14. Imposition of the initial conditions $y(2)=10,\ y'(2)=15$ on the general solution $y(x) = c_1x^2 + c_2x^{-3}$ yields the two equations $4c_1 + c_2/8 = 10,\ 4c_1 - 3c_2/16 = 15$ with solution $c_1 = 3,\ c_2 = -16$. Hence the desired particular solution is $y(x) = 3x^2 - 16/x^3$.

15. Imposition of the initial conditions $y(1)=7,\ y'(1)=2$ on the general solution $y(x) = c_1x + c_2x\ln x$ yields the two equations $c_1 = 7,\ c_1 + c_2 = 2$ with solution $c_1 = 7,\ c_2 = -5$. Hence the desired particular solution is $y(x) = 7x - 5x\ln x$.

16. Imposition of the initial conditions $y(1)=2,\ y'(1)=3$ on the general solution $y(x) = c_1\cos(\ln x) + c_2\sin(\ln x)$ yields the two equations $c_1 = 2,\ c_2 = 3$. Hence the desired particular solution is $y(x) = 2\cos(\ln x) + 3\sin(\ln x)$.

17. If $y = c/x$ then $y' + y^2 = -c/x^2 + c^2/x^2 = c(c-1)/x^2 \neq 0$ unless either $c = 0$ or $c = 1$.

18. If $y = cx^3$ then $yy'' = cx^3 \cdot 6cx = 6c^2x^4 \neq 6x^4$ unless $c^2 = 1$.

19. If $y = 1+\sqrt{x}$ then $yy'' + (y')^2 = (1+\sqrt{x})(-x^{-3/2}/4) + (x^{-1/2}/2)^2 = -x^{-3/2}/4 \neq 0$.

20. Linearly dependent, because

$$f(x) = \pi = \pi(\cos^2 x + \sin^2 x) = \pi\, g(x)$$

21. Linearly independent, because $x^3 = +x^2|x|$ if $x > 0$, whereas $x^3 = -x^2|x|$ if $x < 0$.

22. Linearly independent, because $1 + x = c(1+|x|)$ would require that $c = 1$ with $x = 0$, but $c = 0$ with $x = -1$. Thus there is no such constant c.

23. Linearly independent, because $f(x) = +g(x)$ if $x > 0$, whereas $f(x) = -g(x)$ if $x < 0$.

24. Linearly dependent, because $g(x) = 2f(x)$.

25. $f(x) = e^x \sin x$ and $g(x) = e^x\cos x$ are linearly independent, because $f(x) = k\,g(x)$ would imply that $\sin x = k\cos x$, whereas $\sin x$ and $\cos x$ are linearly independent.

26. To see that $f(x)$ and $g(x)$ are linearly independent, assume that $f(x) = c\,g(x)$, and then substitute both $x = 0$ and $x = \pi/2$.

27. Let $L[y] = y'' + py' + qy$. Then $L[y_c] = 0$ and $L[y_p] = f$, so

$$L[y_c + y_p] = L[y_c] + [y_p] = 0 + f = f.$$

28. If $y(x) = 1 + c_1 \cos x + c_2 \sin x$ then $y'(x) = -c_1 \sin x + c_2 \cos x$, so the initial conditions $y(0) = y'(0) = -1$ yield $c_1 = -2, c_2 = -1$. Hence $y = 1 - 2\cos x - \sin x$.

29. There is no contradiction because if the given differential equation is divided by x^2 to get the form in Equation (8) in the text, then the resulting functions $p(x) = -4/x$ and $q(x) = 6/x^2$ are not continuous at $x = 0$.

30. **(a)** $y_2 = x^3$ and $y_2 = |x^3|$ are linearly independent because $x^3 = c|x^3|$ would require that $c = 1$ with $x = 1$, but $c = -1$ with $x = -1$.

(b) The fact that $W(y_1, y_2) = 0$ everywhere does not contradict Theorem 3, because when the given equation is written in the required form

$$y'' - (3/x)y' + (3/x^2)y = 0,$$

the coefficient functions $p(x) = -3/x$ and $q(x) = 3/x^2$ are not continuous at $x = 0$.

31. $W(y_1, y_2) = -2x$ vanishes at $x = 0$, whereas if y_1 and y_2 were (linearly independent) solutions of an equation $y'' + py' + qy = 0$ with p and q both continuous on an open interval I containing $x = 0$, then Theorem 3 would imply that $W \neq 0$ on I.

32. **(a)** $W = y_1 y_2' - y_1' y_2$, so

$$\begin{aligned} AW' &= A(y_1'y_2' + y_1y_2'' - y_1''y_2 - y_1'y_2') \\ &= y_1(Ay_2'') - y_2(Ay_1'') \\ &= y_1(-By_2' - Cy_2) - y_2(-By_1' - Cy_1) \\ &= -B(y_1y_2' - y_1'y_2) \end{aligned}$$

and thus $AW' = -BW$.

(b) Just separate the variables.

(c) Because the exponential factor is never zero.

In Problems 33–42 we give the characteristic equation, its roots, and the corresponding general solution.

33. $r^2 - 3r + 2 = 0$; $r = 1, 2$; $y(x) = c_1 e^x + c_2 e^{2x}$

34. $r^2 + 2r - 15 = 0$; $r = 3, -5$; $y(x) = c_1 e^{-5x} + c_2 e^{3x}$

35. $r^2 + 5r = 0$; $r = 0, -5$; $y(x) = c_1 + c_2 e^{-5x}$

36. $2r^2+3r = 0; \quad r = 0, -3/2; \quad y(x) = c_1 + c_2e^{-3x/2}$

37. $2r^2-r-2 = 0; \quad r = 1, -1/2; \quad y(x) = c_1e^{-x/2} + c_2e^{x}$

38. $4r^2+8r+3 = 0; \quad r = -1/2, -3/2; \quad y(x) = c_1e^{-x/2} + c_2e^{-3x/2}$

39. $4r^2+4r+1 = 0; \quad r = -1/2, -1/2; \quad y(x) = (c_1 + c_2x)e^{-x/2}$

40. $9r^2-12r+4 = 0; \quad r = -2/3, -2/3; \quad y(x) = (c_1 + c_2x)e^{2x/3}$

41. $6r^2-7r-20 = 0; \quad r = -4/3, 5/2; \quad y(x) = c_1e^{-4x/3} + c_2e^{5x/2}$

42. $35r^2-r-12 = 0; \quad r = -4/7, 3/5; \quad y(x) = c_1e^{-4x/7} + c_2e^{3x/5}$

In Problems 43–48 we first write and simplify the equation with the indicated characteristic roots, and then write the corresponding differential equation.

43. $(r-0)(r+10) = r^2+10r = 0; \quad y''+10y' = 0$

44. $(r-10)(r+10) = r^2-100 = 0; \quad y''-100y = 0$

45. $(r+10)(r+10) = r^2+20r+100 = 0; \quad y''+20y'+100y = 0$

46. $(r-10)(r-100) = r^2-110r+1000 = 0; \quad y''-110y'+1000y = 0$

47. $(r-0)(r-0) = r^2 = 0; \quad y'' = 0$

48. $(r-1-\sqrt{2})(r-1+\sqrt{2}) = r^2-2r-1 = 0; \quad y''-2y'-y = 0$

49. The solution curve with $y(0)=1,\ y'(0)=6$ is $y(x) = 8e^{-x}-7e^{-2x}$. We find that $y'(x)=0$ when $x=\ln(7/4)$ so $e^{-x}=4/7$ and $e^{-2x}=16/49$. It follows that $y(\ln(7/4)) = 16/7$, so the high point on the curve is $(\ln(7/4)), 16/7) = (0.56, 2.29)$, which looks consistent with Fig. 3.1.6.

50. The two solution curves with $y(0)=a$ and $y(0)=b$ (as well as $y'(0)=1$) are

$$y = (2a+1)e^{-x} - (a+1)e^{-2x},$$
$$y = (2b+1)e^{-x} - (b+1)e^{-2x}.$$

7. $W = \begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & 2x \\ 0 & 0 & 2 \end{vmatrix} = 2$ is nonzero everywhere.

8. $W = \begin{vmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{vmatrix} = 2e^{6x}$ is never zero.

9. $W = e^x(\cos^2 x + \sin^2 x) = e^x \neq 0$

10. $W = x^{-7}e^x(x+1)(x+4)$ is nonzero for $x > 0$.

11. $W = x^3 e^{2x}$ is nonzero if $x \neq 0$.

12. $W = x^{-2}[2\cos^2(\ln x) + 2\sin^2(\ln x)] = 2x^{-2}$ is nonzero for $x > 0$.

In each of Problems 13-20 we first form the general solution

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + c_3 y_3(x),$$

then calculate $y'(x)$ and $y''(x)$, and finally impose the given initial conditions to determine the values of the coefficients c_1, c_2, c_3.

13. Imposition of the initial conditions $y(0) = 1$, $y'(0) = 2$, $y''(0) = 0$ on the general solution $y(x) = c_1 e^x + c_2 e^{-x} + c_3 e^{-2x}$ yields the three equations

$$c_1 + c_2 + c_3 = 1, \quad c_1 - c_2 - 2c_3 = 2, \quad c_1 + c_2 + 4c_3 = 0$$

with solution $c_1 = 4/3$, $c_2 = 0$, $c_3 = -1/3$. Hence the desired particular solution is given by $y(x) = (4e^x - e^{-2x})/3$.

14. Imposition of the initial conditions $y(0) = 0$, $y'(0) = 0$, $y''(0) = 3$ on the general solution $y(x) = c_1 e^x + c_2 e^{2x} + c_3 e^{3x}$ yields the three equations

$$c_1 + c_2 + c_3 = 1, \quad c_1 + 2c_2 + 3c_3 = 2, \quad c_1 + 4c_2 + 9c_3 = 0$$

with solution $c_1 = 3/2$, $c_2 = -3$, $c_3 = 3/2$. Hence the desired particular solution is given by $y(x) = (3e^x - 6e^{2x} + 3e^{3x})/2$.

15. Imposition of the initial conditions $y(0) = 2$, $y'(0) = 0$, $y''(0) = 0$ on the general solution $y(x) = c_1 e^x + c_2 x e^x + c_3 x^2 e^{3x}$ yields the three equations

$$c_1 + c_2 = 1, \quad c_1 - 2c_2 + c_3 = 5, \quad 6c_2 - 5c_3 = -11$$

with solution $c_1 = 2,\ c_2 = -1,\ c_3 = 1$. Hence the desired particular solution is given by $y(x) = 2x - x^{-2} + x^{-2} \ln x$.

In each of Problems 21 - 24 we first form the general solution

$$y(x) = y_c(x) + y_p(x) = c_1 y_1(x) + c_2 y_2(x) + y_p(x),$$

then calculate $y'(x)$, and finally impose the given initial conditions to determine the values of the coefficients c_1 and c_2.

21. Imposition of the initial conditions $y(0) = 2,\ y'(0) = -2$ on the general solution $y(x) = c_1 \cos x + c_2 \sin x + 3x$ yields the two equations $c_1 = 2,\ c_2 + 3 = -2$ with solution $c_1 = 2,\ c_2 = -5$. Hence the desired particular solution is given by $y(x) = 2 \cos x - 5 \sin x + 3x$.

22. Imposition of the initial conditions $y(0) = 0,\ y'(0) = 10$ on the general solution $y(x) = c_1 e^{2x} + c_2 e^{-2x} - 3$ yields the two equations $c_1 + c_2 - 3 = 0,\ 2c_1 - 2c_2 = 10$ with solution $c_1 = 4,\ c_2 = -1$. Hence the desired particular solution is given by $y(x) = 4e^{2x} - e^{-2x} - 3$.

23. Imposition of the initial conditions $y(0) = 3,\ y'(0) = 11$ on the general solution $y(x) = c_1 e^{-x} + c_2 e^{3x} - 2$ yields the two equations $c_1 + c_2 - 2 = 3,\ -c_1 + 3c_2 = 11$ with solution $c_1 = 1,\ c_2 = 4$. Hence the desired particular solution is given by $y(x) = e^{-x} + 4e^{3x} - 2$.

24. Imposition of the initial conditions $y(0) = 4,\ y'(0) = 8$ on the general solution $y(x) = c_1 e^x \cos x + c_2 e^x \cos x + x + 1$ yields the two equations $c_1 + 1 = 4,\ c_1 + c_2 + 1 = 8$ with solution $c_1 = 3,\ c_2 = 4$. Hence the desired particular solution is given by $y(x) = e^x(3 \cos x + 4 \sin x) + x + 1$.

25. $L[y] = L[y_1 + y_2] = L[y_1] + L[y_2] = f + g$

26. **(a)** $y_1 = 2$ and $y_2 = 3x$ **(b)** $y = y_1 + y_2 = 2 + 3x$

27. The equations

$$c_1 + c_2 x + c_3 x^2 = 0, \quad c_2 + 2c_3 x + 0, \quad 2c_3 = 0$$

SECTION 5.3

HOMOGENEOUS EQUATIONS WITH CONSTANT COEFFICIENTS

This is a purely computational section devoted to the single most widely applicable type of higher order differential equations — linear ones with constant coefficients. In Problems 1–20, we write first the characteristic equation and its list of roots, then the corresponding general solution of the given differential equation. Explanatory comments are included only when the solution of the characteristic equation is not routine.

1. $r^2-4 = (r-2)(r+2) = 0; \quad r = -2, 2; \quad y(x) = c_1 e^{2x} + c_2 e^{-2x}$

2. $2r^2-3r = r(2r-3) = 0; \quad r = 0, 3/2; \quad y(x) = c_1 + c_2 e^{3x/2}$

3. $r^2+3r-10 = (r+5)(r-2) = 0; \quad r = -5, 2; \quad y(x) = c_1 e^{2x} + c_2 e^{-5x}$

4. $2r^2-7r+3 = (2r-1)(r-3) = 0; \quad r = 1/2, 3; \quad y(x) = c_1 e^{x/2} + c_2 e^{3x}$

5. $r^2+6r+9 = (r+3)^2 = 0; \quad r = -3, -3; \quad y(x) = c_1 e^{-3x} + c_2 x e^{-3x}$

6. $r^2+5r+5 = 0; \quad r = \left(-5 \pm \sqrt{5}\right)/2$

$y(x) = e^{-5x/2}[c_1 \exp(x\sqrt{5}/2) + c_2 \exp(-x\sqrt{5}/2)]$

7. $4r^2-12r+9 = (2r-3)^2 = 0; \quad r = -3/2, -3/2; \quad y(x) = c_1 e^{3x/2} + c_2 x e^{3x/2}$

8. $r^2-6r+13 = 0; \quad r = \left(6 \pm \sqrt{-16}\right)/2 = 3 \pm 2i; \quad y(x) = e^{3x}(c_1 \cos 2x + c_2 \sin 2x)$

9. $r^2+8r+25 = 0; \quad r = \left(-8 \pm \sqrt{-36}\right)/2 = -4 \pm 3i; \quad y(x) = e^{-4x}(c_1 \cos 3x + c_2 \sin 3x)$

10. $5r^4+3r^3 = r^3(5r+3) = 0; \quad r = 0, 0, 0, -3/5; \quad y(x) = c_1 + c_2 x + c_3 x^2 + c_4 e^{-3x/5}$

11. $r^4-8r^3+16r^2 = r^2(r-4)^2 = 0; \quad r = 0, 0, 4, 4; \quad y(x) = c_1 + c_2 x + c_3 e^{4x} + c_4 x e^{4x}$

12. $r^4-3r^3+3r^2-r = r(r-1)^3 = 0; \quad r = 0, 1, 1, 1; \quad y(x) = c_1 + c_2 e^x + c_3 x e^x + c_4 x^2 e^x$

13. $9r^3+12r^2+4r = r(3r+2)^2 = 0; \quad r = 0, -2/3, -2/3$

$y(x) = c_1 + c_2 e^{-2x/3} + c_3 x e^{-2x/3}$

14. $r^4 + 3r^2 - 4 = \left(r^2 - 1\right)\left(r^2 + 4\right) = 0; \quad r = -1, 1, \pm 2i$

$y(x) = c_1 e^x + c_2 e^{-x} + c_3 \cos 2x + c_4 \sin 2x$

15. $4r^4 - 8r^2 + 16 = \left(r^2 - 4\right)^2 = (r-2)^2(r+2)^2 = 0; \quad r = 2, 2, -2, -2$

$y(x) = c_1 e^{2x} + c_2 x e^{2x} + c_3 e^{-2x} + c_4 x e^{-2x}$

16. $r^4 + 18r^2 + 81 = \left(r^2 + 9\right)^2 = 0; \quad r = \pm 3i, \pm 3i$

$y(x) = (c_1 + c_2 x)\cos 3x + (c_3 + c_4 x)\sin 3x$

17. $6r^4 + 11r^2 + 4 = (2r^2 + 1)(3r^2 + 4) = 0; \quad r = \pm i/\sqrt{2}, \pm 2i/\sqrt{3},$

$y(x) = c_1 \cos(x/\sqrt{2}) + c_2 \sin(x/\sqrt{2}) + c_3 \cos(2x/\sqrt{3}) + c_4 \sin(2x/\sqrt{3})$

18. $r^4 - 16 = \left(r^2 - 4\right)\left(r^2 + 4\right) = 0; \quad r = -2, 2, \pm 2i$

$y(x) = c_1 e^{2x} + c_2 e^{-2x} + c_3 \cos 2x + c_4 \sin 2x$

19. $r^3 + r^2 - r - 1 = r\left(r^2 - 1\right) + \left(r^2 - 1\right) = (r-1)(r+1)^2 = 0; \quad r = 1, -1, -1;$

$y(x) = c_1 e^x + c_2 e^{-x} + c_3 x e^{-x}$

20. $r^4 + 2r^3 + 3r^2 + 2r + 1 = (r^2 + r + 1)^2 = 0; \quad \left(-1 \pm \sqrt{3}\, i\right)/2, \left(-1 \pm \sqrt{3}\, i\right)/2$

$y = e^{-x/2}(c_1 + c_2 x)\cos(x\sqrt{3}/2) + e^{-x/2}(c_3 + c_4 x)\sin(x\sqrt{3}/2)$

21. Imposition of the initial conditions $y(0) = 7,\ y'(0) = 11$ on the general solution $y(x) = c_1 e^x + c_2 e^{3x}$ yields the two equations $c_1 + c_2 = 7,\ c_1 + 3c_2 = 11$ with solution $c_1 = 5,\ c_2 = 2$. Hence the desired particular solution is $y(x) = 5e^x + 2e^{3x}$.

22. Imposition of the initial conditions $y(0) = 3,\ y'(0) = 4$ on the general solution $y(x) = e^{-x/3}\left[c_1 \cos\left(x/\sqrt{3}\right) + c_2 \sin\left(x/\sqrt{3}\right)\right]$ yields the two equations $c_1 = 3,\ -c_1/3 + c_2/\sqrt{3} = 4$ with solution $c_1 = 3,\ c_2 = 5\sqrt{3}$. Hence the desired particular solution is $y(x) = e^{-x/3}\left[3\cos\left(x/\sqrt{3}\right) + 5\sqrt{3}\sin\left(x/\sqrt{3}\right)\right]$.

23. Imposition of the initial conditions $y(0) = 3,\ y'(0) = 1$ on the general solution $y(x) = e^{3x}\left(c_1 \cos 4x + c_2 \sin 4x\right)$ yields the two equations $c_1 = 3,\ 3c_1 + 4c_2 = 1$ with solution $c_1 = 3,\ c_2 = -2$. Hence the desired particular solution is

$y(x) = e^{3x}(3 \cos 4x - 2 \sin 4x)$.

14. $r^4 + 3r^2 - 4 = (r^2 - 1)(r^2 + 4) = 0; \quad r = -1, 1, \pm 2i$

$y(x) = c_1 e^x + c_2 e^{-x} + c_3 \cos 2x + c_4 \sin 2x$

15. $4r^4 - 8r^2 + 16 = (r^2 - 4)^2 = (r-2)^2(r+2)^2 = 0; \quad r = 2, 2, -2, -2$

$y(x) = c_1 e^{2x} + c_2 x e^{2x} + c_3 e^{-2x} + c_4 x e^{-2x}$

16. $r^4 + 18r^2 + 81 = (r^2 + 9)^2 = 0; \quad r = \pm 3i, \pm 3i$

$y(x) = (c_1 + c_2 x)\cos 3x + (c_3 + c_4 x)\sin 3x$

17. $6r^4 + 11r^2 + 4 = (2r^2 + 1)(3r^2 + 4) = 0; \quad r = \pm i/\sqrt{2}, \pm 2i/\sqrt{3},$

$y(x) = c_1 \cos(x/\sqrt{2}) + c_2 \sin(x/\sqrt{2}) + c_3 \cos(2x/\sqrt{3}) + c_4 \sin(2x/\sqrt{3})$

18. $r^4 - 16 = (r^2 - 4)(r^2 + 4) = 0; \quad r = -2, 2, \pm 2i$

$y(x) = c_1 e^{2x} + c_2 e^{-2x} + c_3 \cos 2x + c_4 \sin 2x$

19. $r^3 + r^2 - r - 1 = r(r^2 - 1) + (r^2 - 1) = (r-1)(r+1)^2 = 0; \quad r = 1, -1, -1;$

$y(x) = c_1 e^x + c_2 e^{-x} + c_3 x e^{-x}$

20. $r^4 + 2r^3 + 3r^2 + 2r + 1 = (r^2 + r + 1)^2 = 0; \quad \left(-1 \pm \sqrt{3}\,i\right)/2, \left(-1 \pm \sqrt{3}\,i\right)/2$

$y = e^{-x/2}(c_1 + c_2 x)\cos(x\sqrt{3}/2) + e^{-x/2}(c_3 + c_4 x)\sin(x\sqrt{3}/2)$

21. Imposition of the initial conditions $y(0) = 7, \; y'(0) = 11$ on the general solution $y(x) = c_1 e^x + c_2 e^{3x}$ yields the two equations $c_1 + c_2 = 7, \; c_1 + 3c_2 = 11$ with solution $c_1 = 5, \; c_2 = 2$. Hence the desired particular solution is $y(x) = 5e^x + 2e^{3x}$.

22. Imposition of the initial conditions $y(0) = 3, \; y'(0) = 4$ on the general solution $y(x) = e^{-x/3}\left[c_1 \cos\left(x/\sqrt{3}\right) + c_2 \sin\left(x/\sqrt{3}\right)\right]$ yields the two equations $c_1 = 3, \; -c_1/3 + c_2/\sqrt{3} = 4$ with solution $c_1 = 3, \; c_2 = 5\sqrt{3}$. Hence the desired particular solution is $y(x) = e^{-x/3}\left[3\cos\left(x/\sqrt{3}\right) + 5\sqrt{3}\sin\left(x/\sqrt{3}\right)\right]$.

23. Imposition of the initial conditions $y(0) = 3, \; y'(0) = 1$ on the general solution $y(x) = e^{3x}(c_1 \cos 4x + c_2 \sin 4x)$ yields the two equations $c_1 = 3, \; 3c_1 + 4c_2 = 1$ with solution $c_1 = 3, \; c_2 = -2$. Hence the desired particular solution is
$y(x) = e^{3x}(3 \cos 4x - 2 \sin 4x)$.

24. Imposition of the initial conditions $y(0)=1,\ y'(0)=-1,\ y''(0)=3$ on the general solution $y(x) = c_1 + c_2 e^{2x} + c_3 e^{-x/2}$ yields the three equations

$$c_1 + c_2 + c_3 = 1, \quad 2c_2 - c_3/2 = -1, \quad 4c_2 + c_3/4 = 3$$

with solution $c_1 = -7/2,\ c_2 = 1/2,\ c_3 = 4$. Hence the desired particular solution is $y(x) = (-7 + e^{2x} + 8e^{-x/2})/2$.

25. Imposition of the initial conditions $y(0)=-1,\ y'(0)=0,\ y''(0)=1$ on the general solution $y(x) = c_1 + c_2 x + c_3 e^{-2x/3}$ yields the three equations

$$c_1 + c_3 = -1, \quad c_2 - 2c_3/3 = 0, \quad 4c_3/9 = 1$$

with solution $c_1 = -13/4,\ c_2 = 3/2,\ c_3 = 9/4$. Hence the desired particular solution is $y(x) = (-13 + 6x + 9e^{-2x/3})/4$.

26. Imposition of the initial conditions $y(0)=1,\ y'(0)=-1,\ y''(0)=3$ on the general solution $y(x) = c_1 + c_2 e^{-5x} + c_3 x e^{-5x}$ yields the three equations

$$c_1 + c_2 = 3, \quad -5c_2 + c_3 = 4, \quad 25c_2 - 10c_3 = 5$$

with solution $c_1 = 24/5,\ c_2 = -9/5,\ c_3 = -5$. Hence the desired particular solution is $y(x) = (24 - 9e^{-5x} - 25xe^{-5x})/5$.

27. First we spot the root $r = 1$. Then long division of the polynomial $r^3 + 3r^2 - 4$ by $r - 1$ yields the quadratic factor $r^2 + 4r + 4 = (r+2)^2$ with roots $r = -2, -2$. Hence the general solution is $y(x) = c_1 e^x + c_2 e^{-2x} + c_3 x e^{-2x}$.

28. First we spot the root $r = 2$. Then long division of the polynomial $2r^3 - r^2 - 5r - 2$ by the factor $r - 2$ yields the quadratic factor $2r^2 + 3r + 1 = (2r+1)(r+1)$ with roots $r = -1, -1/2$. Hence the general solution is $y(x) = c_1 e^{2x} + c_2 e^{-x} + c_3 e^{-x/2}$.

29. First we spot the root $r = -3$. Then long division of the polynomial $r^3 + 27$ by $r + 3$ yields the quadratic factor $r^2 - 3r + 9$ with roots $r = 3\left(1 \pm i\sqrt{3}\right)/2$. Hence the general solution is $y(x) = c_1 e^{-3x} + e^{3x/2}[c_2 \cos(3x\sqrt{3}/2) + c_3 \sin(3x\sqrt{3}/2)]$.

30. First we spot the root $r = -1$. Then long division of the polynomial

$$r^4 - r^3 + r^2 - 3r - 6$$