Math 2250 – Midterm 2: November 13, 2015

Yet again, this outline is not meant to be an end-all exhaustive study replacement, but rather a structured baseline for your studying. That is, this is a study *companion*. **If something is not on this outline, that does not mean it will not be on the exam!**

The topics covered on the exam are roughly described below. Identify which topics are your own weaknesses. In lab on Thursday, Chee Han and Huachen will review. You should come with questions if you have them. I'll also make an effort to be in my office as much as possible during the week prior to the exam. I'm particularly likely to be in my office if you see nothing on my schedule, found here: `http://www.math.utah.edu/~miles/#contact`.

I strongly recommend looking over the labs and quizzes, as well as making sure you have a good idea of how to solve all of the homework problems listed, even those not collected. Problems from all of these sources are likely to make an appearance in some form on the exam either exactly the same or modified slightly.

The exam will be doable in the 50 minute class period. *I will make this exam slightly shorter than the previous.* You will not be allowed (nor will you need): a calculator, notes, or anything really except for a writing utensil. The intent of the exam is to establish your profiency in the ideas discussed in this class, not to test you on algebra or calculus. That said, these basic mechanics are unavoidable, so being comfortable with them is important.

Chapter 3: Linear Systems and Matrices

3.4: Matrix Operations

- Adding matrices is easy: just add element-wise. Must be the same size.
- Multiplication of two vectors: dot product, sum of products.
- Multiplication of matrices roughly same idea: multiplication of row of one matrix with column of the other.
- Important facts: $m \times n$ matrix times an $n \times p$ matrix produces $m \times p$. You know you can do the multiplication if the "inside" dimension matches.
- Note this immediately tells us that matrix multiplication does NOT commute. $AB \neq AB$ in general.
- We now think of our linear system as $\mathbf{Ax} = \mathbf{b}$, where A is a matrix and x, b are column vectors.
- Other laws of algebra hold: addition commutes and is associative, multiplication is associative and distributive.
- Element-wise if $\mathbf{A} = [a_{ij}]$, an $m \times p$ matrix and $\mathbf{B} = [b_{ij}]$, an $p \times n$ matrix, then $\mathbf{AB} = \mathbf{C} = [c_{ij}]$ has elements where $c_{ij} = \sum_{k=1}^{p} a_{ik} b_{kj}$.
- Other rules don't apply. If $ab = ac$, for numbers, and $a \neq 0$ then $b = c$, but this isn't true for matrices.
- Also if $ab = 0$, then either a or b are 0, but again, not true for matrices.

3.5: Matrix Inverses

- Intuitive idea: we want to "cancel" out a matrix from an equation when we can. In a sense: invert it.
- identity matrix $\mathbf{I}$ is a square $n \times n$matrix with all 0's except along the diagonal, which has 1's.
- In a sense $\mathbf{I}$ serves as the "1" of matrix multiplication. That is, $\mathbf{AI} = \mathbf{A}$.
- If $\mathbf{AB} = \mathbf{BAI}$ for some square matrix $\mathbf{A}$, then $\mathbf{B}$ is said to be the inverse of $\mathbf{A}$ and denoted $\mathbf{A}^{-1}$.
- Note this is NOT an exponent but notation meaning inverse. However, exponent rules still work with it (which is why we do it).

- For a 2×2 matrix with elements $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ the inverse is always $\frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.
- How to remember this: flip diagonals, negate off diagonals, divide by the determinant.
- Note that if the determinant is 0 then this makes no sense. Important idea: not every matrix is invertible!
- Shoes and socks for inverses: $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$, where the order matters!
- For a linear system $\mathbf{Ax} = \mathbf{b}$, then if $\mathbf{A}$ is invertible (the inverse is unique) and therefore we have a unique solution $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$.
- How to compute the inverse in general: write your matrix $\mathbf{A}$ and append the identity matrix $\mathbf{I}$ with a dotted line to the right. Row reduce $\mathbf{A}$ to the identity matrix and do the SAME operations to your appended identity matrix. The matrix you're left with on the right is $\mathbf{A}^{-1}$.
- **Biggest idea from this whole chapter (class?)**: all of these statements are equivalent for an $n \times n$ matrix $\mathbf{A}$:
  1. $\mathbf{A}$ is invertible
  2. $\mathbf{A}$ is row equivalent (can row reduce it to) $\mathbf{I}$
  3. $\mathbf{A}x = 0$ only has the trivial solution
  4. $\mathbf{Ax} = \mathbf{b}$ has a unique solution
  5. $\det \mathbf{A} \neq 0$. (see next section)

3.6: Determinants

- $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$, note that vertical bars mean take the determinant.
- other notations for determinant: $|\mathbf{A}|$ or $\det \mathbf{A}$.
- Cramer's rule allows us to solve $\mathbf{Ax} = \mathbf{b}$ using determinants but honestly I forgot about it until I was writing this document
- Cramer's rule is also VERY computationally inefficient. Nobody solves systems that way.
- For bigger than 2×2, we split into smaller determinants by expanding upon ANY row or column.
- To determine the sign of each term, use the checkerboard.
- If we can get a 0 coefficient for a term (or more) this is often a convenient way of computing the determinant
- Elementary row operations generally **do** change the determinant, however, adding a constant multiple of a row/column to another does not.
- Two copies of the same row (or constant multiples of each other): $\det \mathbf{A} = 0$.
- Upper/lower triangular: zeroes above/below diagonal. Determinant is just the product of the diagonal elements.
- Transpose of a matrix, denoted $\mathbf{A}^T$ is just interchanging rows/columns. That is if $\mathbf{A} = [a_{ij}]$ then $\mathbf{A}^T = [a_{ji}]$.
- Big idea: invertability and $\det \neq 0$ are EQUIVALENT.
- $|\mathbf{AB}| = |\mathbf{A}||\mathbf{B}|$. A pretty crazy fact. Determinants aren't preserved under addition but they are under matrix multiplication.

Chapter 4: Vector Spaces

4.1: $\mathbb{R}^3$ as a vector space

- Vectors as we've thought of them before have tons of nice properties. We study $\mathbb{R}^3$ (three dimensional space that we live in) to understand which of these properties we want to consider in general.

- A vector $\mathbf{u} \in \mathbb{R}^3$ is just a 3-tuple (an ordered list of 3 numbers) $\mathbf{u} = (u_1, u_2, u_3)$.
- Parallelogram law: $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$. Scalar multiplication: $c\mathbf{u} = (cu_1, cu_2, c_u 3$.
- Definition of a vector space: a collection of objects (vectors) that satisfy the following properties:
  1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
  2. $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
  3. $\mathbf{u} + \mathbf{0} = \mathbf{u}$
  4. $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.
  5. $r(\mathbf{u} + \mathbf{v}) = r\mathbf{u} + r\mathbf{v}$
  6. $(r + s)\mathbf{u} = r\mathbf{u} + s\mathbf{u}$
  7. $r(s\mathbf{u}) = (rs)\mathbf{u}$
  8. $1\mathbf{u} = \mathbf{u}$.
- You don't have to memorize the above list but appreciate/be familiar with the properties. They're all things we basically take for granted with what we traditionally think of as vectors, so it's natural to take them as our base properties for more general objects.
- Two vectors are linearly dependent if they are scalar multiples of each other (same direction). We can think of this as there exists some $a, b \neq 0$ such that $a\mathbf{u} + b\mathbf{v} = \mathbf{0}$
- Alternatively, two vectors are linearly INDEPENDENT if when $a\mathbf{u} + b\mathbf{v} = \mathbf{0}$, this implies that $a = b = 0$. That is, there is no non-trivial way to add the two vectors to get $\mathbf{0}$.
- For three vectors, the rough same idea applies. Linear dependence means that one vector can't be written as a linear combination of the other two.
- This boils down to: $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are linearly dependent if there exists some $a, b, c \neq 0$ such that $a\mathbf{u} + b\mathbf{v} + c\mathbf{v} = \mathbf{0}$ or if the only way is $a = b = c = 0$ then they are linearly independent.
- For vectors in $\mathbb{R}^3$, we have a shortcut for computing this. Put the three vectors as columns in a matrix. If the determinant $\neq 0$, they are independent, dependent if it is $= 0$. *Can you prove this?*
- Basis vectors of $\mathbb{R}^3$ are $\mathbf{i} = (1, 0, 0), \mathbf{j} = (0, 1, 0), \mathbf{k} = (0, 0, 1)$.
- These three basis vectors are linearly independent. *Why?*
- We can write any vector in $\mathbb{R}^3$ as $(a, b, c) = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$.
- This gives us the notion of a basis for $\mathbb{R}^3$. Any three vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ such that we can write any vector $\mathbf{t}$ as $\mathbf{t} = a\mathbf{u} + b\mathbf{v} + c\mathbf{w}$.
- Important theorem: any three LINEARLY INDEPENDENT vectors in $\mathbb{R}^3$ constitute a basis. *Can you prove this?*
- Essence of above theorem: linear independence gives us spanning for free in $\mathbb{R}^3$. We'll see this isn't true in general later.
- Subspaces: a subset of a vector space that is ITSELF a vector space.
- Subspaces must contain the zero vector (the origin) *Why?*
- To test whether a set $W$is a subspace in practice two properties must be checked: 1) for all vectors $\mathbf{u}, \mathbf{v}$, $\mathbf{u} + \mathbf{v}$ must be in W. 2) For all vectors $\mathbf{u}$, $c\mathbf{u}$ must be in W for all scalars c.
- If you can find ONE vector that violates either of the above two properties, this is not a subspace. However, proving something IS a subspace means proving those properties hold for ALL subspaces.
- $\mathbb{R}^3$ and $\{\mathbf{0}\}$ are always subspaces of $\mathbb{R}^3$ but are stupid
- Aside from those two: only two possibilities for subspaces are lines and planes through the origin.

4.2: $\mathbb{R}^n$ as a vector space

- $\mathbb{R}^n$ is basically exactly the same as $\mathbb{R}^3$. Everything we said above is true.
- Now vectors are n-tuples, meaning they're just an (ordered) list of n numbers.

- **Huge idea:** If $\mathbf{A}$ is an $m \times n$ matrix, then the set of solutions ($\mathbf{x}$ values) that satisfy $\mathbf{Ax} = \mathbf{0}$ forms a subspace of $\mathbb{R}^n$. **Can you prove this?!**
- We call that subspace the "solution space" of $\mathbf{A}$.
- The above theorem is profound. We aren't explicitly saying what vectors are in our set, but rather implicitly defining the set by the linear system.
- In practice: to find solution space, row reduce to echelon form and parameterize free variables.

4.3: Linear Combinations, Linear Independence

- We already defined linear independence/dependence above for $\mathbb{R}^3$. Same idea for any vector space.
- A set $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is linearly independent if $a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n = \mathbf{0}$ implies that $a_1 = \cdots = a_n = 0$.
- If you can find a non-trivial set of coefficients that allows you to add up the vectors to get $\mathbf{0}$, then this means you can write one vector as a linear combination of the others and they are therefore linearly dependent
- New idea: spanning. Basically: can we write any vector as a linear combination of some base set.
- Expanding on the above, if we can write any $\mathbf{u} \in S$ as $\mathbf{u} = a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n$, then we say that $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ spans S or that $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is a spanning set for S.
- We can go the other direction: start with a set of vectors and ask what we can build with them. That is, $\text{span}\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ = all things we can build with linear combinations of $\mathbf{v}_1, \dots, \mathbf{v}_n$.
- Theorem: $\text{span}\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is always a subspace of whatever vector space the $\mathbf{v}_j$'s live in.
- Observation: any vector in the span of a set is UNIQUELY representable by a linear combination of those vectors.
- Another observation: any subset of a linearly independent set is also linearly independent. (why?)
- Consider k vectors in $\mathbb{R}^n$. We have three possibilities for whether or not these are linearly independent/dependent depending on how k relates to n.
- First case, if $k = n$, this is like our $\mathbb{R}^3$ case. Put all the vectors in a matrix and take the determinant. If it's 0, then they are dependent, and independent if it is $\neq 0$.
- if $k > n$, you can think of this as overpacking. You only have so much freedom being in $\mathbb{R}^n$.
- Theorem from this idea: any set of $n+1$ or greater vectors in $\mathbb{R}^n$ is linearly dependent.
- Weirdest case: if $k < n$, we put the vectors in a (non-square) matrix. Then we check if we can find any square $k \times k$ submatrix (delete rows/columns) that has a non-zero determinant. If we can do this, the vectors are linearly independent.

4.4: Bases, Dimension

- We saw that for $\mathbb{R}^3$, the vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ were fundamental building blocks for all vectors. We want to generalize this idea.
- Two essential components: 1. we want the bare minimum number of blocks (we don't want to include useless stuff) 2. we want to be able to build anything (these are actually the right building blocks)
- Formal definition: a collection of vectors B is called a basis for a vector space V if: 1) the vectors in B are linearly independent, $\text{span}\, B = V$.
- Standard basis vectors for $\mathbb{R}^n$ are just $e_1 = (1, 0, \dots, 0), \dots, e_n = (0, \dots, 1)$.
- Any linearly independent n vectors in $\mathbb{R}^n$ is a basis. That is, we get spanning for free in $\mathbb{R}^n$. This is not true of general vector spaces!
- Theorem: If V has a basis with n elements then any set of more than n vectors is linearly dependent (why?)
- Consequence of the above theorem: **a vector space does not have a unique basis, but all bases have the same number of elements**.

- The number of vectors we need to form a basis for a vector space is what we call the *dimension* of the vector space. This is a number. Denoted $\dim V$
- A vector space does not need to be finite dimensional! Polynomials are an infinite dimensional vector space.
- We can talk about bases for solution spaces: from reduced row echelon form we parameterize and end up with something like $\mathbf{x} = s\mathbf{v}_1 + r\mathbf{v}_2 + \cdots$ where you may have more (or fewer) free variables.
- In the above, $\mathbf{v}_1, \mathbf{v}_2$ are the basis vectors of the solution space! Why? We've just written every solution as a linear combination of them.

Chapter 5: Higher Order Differential Equations

5.1: Intro to Second Order ODEs

- A linear second order ODE is one of the form $A(x)y'' + B(x)y' + C(x)y = F(x)$. Basically, we have no y or derivative of it doing anything except being multiplied by a function of x.
- Homogeneous equation: $F(x) = 0$ in the above.
- For the ODE $y'' + p(x)y' + q(x)y = 0$, theorem: if y_1, y_2 are any two solutions, $y = c_1y_1 + c_2y_2$ is always a solution. This is called the principle of superposition (we can superimpose any solutions to get a new solution)
- Two FUNCTIONS (these are vectors!) are linearly independent if they are not constant multiples of each other. In practice, you can check their ratio to see if it is just a number.
- If f, g are two functions, define the Wronskian $W(f,g) = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix}$.
- Basically, $W \neq 0$ is equivalent to linear independence and $W = 0$ is equivalent with linear dependence.
- Subtle but important theorem: If y_1, y_2 are two linearly independent solutions, then ANY solution y can be written as $y = c_1y_1 + c_2y_2$.
- Interpretation: if y_1, y_2 are linearly independent, they form a basis for our solution space (this is kinda like $\mathbb{R}^2$, where we get spanning for free once we have linear independence)
- Constant coefficients: $ay'' + b' + cy = 0$ we guess e^{rx} and we indeed have a solution if $ar^2 + br + c = 0$. We've reduced a ODE to a quadratic equation. This is called the characteristic equation
- If this equation has two real roots, say r_1, r_2, solution is easy: $y = c_1e^{r_1x} + c_2e^{r_2x}$.

5.2: General Solutions of Linear ODEs

- Can generalize the previous section: nth order linear homogeneous ODE: $y^{(n)} + p_1(x)y^{(n-1)} + \cdots p_n(x)y = 0$.
- If $y_1, \ldots, y_n$ are solutions, any linear combination of them is (superposition)
- Linear independence/dependence the same idea: if we can write one as a linear combination of the others, they are dependent, otherwise, independent.
- Wronskian still works to check (just take as many derivatives as you have functions)
- Basis idea still applies: if we have n linearly independent solutions to our homogeneous equation, ANY solution can be written as a linear combination of these. That is, we have a basis.
- Non-homogeneous equations are trickier. **Huge idea:** solutions Y of the non-homogeneous problem can be written as $Y = y_c + y_p$, where y_c is the complementary solution (which satisfies the homogeneous equation) and y_p is SOME other particular solution to the non-homogeneous equation

5.3: Homogeneous, Constant Coefficients

- nth order linear constant coefficient homogeneous ODE: $a_ny^{(n)} + \cdots a_1y' + a_0y = 0$.
- Same idea: look for solutions of the form e^{rx}. We get the characteristic equation $a_nr^n + \cdots a_1r + a_0 = 0$. Must have n roots in total

- n distinct real roots: basis of solutions is just $\{e^{r_1 x}, \ldots, e^{r_n x}$ so solutions look like $y = c_1 e^{r_1 x} + \cdots + c_n e^{r_n x}$, where the c_j are determined by initial conditions
- Differential operator: $L = a_n \frac{\mathrm{d}^n}{\mathrm{d}x^n} + \cdots + a_1 \frac{\mathrm{d}}{\mathrm{d}x} + a_0$, is something we APPLY to $y(x)$, that is, $Ly = a_n y^{(n)} + \cdots a_1 y' + a_0 y$
- Can also call the derivative an operator $D = \frac{\mathrm{d}}{\mathrm{d}x}$ so D^2 is the second derivative and we can rewrite the above as $L = a_n D^n + \cdots a_1 D + a_0$. Convenient for discussing roots.
- Consider $L = (D - r_1)^k (D - r_0)$, that is, r_0 is a regular root and r_1 is a root repeated k times (multiplicity k). We want $e^{r_1 x}$ to be our basis element but we need to add k linearly independent copies of it
- How do we make things that look like $e^{r_1 x}$ linearly independent? Throw an x in front of them. Basis: $\{e^{r_1 x}, xe^{r_1 x}, \ldots, x^{k-1} e^{r_1 x}\}$.
- Complex roots: motivation is $e^{i\theta} = \cos\theta + i \sin\theta$, thus we'll end up with trig functions.
- Complex roots always come in complex conjugate pairs! That is, if $a + bi$ is a root so is $a - bi$. We lump these together when we write our solution.
- Basis contribution from $a \pm bi$: $e^{ax}(c_1 \cos bx + c_2 \sin bx)$.
- Polar form for complex values: $z = x + iy = re^{i\theta}$.
- If $a \pm bi$ is repeated k times, our solution is: $\sum_{p=0}^{k-1} x^p e^{ax} (c_p \cos bx + d_p \sin b_x)$. In other words, we need to repeat coefficients in front of BOTH sine and cosine.
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5.5: Non-homogeneous equations, Undetermined Coefficients

- Whole idea: if we have $a_n y^{(n)} + \cdots a_1 y' + a_0 y = f(x)$, we can guess y_p, a particular solution for certain $f(x)$.
- Generally, what type of objects, when you take the derivative of them, do you get a constant times something that looks roughly like the original object? Polynomials, trig functions, exponentials.
- If $f(x) = p_0 + \cdots p_m x^m$, our guess is $y_p = A_0 + \cdots + A_m p^m$. The same degree polynomial.
- If $f(x) = \alpha e^{rx}$, we guess $y_p = Ae^{rx}$.
- If $f(x) = \sin kx$ or $\cos kx$, we guess $y_p = A \cos kx + B \sin kx$.
- This process is called the Method of Undetermined Coefficients. How do we determine them? Plug into the ODE (non-homogeneous) and see which values satisfy the equation
- Annoying issue: we don't want to repeat terms from y_c, the homogeneous part. So if one of the basis elements looks like a term from y_c, multiply by as many x's as you need in your guess so that there is no duplication
- In general, if $f(x) = P_m(x) e^{rx} \sin kx$ or $\cos kx$, then our guess is $y_p = x^s \left[(A_0 + A_1 x + \cdots A_m x^m) e^{rx} \cos kx + (B_0 + \cdots B_m x^M) e^{rx} \sin kx \right]$, where s is the minimum number to avoid duplicates from y_c and P_m is any degree m polynomial.

5.4: Mechanical Vibrations

- Classic case study, spring, mass, dashpot system: $m\ddot{x} + c\dot{x} + kx = F(x)$, where m is the mass, c is the dashpot drag coefficient and k is the spring stiffness. $F(x)$ is some external forcing. This just comes from $\sum F = ma$.
- Undamped: $c = 0$, unforced or free: $F = 0$.
- Not the only way this type of ODE occurs: circuits, pendulums
- Free undamped motion: $m\ddot{x} + kx = 0$, define $\omega_0 = \sqrt{k/m}$, motion is $x(t) = A \cos \omega_0 t + B \sin \omega_0 t$.
- We can rewrite this to study it easier: $x(t) = C \cos(\omega_0 t - \alpha)$ where $C = \sqrt{A^2 + B^2}$ and $\alpha = \tan^{-1}(B/A)$ but must be between 0 and 2π.
- Can also write $x(t) = C \cos(\omega(0)(t - \delta))$ where $\delta = \alpha_0 / \omega_0$, a time lag measured in seconds.

- ω_0 is an angular frequency (rads/second), $T = 2\pi/\omega_0$ is the period of oscillation. Actual frequency (in Hz = 1/s) is $v = 1/T$.
- Consider adding damping back: $m\ddot{x} + c\dot{x} + kx = 0 = \ddot{x} + 2p\dot{x} + \omega_0^2 x = 0$ for convenience again
- Constant coefficients, roots are: $r_1, r_2 = -p \pm \sqrt{p^2 - \omega_0^2}$. Three possible cases depending on $p^2 - \omega_0^2 = \frac{c^2-4km}{4m^2}$.
- Call the boundary case: $c_{cr} = \sqrt{4km}$ critical damping. In this regime, we have repeated roots at $r_= - p$ so our solution is $x = e^{-pt}(c_1 + c_2 t)$. Not very interesting behavior. Dies out.
- Similar if $c > c_{cr}$, overdamping. Solutions are two negative real roots. Die out.
- If $c > c_{cr}$: we call this underdamping and our solution is $x = e^{-pt}(A\cos\omega_1 t + B\sin\omega_1 t$, where $\omega_1 = \sqrt{\omega_0^2 - p^2}$.
- This behavior: oscillations with a decaying envelope. Frequency ω_1 is LESS than natural frequency ω_0. We call this the pseudofrequency. Can find pseudoperiod $T = 2\pi/\omega_1$.

5.6: Forced Oscillations, Resonance

- Consider undamped forced oscillations $m\ddot{x} + kx = F_0\cos\omega t$. Assuming $\omega \neq \omega_0$, our natural frequency, we get particular solutions are $x = A\cos\omega t$ where $A = \frac{F_0/m}{\omega_0^2-\omega^2}$.
- With the general solution, we have $x = c_1\cos\omega_0 t + c_2\sin\omega_0 t + A\cos\omega t$.
- Note we can always combine $\sin, \cos$ into a phase shifted cos.
- If $x(0) = x'(0) = 0$, our solution reduces to $D\sin\frac{(w_0-\omega)t}{2}\sin\frac{(\omega_0+\omega)t}{2}$. If $\omega \approx \omega_0$, the first sin is slowly oscillating and the second is rapidly. This is phenomenon of beats.
- Notice if $\omega \approx \omega_0$, oscillations get larger and larger.
- Consider $\ddot{x} + \omega_0^2 x = \frac{F_0}{m}\cos\omega_0 t$. That is, forcing at our natural frequency. We now need to correct our particular solution!
- $x_p = \frac{F_0}{2momega_0} t\sin\omega_0 t$. Grows with time! This is resonance. Oscillations with growing envelope t.
- Damped forced oscillations: add dashpot back, $m\ddot{x} + c\dot{x} + kx = F_0\cos\omega t$.
- In this case, $x_p = C\cos(\omega t - \alpha)$.
- What we find, depending on ω, C has a maximum, we call this practical resonance. If ω is very small, the amplitude approaches F_0/k. If ω is very large, amplitude goes to 0.
- Modeling in general: often good to write down conservation of energy and take some derivatives.