Final Exam
Math 2250 - Differential Equations & Linear Algebra
December 15, 2015

Answer each question completely in the area below. Show all work and explain your reasoning. If the work is at all ambiguous, it is considered incorrect. No phones, calculators, or notes are allowed. Anyone found violating these rules or caught cheating will be asked to leave immediately. Point values are in the square to the left of the question. **If there are any other issues, please ask the instructor.**

By signing below, you are acknowledging that you have read and agree to the above paragraph, as well as agree to abide University Honor Code:

Name: ______________________________

Signature: ______________________________

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Solutions

Question	Points	Score
1	15	
2	20	
3	40	
4	15	
5	15	
6	10	
7	10	
8	20	
9	25	
10	30	
Total:	200	

Note: There are 9 questions on the exam with 200 points available.

15 1. Solve the initial value problem for $x(t)$:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\frac{x}{t} + e^{-t}, \quad x(1) = 1.$$

Solution: We see that this a linear, first order differential equation of the form

$$\frac{\mathrm{d}x}{\mathrm{d}t} + P(t)x = Q(t),$$

which means we have a technique for solving this: **integrating factor**!

$$\begin{aligned}
\mu = e^{\int \frac{1}{t}\mathrm{d}t} &= e^{\ln(t)} = t \\
t\frac{\mathrm{d}x}{\mathrm{d}t} + x &= te^{-t} \\
\frac{\mathrm{d}}{\mathrm{d}t}\{tx\} &= te^{-t} \\
tx(t) &= \int te^{-t}dt + C \\
tx(t) &= -te^{-t} - e^{-t} + C \\
x(t) &= \frac{C}{t} - e^{-t} - \frac{e^{-t}}{t} \\
x(1) &= C - 2e^{-1} = 1 \\
C &= 1 + 2e^{-1}.
\end{aligned}$$

______/15 pts

2. Consider the matrix

$$A = \begin{bmatrix} 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 \end{bmatrix}$$

10 (a) Find a basis that spans the solution space of the homogeneous equation $A\mathbf{x} = \mathbf{0}$. *Hint:* reduce this matrix to reduced row echelon form first.

5 (b) What is the dimension of the solution space ?

5 (c) Is the vector $\mathbf{v} = [1, 0, 0, 0, 1]^T$ is in the spanning set of the basis you found above?

Solution: **This is a slightly modified question from exam II and a quiz.**

(a) This is already pretty close to reduced row echelon form. We can say, add the second equation to the first and then add the third equation to the second to yield:

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & 1 \end{bmatrix}$$

We now parameterize our non-leading variables x_4, x_5 and write the other variables in terms of these two.

$$x_4 = s, \quad x_5 = t$$
$$x_3 = s - t$$
$$x_2 = -t$$
$$x_1 = -s$$
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} s + \begin{bmatrix} 0 \\ -1 \\ -1 \\ 0 \\ 1 \end{bmatrix} t$$

Thus, the basis is:

$$B = \left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

(b) We have two basis vectors, so the dimension is two.

(c) No. If we try to write $\mathbf{v}$ as a linear combination of our two basis vectors:

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \\ 1 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

no combination of s and t can solve this system.

3. For both parts of this problem, solve the linear differential system

$$\mathbf{x}' = A\mathbf{x}$$

with

20 (a)

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \qquad \mathbf{x}(0) = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

20 (b)

$$A = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}$$

(no initial condition for this one, just provide a general solution).

Solution:

(a) The eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = 1$, The eigenvectors are

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\mathbf{x}(t) = c_1\mathbf{v}_1 e^{3t} + c_2\mathbf{v}_2 e^{t}$$

$$\mathbf{x}(0) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$
$$\implies c_1 = -0.5,\ c_2 = -1.5$$

(b) **This was taken directly from the practice exam. There was also a quiz problem nearly identical to this.**

The eigenvalues are $\lambda = -1 \pm i = a \pm ib$. The eigenvectors are

$$\mathbf{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \pm i \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \mathbf{a} \pm i\mathbf{b}$$

The solution is then, after taking the real and imaginary parts of $\mathbf{v}e^{\lambda t}$, we find:

$$\mathbf{x}(t) = c_1 e^{-t}[(\mathbf{a}\cos(t) - \mathbf{b}\sin(t) + c_2 e^{-t}[\mathbf{b}\cos(bt) + \mathbf{a}\sin(bt)]$$

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15 4. Using the **Laplace transform**, solve the following differential equation:

$$x'' + 2x' + 2x = e^{-t}, \qquad x(0) = 1, \qquad x'(0) = 1.$$

Hint: you should end up with partial fractions with two factors, one of degree one and the other of degree two.

Solution: This was taken directly from the practice exam.

When we Laplace transform both sides of the equation, we get

$$\left\{s^2 X - s - 1\right\} + 2\left\{sX - 1\right\} + 2X = \frac{1}{s+1}.$$

Rearranging, this becomes

$$\left\{s^2 + 2s + 2\right\} X - 3 - s = \frac{1}{s+1},$$

which we can move to the other side to find

$$\left\{s^2 + 2s + 2\right\} X = \frac{s^2 + 4s + 4}{s+1},$$

and now solving for X:

$$X = \frac{(s+2)^2}{(s+1)(s^2+2s+2)},$$

which we must use partial fractions on:

$$X = \frac{A}{s+1} + \frac{Bs + C}{s^2 + 2s + 2},$$

through any variety of techniques, we find $A = 1, B = 0, C = 2$. We're then left with (after completing the square on the second term):

$$X = \frac{1}{s+1} + \frac{2}{(s+1)^2 + 1}.$$

We can invert the Laplace transform on this directly to find

$$x(t) = 2e^{-t}\sin(t) + e^{-t}.$$

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15 5. Consider the 5 great lakes, diagrammed below:

Denote the amount of polution in lake i as p_i (measured in pu) and consider the vector $\mathbf{p}(t) = [p_i]$.

In the above diagram, the flow rates (and directions) between the lakes are designated by the r_i terms. You can assume the volume of each lake is a constant, V_i. The system is mostly polluted from once source: Chicago.

Write down but do not solve a linear differential system describing the amount of pollutant $\mathbf{p}(t)$ in the great lakes. Your answer should look something like $\dot{\mathbf{p}} = A\mathbf{p} + \mathbf{f}(t)$.

Solution: It doesn't matter which order we place the elements of the vector. Let's go from left to right, so:

$$\mathbf{p} = [p_S \quad p_M \quad p_H \quad p_E \quad p_O]^T,$$

where p_i denotes the amount of pollution (measured in pu) in lake i.

We need to just figure out the fluxes into each of the lake. Notice that $\frac{dp_i}{dt}$ has units (change in pollution)/time. These are the units we need for each of the terms in our differential equation. How do we achieve this? Consider the first lake, Superior. The corresponding DE is:

$$\frac{dp_S}{dt} = \frac{p_H}{V_H}\,[\text{pu/L}] \times r_2\,[\text{L/year}] = \frac{p_H r_2}{V_H}\,[\text{pu/year}].$$

From this, we can see that in general the flux terms will look like concentration $c_i = p_i/v_i$ multiplied by the flow rate r_i, which produces the units we desire.

The linear system is then:

$$\begin{aligned}
\frac{\mathrm{d}p_S}{\mathrm{d}t} &= \frac{p_H}{V_H} r_2 \\
\frac{\mathrm{d}p_M}{\mathrm{d}t} &= -\frac{p_M}{V_M} r_1 + p_{in} \\
\frac{\mathrm{d}p_H}{\mathrm{d}t} &= -\frac{p_H}{V_H} r_2 + \frac{p_M}{v_M} r_1 - \frac{p_H}{V_H} r_4 + \frac{p_E}{V_E} r_3 \\
\frac{\mathrm{d}p_E}{\mathrm{d}t} &= \frac{p_H}{V_H} r_4 - \frac{p_E}{V_E} r_3 + \frac{p_O}{V_O} r_6 - \frac{p_E}{V_E} r_5 \\
\frac{\mathrm{d}p_O}{\mathrm{d}t} &= -\frac{p_O}{V_O} r_6 + \frac{p_E}{V_E} r_5.
\end{aligned}$$

which in linear system form looks like

$$\dot{\mathbf{p}} = \mathsf{A}\mathbf{p} + \mathbf{f}(t), \qquad \mathbf{p} = \begin{bmatrix} p_S \\ p_M \\ p_H \\ p_E \\ p_O \end{bmatrix}, \qquad \mathsf{A} = \begin{bmatrix} 0 & 0 & \frac{r_2}{V_H} & 0 & 0 \\ 0 & -\frac{r_1}{V_M} & 0 & 0 & 0 \\ 0 & \frac{r_1}{V_M} & -\frac{r_2}{V_H} - \frac{r_4}{V_H} & \frac{r_3}{V_E} & 0 \\ 0 & 0 & \frac{r_4}{V_H} & -\frac{r_3}{V_E} - \frac{r_5}{V_E} & \frac{r_6}{V_O} \\ 0 & 0 & 0 & \frac{r_5}{V_E} & -\frac{r_6}{V_O} \end{bmatrix}, \qquad \mathbf{f} = \begin{bmatrix} 0 \\ p_{in} \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

10 6. Provide a general solution for the differential equation

$$y^{(4)} + 18y'' + 81y = 0.$$

Solution: **This was taken directly from exam II.**

This is a constant coefficient, linear differential equation, meaning we get the characteristic equation:

$$r^4 + 18r^2 + 81 = (r^2 + 9)^2 = 0.$$

Thus, we have 4 roots, total, $\pm 3i$ repeated twice (multiplicity 2). Our basis of solutions is then:

$$B = \{\sin 3x, \cos 3x, x \sin 3x, \cos 3x\},$$

which provides general solutions of the form

$$y(x) = (c_1 + c_2 x)\cos 3x + (c_3 + c_4 x)\sin 3x.$$

Note we always have the same number of unknowns as total roots (or order of our differential equation). We can't combine the linear terms into a single and distribute it. We need all 4 coefficients.

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10 7. Thinking of each of the systems below as a spring-mass system, which of the following has resonance? Why?

(i) $\ddot{x} + 4\dot{x} + x = \sin 2t$.

(ii) $\ddot{x} + 2x = \sin 2t$.

(iii) $\ddot{x} + 2x = 0$.

(iv) $\ddot{x} + 4x = \sin 2t$.

Solution: **(iv)** We know that the defining characteristic of resonance is that the system is being forced at its natural frequency. Thus, we can immediately eliminate (iii) since it has no forcing.

In class, we also saw that although damped systems can display "practical resonance", a damped system cannot display true resonance. Thus, we can eliminate (i), a damped system.

This leaves the two similar systems, (ii) and (iv), both being forced at $\omega = 2$. Which has this as a natural frequency? If we think in terms of the roots of the characteristic polynomial, (ii) has roots $r^2 = -2 \implies r = \pm i\sqrt{2}$, thus solutions look like $\cos\sqrt{2}t$ and $\sin\sqrt{2}t$. This isn't quite right.

For (iv), we see that the roots of the characteristic polynomial of $r^2 = -4 \implies r = \pm i2$ which has solutions of the form $\sin 2t$ and $\cos 2t$. Thus, this has a natural frequency of $\omega_0 = 2$ and therefore this system will display resonance. s

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20 8. Find the solution to

$$y''' + y'' = 3e^x + 4x.$$

Solution: **This was taken directly from exam II.**

We have a non-homogeneous, linear (constant coefficient) differential equation. Thus, we know that solutions are of the form

$$y(x) = y_c + y_p,$$

where y_c are complementary solutions, or solutions to the homogeneous problem and y_p is a particular solution to the non-homogeneous problem.

We first try to find y_c, which can be found by solving the homogeneous equation

$$y_c''' + y_c'' = 0,$$

which, since we have a constant coefficient differential equation, yields the characteristic equation

$$r^3 + r^2 = r^2(r+1) = 0.$$

We have roots $r = 0$ (twice) and $r = -1$, meaning the basis for our complementary solutions is

$$B = \left\{1, x, e^{-x}\right\},$$

and we have the form

$$y_c(x) = c_1 + c_2 x + c_3 e^{-x}.$$

We now turn to the particular solution, for which our naive guess would be:

$$(A + Bx + Cx^2) + De^x,$$

however, we see that we have duplication from our homogeneous part (specifically, that a constant and x are repeated), so our actual guess is of the form

$$y_p(x) = x^2(A + Bx) + De^x.$$

When we plug this into the ODE, we find

$$y_p''' + y_p'' = (6B + De^x) + (2A + 6Bx + De^x) = 3e^x + 4x^2.$$

Equating powers of x and the exponential, we get:

$$\begin{aligned} 2D &= 3 & \implies D &= 3/2 \\ 6B &= 4 & \implies B &= 2/3 \\ 6B + 2A &= 0 & \implies A &= -2. \end{aligned}$$

Thus, our total solution is

$$y(x) = y_c(x) + y_p(x) = c_1 + c_2 x + c_3 e^{-x} - 2x^2 + \frac{2}{3}x^3 + \frac{3}{2}e^x.$$

9. Vector spaces of functions: Let P be a set of polynomials inside the vector space of functions:

$$P = \{\mathbf{p}_1(x),\ \mathbf{p}_2(x),\ \mathbf{p}_3(x),\ \mathbf{p}_4(x)\} = \{1,\ x,\ 3x^2 - 1,\ \ 5x^3 - 3x\}$$

10 (a) Determine if the set P is linearly independent or not.

10 (b) Does P span the vector space V of all polynomials of third-degree or less? Justify your answer. (*Hint: consider setting up a linear system (matrix) derived from writing a general third-order degree polynomial. What can you say about this matrix?*

5 (c) Is P a basis for the vector space of third-degree polynomials? Why or why not?

Solution: This was taken directly from lab.

(a) Linear independence means that the only solution (c_1, c_2, c_3, c_4) to the homogeneous equation

$$c_1\mathbf{p}_1(x) + c_2\mathbf{p}_2(x) + c_3\mathbf{p}_3(x) + c_4\mathbf{p}_4(x) = 0$$

is $(0, 0, 0, 0)$ for any x-value. Setting up, we get

$$\begin{aligned} 0 &= c_1 + c_2 x + c_3(3x^2 - 1) + c_4(5x^3 - 3x) \\ &= (c_1 - c_3) + x(c_2 - 3c_4) + x^2 3c_3 + 5x^3 c_4 \end{aligned}$$

The last two terms in the above can only be zero if $c_3 = 0$ and $c_4 = 0$. The first two terms then are only zero if $c_1 = 0$ and $c_2 = 0$, which verifies that the set P is linearly independent.

(b) The set of all polynomials of third degree or less are of the form

$$\alpha + \beta x + \gamma x^2 + \delta x^3$$

where α, β, γ, δ are any real numbers, which are found by solving the matrix equation

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{bmatrix}$$

which has a general solution only if the matrix is invertible, which is guaranteed if the determinant is non-zero:

$$\det\left(\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}\right) = 15 \neq 0.$$

Hence, the set P spans all third-degree polynomials.

Fun fact: The set P are called Legendre polynomials, which are not only linearly independent, but are also orthogonal, meaning that in some sense, these polynomials are "at 90-degrees" from one another, making them useful as a coordinate system—the curious student can see section 4.10 for more details.

(c) Yes. The definition of a basis is a linearly independent spanning set, which we've established from parts (a) and (b) and therefore we can immediately conclude that it is a basis.

10. Consider the following non-linear system

$$\dot{x} = x^2 + xy - x, \qquad \dot{y} = yx - y^2 - y.$$

15 (a) Draw the nullclines of the non-linear system and use this information to determine equilibrium solutions.

Solution: **This was taken directly from a quiz.**
For the sake of convenience, call the functions

$$\dot{x} = x^2 + xy - x = f(x, y), \qquad \dot{y} = xy - y^2 - y = g(x, y).$$

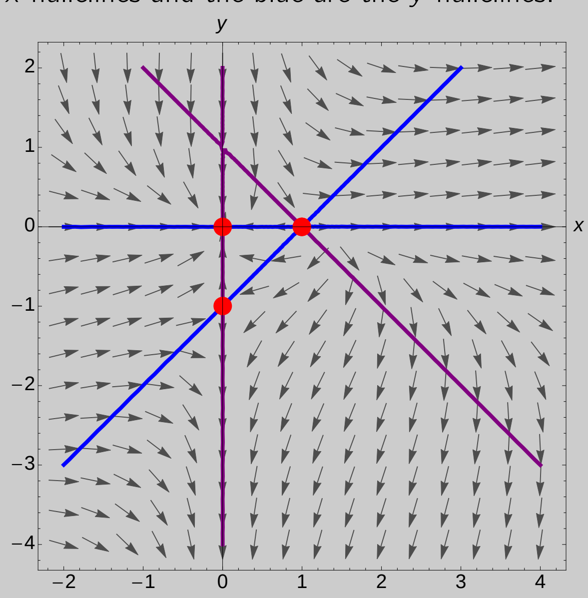
The nullclines of the system are therefore where $f(x, y) = 0$ or where $g(x, y) = 0$. We'll first look at the x nullclines:

$$f(x, y) = 0 = x^2 + xy - x = x(x + y - 1).$$

Thus, this corresponds to two components: either $x = 0$ or $x + y - 1 = 0 \implies y = 1 - x$. We can do the same for the y nullcline:

$$g(x, y) = 0 = xy - y^2 - y = y(x - y - 1),$$

which implies that either $y = 0$ or $y = x - 1$. We can plot these below (along with the slope field, although this would be a bit of a nightmare to draw by hand and is also unnecessary) where the purple lines are the x nullclines and the blue are the y nullclines.



A critical point occurs when all of the derivatives are zero, or, at the intersection of nullclines, which we see here correspond to: $(0, 0)$, $(1, 0)$, and $(0, -1)$. Thus, there are three critical points and therefore three equilibrium solutions.

15 (b) Classify the behavior of the equilibrium solution $(x_\star, y_\star) = (0, 0)$ by linearizing the system and analyzing the Jacobian.

Solution: If we didn't have the slope field above, it would be unclear what type of equilibrium the origin is. Thus, we can use *local* analysis to determine its stability. In other words, consider perturbing slightly away from the origin: what happens? We know that this entirely boils down to the eigenvalues of the Jacobian, which is defined to be:

$$J(x_\star, y_\star) = \begin{bmatrix} \frac{\partial f}{\partial x}(x_\star, y_\star) & \frac{\partial f}{\partial y}(x_\star, y_\star) \\ \frac{\partial g}{\partial x}(x_\star, y_\star) & \frac{\partial g}{\partial y}(x_\star, y_\star) \end{bmatrix}.$$

Computing these partial derivatives:

$$\frac{\partial f}{\partial x} = 2x + y - 1, \qquad \frac{\partial f}{\partial y} = x, \qquad \frac{\partial g}{\partial x} = y, \qquad \frac{\partial g}{\partial y} = x - 2y - 1.$$

Evaluating the Jacobian here:

$$J(0,0) = \begin{bmatrix} \frac{\partial f}{\partial x}(0,0) & \frac{\partial f}{\partial y}(0,0) \\ \frac{\partial g}{\partial x}(0,0) & \frac{\partial g}{\partial y}(0,0) \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Here, we don't even need to compute the eigenvalues because we can just read them off $\lambda_1 = -1, \lambda_2 = -1$. Since we have two real eigenvalues with negative parts, the solution to the linear system for the perturbation looks like decaying exponentials, meaning this is a **stable node** or a **sink**, which is also clear from the slope field by noting that trajectories tend toward the origin.

Bonus Questions

11. **(Current Events)** In the script of the original Star Wars movies, George Lucas named a character "Buffy" (not to be confused with the vampire slayer) but later renamed this character. Which character originally had this name?

Solution:

12. **(Cats)** What is the term for a group of cats?

Solution: Acceptable answers: "clowder", "glaring".

13. **(Math)** Although we only discussed the eigenvalue method for solving systems of differential equations, Laplace transforms are perfectly suitable.

Using Laplace transforms, solve the system of differential equations for $x(t)$ and $y(t)$:

$$6x + 6y' + y = 2e^{-t}, \qquad 2x - y = 0, \qquad x(0) = 1, \qquad y(0) = 2.$$

Solution: When we Laplace transform both equations, we get the transformed system

$$6X - 6sY - 12 + Y = \frac{2}{s+1}, \qquad 2X - Y = 0.$$

Note, we now have an algebraic (linear) system of equations for X, Y, which we can solve, for instance by taking $-3\times$ the second equation and adding it to the first to yield

$$Y(6s+4) = \frac{2}{s+1} + 12 = \frac{12s+14}{s+1} \qquad \Longrightarrow Y = \frac{12s+14}{(s+1)(6s+4)}.$$

Performing partial fractions, we find:

$$Y(s) = -\frac{1}{s+1} + \frac{18}{6s+4} = -\frac{1}{s+1} + 3\frac{1}{s+\frac{2}{3}}.$$

Computing the inverse Laplace transform of this:

$$y(t) = -e^{-t} + 3e^{-2t/3}.$$

Recall that $2x - y = 0$ so $X = Y/2$, meaning that $x(t) = y(t)/2$, so:

$$x(t) = -\frac{1}{2}e^{-t} + \frac{3}{2}e^{-2t/3}.$$