44. This follows immediately from the result in Problem 43, because an invertible matrix is row-equivalent to the identity matrix.

45. One can simply photocopy the portion of the proof of Theorem 7 that follows Equation (20). Starting only with the assumption that $\mathbf{A}$ and $\mathbf{B}$ are square matrices with $\mathbf{AB} = \mathbf{I}$, it is proved there that $\mathbf{A}$ and $\mathbf{B}$ are then invertible.

46. If $\mathbf{C} = \mathbf{AB}$ is invertible, so $\mathbf{C}^{-1}$ exists, then $\mathbf{A}(\mathbf{BC}^{-1}) = \mathbf{I}$ and $(\mathbf{C}^{-1}\mathbf{A})\mathbf{B} = \mathbf{I}$. Hence the fact that $\mathbf{A}$ and $\mathbf{B}$ are invertible follows immediately from Problem 45.

SECTION 3.6

DETERMINANTS

1. $$\begin{vmatrix} 0 & 0 & 3 \\ 4 & 0 & 0 \\ 0 & 5 & 0 \end{vmatrix} = +(3)\begin{vmatrix} 4 & 0 \\ 0 & 5 \end{vmatrix} = 3 \cdot 4 \cdot 5 = 60$$

2. $$\begin{vmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{vmatrix} = +(2)\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} - (1)\begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = 2(4-1)-(2-0) = 4$$

3. $$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 2 & 0 & 5 & 0 \\ 3 & 6 & 9 & 8 \\ 4 & 0 & 10 & 7 \end{vmatrix} = +(1)\begin{vmatrix} 0 & 5 & 0 \\ 6 & 9 & 8 \\ 0 & 10 & 7 \end{vmatrix} = -(5)\begin{vmatrix} 6 & 8 \\ 0 & 7 \end{vmatrix} = -5(42-0) = -210$$

4. $$\begin{vmatrix} 5 & 11 & 8 & 7 \\ 3 & -2 & 6 & 23 \\ 0 & 0 & 0 & -3 \\ 0 & 4 & 0 & 17 \end{vmatrix} = -(-3)\begin{vmatrix} 5 & 11 & 8 \\ 3 & -2 & 6 \\ 0 & 4 & 0 \end{vmatrix} = 3(-4)\begin{vmatrix} 5 & 8 \\ 3 & 6 \end{vmatrix} = -12(30-24) = -72$$

5. $$\begin{vmatrix} 0 & 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 5 & 0 & 0 & 0 \end{vmatrix} = +1\begin{vmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 5 & 0 & 0 \end{vmatrix} = +2\begin{vmatrix} 0 & 3 & 0 \\ 0 & 0 & 4 \\ 5 & 0 & 0 \end{vmatrix} = 2(+5)\begin{vmatrix} 3 & 0 \\ 0 & 4 \end{vmatrix} = 2 \cdot 5 \cdot 3 \cdot 4 = 120$$

14. $\begin{vmatrix} 4 & 2 & -2 \\ 3 & 1 & -5 \\ -5 & -4 & 3 \end{vmatrix} \overset{R1-R2}{=} \begin{vmatrix} 1 & 1 & 3 \\ 3 & 1 & -5 \\ -5 & -4 & 3 \end{vmatrix} \overset{\substack{R2-3R1 \\ R3+5R1}}{=} \begin{vmatrix} 1 & 1 & 3 \\ 0 & -2 & -14 \\ 0 & 1 & 18 \end{vmatrix} = +1\begin{vmatrix} -2 & -14 \\ 1 & 18 \end{vmatrix} = -22$

15. $\begin{vmatrix} -2 & 5 & 4 \\ 5 & 3 & 1 \\ 1 & 4 & 5 \end{vmatrix} \overset{R1+2R3}{=} \begin{vmatrix} 0 & 13 & 14 \\ 5 & 3 & 1 \\ 1 & 4 & 5 \end{vmatrix} \overset{R2-5R3}{=} \begin{vmatrix} 0 & 13 & 14 \\ 0 & -17 & -24 \\ 1 & 4 & 5 \end{vmatrix} = +1\begin{vmatrix} 13 & 14 \\ -17 & -24 \end{vmatrix} = -74$

16. $\begin{vmatrix} 2 & 4 & -2 \\ -5 & -4 & -1 \\ -4 & 2 & 1 \end{vmatrix} \overset{R1-2R3}{=} \begin{vmatrix} 10 & 0 & -4 \\ -5 & -4 & -1 \\ -4 & 2 & 1 \end{vmatrix} \overset{R2+2R3}{=} \begin{vmatrix} 10 & 0 & -4 \\ -13 & 0 & 1 \\ -4 & 2 & 1 \end{vmatrix} = -2\begin{vmatrix} 10 & -4 \\ -13 & 1 \end{vmatrix} = 84$

17. $\begin{vmatrix} 2 & 3 & 3 & 1 \\ 0 & 4 & 3 & -3 \\ 2 & -1 & -1 & -3 \\ 0 & -4 & -3 & 2 \end{vmatrix} \overset{R3-R1}{=} \begin{vmatrix} 2 & 3 & 3 & 1 \\ 0 & 4 & 3 & -3 \\ 0 & -4 & -4 & -4 \\ 0 & -4 & -3 & 2 \end{vmatrix} = 2\begin{vmatrix} 4 & 3 & -3 \\ -4 & -4 & -4 \\ -4 & -3 & 2 \end{vmatrix} \overset{\substack{R2+R1 \\ R3+R1}}{=} 2\begin{vmatrix} 4 & 3 & -3 \\ 0 & -1 & -7 \\ 0 & 0 & -1 \end{vmatrix} = 8$

18. $\begin{vmatrix} 1 & 4 & 4 & 1 \\ 0 & 1 & -2 & 2 \\ 3 & 3 & 1 & 4 \\ 0 & 1 & -3 & -2 \end{vmatrix} \overset{R3-3R1}{=} \begin{vmatrix} 1 & 4 & 4 & 1 \\ 0 & 1 & -2 & 2 \\ 0 & -9 & -11 & 1 \\ 0 & 1 & -3 & -2 \end{vmatrix} = 1\begin{vmatrix} 1 & -2 & 2 \\ -9 & -11 & 1 \\ 1 & -3 & -2 \end{vmatrix} \overset{\substack{R2+9R1 \\ R3-R1}}{=} \begin{vmatrix} 1 & -2 & 2 \\ 0 & -29 & 19 \\ 0 & -1 & -4 \end{vmatrix} = 135$

19. $\begin{vmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & -2 & 0 \\ -2 & 3 & -2 & 3 \\ 0 & -3 & 3 & 3 \end{vmatrix} \overset{R3+2R1}{=} \begin{vmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & -2 & 0 \\ 0 & 3 & -2 & 9 \\ 0 & -3 & 3 & 3 \end{vmatrix} = 1\begin{vmatrix} 1 & -2 & 0 \\ 3 & -2 & 9 \\ -3 & 3 & 3 \end{vmatrix} \overset{C2+2C1}{=} \begin{vmatrix} 1 & 0 & 0 \\ 3 & 4 & 9 \\ -3 & -3 & 3 \end{vmatrix} = 39$

20. $\begin{vmatrix} 1 & 2 & 1 & -1 \\ 2 & 1 & 3 & 3 \\ 0 & 1 & -2 & 3 \\ -1 & 4 & -2 & 4 \end{vmatrix} \overset{\substack{R2-2R1 \\ R4+R1}}{=} \begin{vmatrix} 1 & 2 & 1 & -1 \\ 0 & -3 & 1 & 5 \\ 0 & 1 & -2 & 3 \\ 0 & 6 & -1 & 3 \end{vmatrix} = 1\begin{vmatrix} -3 & 1 & 5 \\ 1 & -2 & 3 \\ 6 & -1 & 3 \end{vmatrix} \overset{\substack{R2+2R1 \\ R3+R1}}{=} \begin{vmatrix} -3 & 1 & 5 \\ -5 & 0 & 13 \\ 3 & 0 & 8 \end{vmatrix} = 79$

21. $\Delta = \begin{vmatrix} 3 & 4 \\ 5 & 7 \end{vmatrix} = 1; \quad x = \frac{1}{\Delta}\begin{vmatrix} 2 & 4 \\ 1 & 7 \end{vmatrix} = 10, \quad y = \frac{1}{\Delta}\begin{vmatrix} 3 & 2 \\ 5 & 1 \end{vmatrix} = -7$

22. $\Delta = \begin{vmatrix} 5 & 8 \\ 8 & 13 \end{vmatrix} = 1; \quad x = \frac{1}{\Delta}\begin{vmatrix} 3 & 8 \\ 5 & 13 \end{vmatrix} = -1, \quad y = \frac{1}{\Delta}\begin{vmatrix} 5 & 3 \\ 8 & 5 \end{vmatrix} = 1$

5. $\mathbf{v} = \frac{3}{2}\mathbf{u}$, so the vectors $\mathbf{u}$ and $\mathbf{v}$ are linearly dependent.

6. $a\mathbf{u}+b\mathbf{v} = a(0,2)+b(3,0) = (3b,2a) = \mathbf{0}$ implies $a=b=0$, so the vectors $\mathbf{u}$ and $\mathbf{v}$ are linearly independent.

7. $a\mathbf{u}+b\mathbf{v} = a(2,2)+b(2,-2) = (2a+2b,2a-2b) = \mathbf{0}$ implies $a=b=0$, so the vectors $\mathbf{u}$ and $\mathbf{v}$ are linearly independent.

8. $\mathbf{v} = -\mathbf{u}$, so the vectors $\mathbf{u}$ and $\mathbf{v}$ are linearly dependent.

In each of Problems 9–14, we set up and solve (as in Example 2 of this section) the system

$$a\mathbf{u}+b\mathbf{v} = \begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \mathbf{w}$$

to find the coefficient values a and b such that $\mathbf{w}=a\mathbf{u}+b\mathbf{v}$,

9. $\begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow a=3,\ b=2$ so $\mathbf{w} = 3\mathbf{u}+2\mathbf{v}$

10. $\begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \Rightarrow a=2,\ b=-3$ so $\mathbf{w} = 2\mathbf{u}-3\mathbf{v}$

11. $\begin{bmatrix} 5 & 2 \\ 7 & 3 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow a=1,\ b=-2$ so $\mathbf{w} = \mathbf{u}-2\mathbf{v}$

12. $\begin{bmatrix} 4 & -2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix} \Rightarrow a=3,\ b=5$ so $\mathbf{w} = 3\mathbf{u}+5\mathbf{v}$

13. $\begin{bmatrix} 7 & 3 \\ 5 & 4 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \end{bmatrix} \Rightarrow a=2,\ b=-2$ so $\mathbf{w} = 2\mathbf{u}-3\mathbf{v}$

14. $\begin{bmatrix} 5 & -6 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix} \Rightarrow a=7,\ b=5$ so $\mathbf{w} = 7\mathbf{u}+5\mathbf{v}$

In Problems 15–18, we calculate the determinant $\left|\mathbf{u} \quad \mathbf{v} \quad \mathbf{w}\right|$ so as to determine (using Theorem 4) whether the three vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are linearly dependent (det = 0) or linearly independent (det ≠ 0).

The nontrivial solution $a = 11,\ b = 4,\ c = -1$ gives $11\mathbf{u} + 4\mathbf{v} - \mathbf{w} = \mathbf{0}$, so the three vectors are linearly dependent.

22. $$\mathbf{A} = \begin{bmatrix} 1 & 5 & 0 \\ 1 & 1 & 1 \\ 0 & 3 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{E}$$

The system $\mathbf{Ax} = \mathbf{0}$ has only the trivial solution $a = b = c = 0$, so the vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are linearly independent.

23. $$\mathbf{A} = \begin{bmatrix} 2 & 5 & 2 \\ 0 & 4 & -1 \\ 3 & -2 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{E}$$

The system $\mathbf{Ax} = \mathbf{0}$ has only the trivial solution $a = b = c = 0$, so the vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are linearly independent.

24. $$\mathbf{A} = \begin{bmatrix} 1 & 4 & -3 \\ 4 & 2 & 3 \\ 5 & 5 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{E}$$

The system $\mathbf{Ax} = \mathbf{0}$ has only the trivial solution $a = b = c = 0$, so the vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are linearly independent.

In Problems 25–28, we solve the nonhomogeneous system $\mathbf{Ax} = \mathbf{t}$ by reducing the augmented coefficient matrix $\mathbf{A} = \begin{bmatrix} \mathbf{u} & \mathbf{v} & \mathbf{w} & \mathbf{t} \end{bmatrix}$ to echelon form $\mathbf{E}$. The solution vector $\mathbf{x} = \begin{bmatrix} a & b & c \end{bmatrix}^T$ appears as the final column of $\mathbf{E}$, and provides us with the desired linear combination $\mathbf{t} = a\mathbf{u} + b\mathbf{v} + c\mathbf{w}$.

25. $$\mathbf{A} = \begin{bmatrix} 1 & 3 & 1 & 2 \\ -2 & 0 & -1 & -7 \\ 2 & 1 & 2 & 9 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{bmatrix} = \mathbf{E}$$

Thus $a = 2,\ b = -1,\ c = 3$ so $\mathbf{t} = 2\mathbf{u} - \mathbf{v} + 3\mathbf{w}$.

26. $$\mathbf{A} = \begin{bmatrix} 5 & 1 & 5 & 5 \\ 2 & 5 & -3 & 30 \\ -2 & -3 & 4 & -21 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & -1 \end{bmatrix} = \mathbf{E}$$

Thus $a = 1,\ b = 5,\ c = -1$ so $\mathbf{t} = \mathbf{u} + 5\mathbf{v} - \mathbf{w}$.

so their sum $(x+u, y+v, z+w)$ is in V. Similarly,

$$cz = c(2x+3y) = 2(cx)+3(cy),$$

so the scalar multiple (cx, cy, cz) is in V.

33. $(0,1,0)$ is in V but the sum $(0,1,0)+(0,1,0) = (0,2,0)$ is not in V; thus V is not closed under addition. Alternatively, $2(0,1,0) = (0,2,0)$ is not in V, so V is not closed under multiplication by scalars.

34. $(1,1,1)$ is in V, but

$$2(1,1,1) = (1,1,1)+(1,1,1) = (2,2,2)$$

is not, so V is closed neither under addition of vectors nor under multiplication by scalars.

35. Evidently V is closed under addition of vectors. However, $(0,0,1)$ is in V but $(-1)(0,0,1) = (0,0,-1)$ is not, so V is not closed under multiplication by scalars.

36. $(1,1,1)$ is in V, but

$$2(1,1,1) = (1,1,1)+(1,1,1) = (2,2,2)$$

is not, so V is closed neither under addition of vectors nor under multiplication by scalars.

37. Pick a fixed element $\mathbf{u}$ in the (nonempty) vector space V. Then, with $c = 0$, the scalar multiple $c\mathbf{u} = 0\mathbf{u} = \mathbf{0}$ must be in V. Thus V necessarily contains the zero vector $\mathbf{0}$.

38. Suppose $\mathbf{u}$ and $\mathbf{v}$ are vectors in the subspace V of $\mathbf{R}^3$ and a and b are scalars. Then $a\mathbf{u}$ and $b\mathbf{v}$ are in V because V is closed under multiplication by scalars. But then it follows that the linear combination $a\mathbf{u}+b\mathbf{v}$ is in V because V is closed under addition of vectors.

39. It suffices to show that every vector $\mathbf{v}$ in V is a scalar multiple of the given nonzero vector $\mathbf{u}$ in V. If $\mathbf{u}$ and $\mathbf{v}$ were linearly independent, then — as illustrated in Example 2 of this section — every vector in $\mathbf{R}^2$ could be expressed as a linear combination of $\mathbf{u}$ and $\mathbf{v}$. In this case it would follow that V is all of $\mathbf{R}^2$ (since, by Problem 38, V is closed under taking linear combinations). But we are given that V is a proper subspace of $\mathbf{R}^2$, so we must conclude that $\mathbf{u}$ and $\mathbf{v}$ are linearly dependent vectors. Since $\mathbf{u} \neq 0$, it follows that the arbitrary vector $\mathbf{v}$ in V is a scalar multiple of $\mathbf{u}$, and thus V is precisely the set of all scalar multiples of $\mathbf{u}$. In geometric language, the subspace V is then the straight line through the origin determined by the nonzero vector $\mathbf{u}$.