

# Math 2250 – Final Exam

## December 15, 2015 – 10:30 AM

As with the previous outline, this outline is not meant to be an end-all exhaustive study replacement, but rather a structured baseline for your studying. That is, this is a study *companion*. **If something is not on this outline, that does not mean it will not be on the exam!**

The final exam is **cumulative**, meaning it will draw from the topics discussed over the entirety of the course. The breakdown will be roughly:

- $\frac{1}{3}$  exam 1 material
- $\frac{1}{3}$  exam 2 material
- $\frac{1}{3}$  material covered since exam 2.

For the first two exams, you already have outlines discussing the topics. The remaining topics are outlined below. Since you have slightly more than double the time of the previous exams (2 hours), expect the length to be approximately twice that of the midterm exams.

On the Thursday before the exam, Cheehan and Huachen will review. I'll do my best to be around as much as possible the week before finals as well as Monday (the day before the final). If there is enough interest, I could also come in at some point during the weekend: just send me an email.

## Chapter 10: Transform Methods

### 10.1: Laplace Transform and its Inverse

- Power series motivation: if we have some  $\sum_{n=0}^{\infty} a(n)x^n = f(x)$ , then we can associate  $f(x) \leftrightarrow a(n)$ .
- In other words: we can associate two functions of different variables. This is the idea of a *transform*
- Operators take  $f(x) \rightarrow g(x)$  (for example, a derivative). Transforms take  $f(x) \rightarrow F(y)$ . They change the variable.
- Definition of the Laplace transform of  $f(t)$ :  $F(s) = \mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st}f(t) dt$ .
- In the language above: we associate  $F(s) \leftrightarrow f(t)$ . Why? Some problems (DEs) are easier to solve once we've transformed them.
- **You'll be given the table of transforms**, but be familiar with how to derive Laplace transforms from the definition. For example,  $\mathcal{L}\{\exp(at)\} = 1/(s - a)$ .
- Very useful theorem: linearity of the transform.  $\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f\} + b\mathcal{L}\{g\}$ .
- Often have to make clever substitutions to evaluate, like:  $\cos kt = \frac{1}{2}(\exp ikt - \exp -ikt)$ .
- Inverse transform: going the other way,  $f(t) = \mathcal{L}^{-1}\{F(s)\}$ .
- Unit step function:  $u(t - c) = u_c(t)$  is 0 before  $t = c$  and 1 after.

- Sometimes useful fact: if  $F(s)$  is the Laplace transform of some “nice” function  $f(t)$  then  $F(s) \rightarrow 0$  as  $s \rightarrow \infty$ . Therefore, we can eliminate insane  $F(s)$  as impossible for physical problems.

## 10.2: Transformations of IVPs

- We assume that we’re taking the Laplace transform of things that don’t blow up.
- Using this assumption,  $\mathcal{L}\{f'(t)\} = sF(s) - f(0)$ . Can you prove this? Integration by parts.
- Similarly,  $\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$ .
- In other words: Laplace transform of derivatives depends on the initial conditions (in time!), making it very useful for initial value problems
- General procedure: start with some differential equation  $x'(t) = \dots$ , transform it and solve for  $X(s) = \dots$ , invert the Laplace transform to explicitly get  $x(t)$ .
- Often this involves partial fractions and other clever manipulations. Doing many problems is the only way to get the hang of these!
- Overall: we should be thankful, we’ve turned a (potentially difficult) differential problem into an algebraic one.
- Additional property (idea): if multiplying by  $s$  in  $s$ -space corresponds to a derivative in time, dividing by  $s$  corresponds to an integral in time. In other words,  $\mathcal{L}\left\{\int_0^t f(\tau) d\tau\right\} = F(s)/s$ .

## 10.3: Translation, Partial Fractions

- We didn’t review partial fractions in depth, know how to do them! They show up a ton in Laplac transforms.
- Translation idea:  $\mathcal{L}\{e^{at}f(t)\} = F(s - a)$ .
- Common trick: rewrite denominators with complex roots as  $(s + a)^2 + k^2$  by completing the square

## 10.4: Derivatives, Integrals, Products of Transforms

- Motivating question: is  $\mathcal{L}\{f(t) \cdot g(t)\} = F(s) \cdot G(s)$ ? No! We need to modify our definition of multiplication for this to be true.
- Actual statement:  $\mathcal{L}\{f(t) \cdot g(t)\} = F * G(s)$  and  $\mathcal{L}\{f * g\} = F(s) \cdot G(s)$
- What is  $*$ ? The convolution. Like multiplication (two functions in, one out) but more complicated. Defined by:  $(f * g)(t) = \int_0^t f(\tau)g(t - \tau) d\tau$ .
- $f * g = g * f$ , doesn’t matter which term you put the  $t - \tau$  on.
- Duality: multiplying by the variable  $s$  (in  $s$ -space) gave us derivatives in time ( $t$ -space): multiplying by  $t$  in time gives us derivatives in  $s$ -space.
- Actual theorem:  $\mathcal{L}\{-tf(t)\} = F'(s)$ . Useful version:  $f(t) = -\frac{1}{t}\mathcal{L}^{-1}\{F'(s)\}$ .
- Practicality: sometimes its easier to take the inverse Laplace transform of the derivative so now we can just do that and correct by a factor of  $t$ .
- Similar idea: dividing by  $s$  gave us integrals in time, so dividing by  $t$  gives us integrals in  $s$ .

- Actual theorem:  $\mathcal{L}\{f(t)/t\} = \int_s^\infty F(\sigma) d\sigma$ . Practical version:  $f(t) = t\mathcal{L}^{-1}\{\int_s^\infty F(\sigma) d\sigma\}$ .
- Again: sometimes it's easier to invert the integral of whatever we have, so now we can do that and correct by a factor of  $t$ .

## 10.5: Periodic, Piecewise Continuous Inputs

- Recall: exponential in time produced a shift in  $s$ , the opposite is also true:  $\mathcal{L}\{u(t-a)f(t-a)\} = e^{-as}F(s)$ .
- If  $f(t+p) = f(t)$ , that is, we have a repeating (periodic) source with period  $p$  then we can take the Laplace transform of just one period:  $F(s) = \frac{1}{1-e^{-ps}} \int_0^p e^{-st}f(s) ds$

# Chapter 6: Eigenvalues, Eigenvectors

## 6.1: Introduction

- Definition of an eigenvalue/eigenvector for a matrix  $A$ : a number (scalar)  $\lambda$  and vector  $\mathbf{v}$  such that  $A\mathbf{v} = \lambda\mathbf{v}$ .
- Note: eigenvalues and eigenvectors are associated with each other. You can't just freely mix and match.
- If  $\mathbf{v}$  is an eigenvector, so is  $c\mathbf{v}$ , thus we typically just take one representative scaling
- We want to solve  $A\mathbf{v} = \lambda\mathbf{v}$  which is equivalent to  $(A - \lambda I)\mathbf{v} = \mathbf{0}$ . When does this have non-trivial solutions for  $\mathbf{v}$ ? When the matrix  $A - \lambda I$  is singular!
- How do we force that matrix to be singular?  $\det(A - \lambda I) = 0$ . This gives us a polynomial in  $\lambda$  called the **characteristic polynomial**. We find the roots of it, which correspond to eigenvalues.
- Procedure: find the  $\lambda$ 's from the roots of the characteristic polynomial, for each  $\lambda_j$  solve the linear system  $(A - \lambda_j I)\mathbf{v}_j = \mathbf{0}$  for the corresponding eigenvector(s)  $\mathbf{v}_j$ .
- We saw many interesting possibilities for a  $2 \times 2$  matrix: two distinct eigenvalues each with distinct eigenvectors, complex conjugate pair of eigenvalues with a complex conjugate pair of eigenvectors, repeated eigenvalue with corresponding 1 or 2 eigenvectors (this is interesting!).
- We note that we  $(A - \lambda_j I)\mathbf{v}_j = \mathbf{0}$  is a homogeneous linear system, thus the solutions produce a solution space. We call this the eigenspace corresponding to  $\lambda_j$ .

## 6.2: Diagonalization

- Big idea: assuming we have  $n$  linearly independent eigenvectors of  $A$ , we can decompose  $A = PDP^{-1}$ , where  $P$  is a matrix consisting of the eigenvectors and  $D$  is a diagonal matrix of the eigenvalues.
- In other words: the eigenvalues/eigenvectors tell us everything about the matrix!
- A matrix is diagonalizable if and only if it has  $n$  linearly independent eigenvectors
- One sufficient (but not necessary) condition to satisfy this is:  $n$  distinct eigenvalues.

# Chapter 7: Linear Systems of Differential Equations

## 7.1: First Order Systems

- We now examine **coupled** (they depend on each other) differential equations. For instance,  $x' = x + y, y' = x - y$ .
- Where do these arise? One example: spring mass systems,  $F = ma$  still but we get a system for each mass's displacement
- Another example: tank problems, each  $x_i$  is the amount of salt in a tank.
- Huge idea: **we can rewrite any linear system of differential equations as a linear first order system!**
- How do we do this? If we had say,  $x'' = \dots, y'' = \dots$ , we can introduce new variables, say  $x' = w, y' = z$ , so now we have a system  $x' = w, y' = z, w' = \dots, z' = \dots$
- Technique worth noting: if we have  $x' = f(x, y), y' = g(x, y)$ , we can study trajectories (that is, not how they change in time but where they move by noticing)  $dy/dx = g(x, y)/f(x, y)$ , which might be a separable differential equation.

## 7.2: Matrices, Linear Systems

- Due to the result above, we can focus on studying linear first order differential equations.
- In general, these can be formulated as  $\mathbf{x}'(t) = \mathbf{A}\mathbf{x} + \mathbf{f}(t)$ .
- What do we mean by  $\mathbf{x}'(t)$ ? Just take derivative of each element.
- Basically, our solution to our DE is now a vector of  $n$  functions, not just one function!
- Note that even if we had higher order derivatives, we can eliminate them (and add more equations) by the process described in section 7.1.
- Homogeneous equation:  $\mathbf{f}(t) = 0$ .
- Principle of superposition: If we have say,  $\mathbf{x}_1, \dots, \mathbf{x}_n$  as solutions to our differential equation, any linear combination  $\mathbf{x} = c_1\mathbf{x}_1 + \dots + c_n\mathbf{x}_n$  is also a solution. This always happens with linear systems!
- We want to find a basis of solutions. How do we know if we have a basis? Spanning and linear independence.
- To check linear independence: use the Wronskian. No longer need derivatives. The Wronskian is just  $|\mathbf{x}_1 \cdots \mathbf{x}_n|$ .
- How do we use the Wronskian? If the functions are linearly dependent  $W = 0$  at every point. If they are independent then  $W \neq 0$  at each point. Very similar to vectors in  $\mathbb{R}^n$ .
- Theorem: if we have  $n$  linearly independent solutions  $\mathbf{x}_1, \dots, \mathbf{x}_n$  to the homogeneous equation then **any** solution can be written as  $\mathbf{x} = c_1\mathbf{x}_1 + \dots + c_n\mathbf{x}_n$ .
- In other words, if we have  $n$  linearly independent solutions, they span the solution space (can generate any solution) and therefore constitute a basis. (This is again similar to  $\mathbb{R}^n$ .)
- In the above, the  $c_i$ 's are determined by the initial conditions

- Identical to our 1D differential equations: if we have the general solution for the homogeneous problem, all solutions to the non-homogeneous problem are  $\mathbf{x} = \mathbf{x}_c + \mathbf{x}_p$ , where  $\mathbf{x}_p$  is any other particular solution to the non-homogeneous problem.

### 7.3: Eigenvalue Method

- We know from the previous section if we find  $n$  linearly independent solutions to the homogeneous equation, we have a basis and we're effectively done. How do we do this?
- Note that  $\mathbf{x}' = A\mathbf{x}$ , if you squint, looks like  $x' = Ax$  which has solutions  $e^{Ax}$ , so let's guess solutions of the form  $\mathbf{x} = \mathbf{v}e^{\lambda t}$ .
- When we plug this in, we find this is indeed a solution if  $A\mathbf{v} = \lambda\mathbf{v}$ . In other words, eigenvalues/eigenvectors tell us our basis elements!
- Procedure for solving  $\mathbf{x}' = A\mathbf{x}$ : find  $\lambda$ 's of  $A$ , then find  $\mathbf{v}$ 's corresponding to them. Combine in a linear combination to get the general solution. That is,  $\mathbf{x}(t) = c_1\mathbf{v}_1e^{\lambda_1 t} + \dots + c_n\mathbf{v}_ne^{\lambda_n t}$ .
- Real, distinct eigenvalues: this easy. Just regular exponentials that grow or decay.
- Say we have a complex conjugate pair of eigenvalues  $\lambda = p \pm qi$ , we know we have a conjugate pair of eigenvectors  $\mathbf{v} = \mathbf{a} + \mathbf{b}i$ , thus our solution is  $\tilde{\mathbf{x}}(t) = \mathbf{v}e^{\lambda t} = (\mathbf{a} + \mathbf{b}i)e^{pt}(\cos qt + i \sin qt)$ .
- Big idea: how do we get two *real* eigenfunctions out of this? Take the real and imaginary parts of  $\tilde{\mathbf{x}}$ !  $\mathbf{x}_1 = e^{pt}(\mathbf{a} \cos qt - \mathbf{b} \sin qt)$  and  $\mathbf{x}_2 = e^{pt}(\mathbf{b} \cos qt + \mathbf{a} \sin qt)$ .
- Note, it doesn't matter if we start with the + or - in the conjugate, as the final answer will just differ by a - sign and will be absorbed into the constants anyway.

### 7.4: Second Order Systems

- Although we can reduce everything to a first order system, second order systems are pretty common. Can we say anything about them?
- Motivating example, spring mass system, we end up with:  $M\ddot{\mathbf{x}} = K\mathbf{x}$  and since  $M$  is diagonal, its invertible so we effectively have  $\ddot{\mathbf{x}} = A\mathbf{x}$ .
- If we guess a solution  $\mathbf{x} = \mathbf{v}e^{\alpha t}$  we find that  $A\mathbf{v} = \alpha^2\mathbf{v}$ . In other words, our solutions are growing exponentials where  $\alpha = \sqrt{\lambda}$ , where  $\lambda$  is an eigenvalue of  $A$ .
- For physical systems, its very common for  $A$  to have negative eigenvalues, meaning  $\sqrt{\lambda} = \text{complex number}$ .
- From this observation, we get the result, if  $A$  has eigenvalues of the form  $-\omega_1^2, \dots, -\omega_n^2$  then the solution to the above 2nd order system is  $\mathbf{x}(t) = \sum_{j=1}^n \mathbf{v}_j [a_j \cos \omega_j t + b_j \sin \omega_j t]$ , where  $a_j, b_j$  are arbitrary constant
- addendum to the above: if  $\omega_0 = 0$ , the corresponding term is  $\mathbf{x}_0 = (a_0 + b_0 t)\mathbf{v}_0$ .
- In other words, for the (undamped) spring mass system, we get oscillatory behavior
- Each term in this series can be called a "mode" of vibration or motion. The components of  $\mathbf{v}_j$  tell you how the masses want to move relative to each other,  $\omega$  tells you how quickly they do this.

- We can also consider forced systems of the form  $M\ddot{\mathbf{x}} = K\mathbf{x} + \mathbf{F}$  but again since  $M$  is invertible, we really have  $\ddot{\mathbf{x}} = A\mathbf{x} + \mathbf{f}$ .
- If in particular,  $\mathbf{f} = \mathbf{F}_0 \cos \omega t$ , then we guess a particular solution  $\mathbf{x}_p = \mathbf{c} \cos \omega t$ , which leads us to  $(A + \omega^2 I)\mathbf{c} = -\mathbf{F}_0$ .
- Interpretation of the above: unless  $-\omega^2 = \lambda$ , that is, we're forcing at one of the natural frequencies, then the above can be solved.
- As  $\omega \rightarrow \lambda$ , we get resonance (our solution blows up)
- Our solutions look like  $\mathbf{x}(t) = \mathbf{x}_c + \mathbf{x}_p$ . If we include damping,  $\mathbf{x}_c \rightarrow 0$  and  $\mathbf{x} \rightarrow \mathbf{x}_p$ , the steady state.

## Chapter 9: Non-linear Systems

### 9.1: Stability, Phase Plane

- In general, non-linear systems are way way harder than linear systems. Far fewer systematic tools exist for studying.
- Suppose we have the autonomous system  $y' = G(x, y), x' = F(x, y)$ . One tool for studying is drawing the slope field:  $dy/dx = G(x, y)/F(x, y)$ . Trajectories form a "phase portrait"
- If these equations weren't autonomous, the picture would change over time: completely screwed, basically.
- Null clines:  $F = 0$  or  $G = 0$ . What do the intersections of null-clines tell us? BOTH  $F = G = 0$ , we call these critical points.
- Say  $F(x_*, y_*) = G(x_*, y_*) = 0$  then  $x \equiv x_*, y \equiv y_*$  is a constant (equilibrium) solution
- What is the behavior of equilibria? That is, how do solutions tend toward or away from them? Hard to tell! Will address in 9.2.
- Some possibilities: "node" when trajectories all go into or out of this point. "sink" if they go in, "source" if they go out.
- idea of stability: start near an equilibria, stay near it. unstable if not stable
- saddle node: direction approaching point determines whether you approach it or are repulsed by it
- center: oscillatory trajectories around the point
- spiral sink/source node: same idea except that you spiral inward or outward.
- can spiral in/out toward a limit cycle (oscillation)

### 9.2: Linear, Almost Linear Systems

- How can we get a better handle on the behavior of critical points?
- Perturb away slightly:  $u = x - x_*, v = y - y_*$ , then we can come up with ODEs:
- $u' \approx f_x(\mathbf{x}_*, y_*)u + f_y(\mathbf{x}_*, y_*)v$ . This is a linear system! Eigenvalues of "linearized" system tell us the behavior locally!
- Jacobian matrix = matrix of partial derivatives evaluated at critical point.

- Real negative eigenvalues: stable, real positive eigenvalues: outward. Opposite signs: saddle point (direction matters).
- Pure imaginary: center. Complex: stable/unstable spiral.

### **9.3/4: Applications**

- predator-prey, infectious diseases, non-perfect springs.
- lots of other exciting things we don't have time to cover but I'll gladly ramble about in my office anytime 😊