 2$. Note that for problems like this, the number of free variables you have will always be the dimension of the solution space.