Midterm Exam II
Math 2250 - Differential Equations & Linear Algebra
November 13, 2015

Answer each question completely in the area below. Show all work and explain your reasoning. If the work is at all ambiguous, it is considered incorrect. No phones, calculators, or notes are allowed. Anyone found violating these rules or caught cheating will be asked to leave immediately. Point values are in the square to the left of the question. **If there are any other issues, please ask the instructor.**

By signing below, you are acknowledging that you have read and agree to the above paragraph, as well as agree to abide University Honor Code:

Name: ______________________________

Signature: ______________________________

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Solutions

Question	Points	Score
1	12	
2	21	
3	12	
4	20	
5	10	
6	15	
7	20	
Total:	110	

Note: There are 7 questions on the exam with 110 points available, but the exam will be graded out of 100.

1. Consider the matrix:

$$\mathbf{A} = \begin{bmatrix} -1 & 3 & 0 \\ -1 & 2 & 1 \\ -3 & 5 & 4 \end{bmatrix}.$$

7 (a) Find $\det \mathbf{A}$.

5 (b) In general, what does $\det \mathbf{A}$ tell you about the number of solutions to the linear system $\mathbf{A}\mathbf{x} = \mathbf{0}$?

Solution: This problem was taken directly from the practice exam.

(a) Although we can expand via any row or column, we see that the natural choice is the first row, as we have a 0 in it that makes the calculation slightly easier:

$$\begin{vmatrix} -1 & 3 & 0 \\ -1 & 2 & 1 \\ -3 & 5 & 4 \end{vmatrix} = (-1)\begin{vmatrix} 2 & 1 \\ 5 & 4 \end{vmatrix} - (3)\begin{vmatrix} -1 & 1 \\ -3 & 4 \end{vmatrix} + (0)\begin{vmatrix} \text{doesn't matter} \end{vmatrix}$$
$$= (-1)(8-5) - 3(-4+3) = 0.$$

(b) In general, if $\det \mathbf{A} \neq 0$, we know that $\mathbf{A}$ is invertible and therefore $\mathbf{x} = \mathbf{A}^{-1}\mathbf{0} = \mathbf{0}$, and since this the inverse is unique, the trivial solution is the only solution to the problem. If $\det \mathbf{A} = 0$, we don't have this uniqueness and therefore the remaining possibility is that there are infinitely many solutions.

2. For each of the subsets below, prove or disprove whether they are subspaces of the corresponding $\mathbb{R}^n$.

7 (a)
$$W_1 = \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 = 7x_3\right\}.$$

7 (b)
$$W_2 = \left\{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0, x_2 \geq 0\right\}.$$

7 (c)
$$W_3 = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{A}\mathbf{x} = k\mathbf{x}\}.$$

That is, $\mathbf{A}$ is fixed $n \times n$ matrix, and k is a fixed scalar.

Solution: Each part of this question were taken directly from a quiz, homework, and lab respectively.

(a) *Copied from the quiz solution:* The first issue is: should we try to prove or disprove this being a subspace? While your intuition should hopefully be building, we actually know for sure that this **is** a subspace. Why? What shape can we think of this object as? It's a line through the origin! Thus, we already know it's a subspace, but I wanted you to prove this directly, meaning we must show two properties.

1. The first property we must show is that if $\mathbf{u}, \mathbf{v}$ are both in W_1 then $\mathbf{w} = \mathbf{u} + \mathbf{v}$ is also in W_1. Note, I'm just giving the new vector you get a name, but it's still generic. I'm not saying what these particular vectors actually ARE. I'm just generically saying: consider adding two vectors and call the result $\mathbf{w}$.

   If $\mathbf{u}, \mathbf{v}$ are both in W, note that we can write them in the following way:
   $$\mathbf{u} = (7u_3,\ u_2,\ u_3), \qquad \mathbf{v} = (7v_3,\ v_2,\ v_3).$$
   When we add them, we see
   $$\begin{aligned}\mathbf{u} + \mathbf{v} &= (7u_3,\ u_2,\ u_3) + (7v_3,\ v_2,\ v_3) = (7u_3 + 7v_3,\ u_2 + v_2,\ u_3 + v_3) \\ &= (7\{u_3 + v_3\},\ u_2 + v_2,\ u_3 + v_3) = (7w_3,\ w_2,\ w_3).\end{aligned}$$
   Where again, I've just called $u_2 + v_2 = w_2$ and $u_3 + v_3 = w_3$ for convenience. Note that this looks EXACTLY like a vector in W_1. In fact, looking like this (that 7 times your third element is your first element) is what defines being in W_1. Thus, we can conclude $\mathbf{w} \in W_1$.

2. Similarly, our second condition is that if $\mathbf{u} \in W_1$, then $c\mathbf{u} \in W_1$ as well, where c can be ANY scalar. That is, if we take any vector in our set and re-scale it by any scalar, do we stay in our set? To see this, let
   $$\mathbf{u} = (7u_3, u_2, u_3),$$
   in which case this suggests
   $$c\mathbf{u} = (7cu_3, cu_2, cu_3) = (7\tilde{u}_3,\ \tilde{u}_2,\ \tilde{u}_3),$$
   where I am just giving the names $\tilde{u}_2 = cu_2$ and $\tilde{u}_3 = cu_3$. Note, the thing we've ended up with is again exactly the form of a vector in W_1, so we can conclude $c\mathbf{u} \in W_1$.

Thus, we've shown these properties to hold for ALL vectors in W_1, as I never specified what $\mathbf{u}$ or $\mathbf{v}$ were, so this is indeed a subspace.

(b) By similar logic, we see that this is *not* a line the origin, so our suspicion should be that this is not a subspace. How do we prove that? We find a single counterexample that violates one of the two properties we proved above for the other subset. This is surprisingly straightforward. Consider a particular vector $\mathbf{u} \in W_2$, say $(1,1)$. Since $\mathbf{u} \in W_2$, we need $c\mathbf{u} \in W_2$ as well if W_2 is a subspace. However, consider $c = -1$, so

$$(-1)\mathbf{u} = (-1,1) \notin W_2.$$

Thus, we do **not** have closure under scalar multiplication for this particular vector and therefore isn't true for all vectors, meaning this is **not** a subspace.

(c) We prove this one in a different manner. Recall a very important fact: if $\mathbf{B}$ is a fixed matrix, then the set of solutions to the linear homogeneous system:

$$\mathbf{B}\mathbf{x} = \mathbf{0}.$$

always forms a subspace. However, we don't have an equation of this form. Can we rearrange to get one? Yes! Note:

$$\mathbf{A}\mathbf{x} = k\mathbf{x} \qquad \Longrightarrow \qquad (\mathbf{A} - k\mathbf{I})\,\mathbf{x} = \mathbf{0} = \mathbf{B}\mathbf{x} = \mathbf{0},$$

where we have defined $\mathbf{B} = \mathbf{A} - k\mathbf{I}$. Thus, we've obtained a linear homogeneous system, which we know has solutions that form a subspace and we're done.

5 3. (a) Define what it means for these vectors to be linearly independent.

For concreteness, you can consider a collection of n vectors: $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$, each of which is in some vector space V.

7 (b) Are the following vectors linearly independent or dependent? Why?

$$\mathbf{v}_1 = \begin{bmatrix} -1 \\ -1 \\ -3 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}, \qquad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix}.$$

Hint: use your calculation from Problem 1(a).

Solution:

(a) Informally, vectors are linearly independent if there is no way to add any collection of them (in a linear combination) to obtain one of the others. The precise definition is that if:

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = \mathbf{0},$$

then $c_1 = c_2 = \cdots = 0$. That is, the only way to add these vectors in a linear combination to get the zero vector is the trivial combination (zero coefficients).

(b) **This question was asked in a slightly modified form on a quiz.** We know that for three vectors in $\mathbb{R}^3$, we can take the shortcut of checking linear independence by putting these vectors in a matrix and computing the determinant. **We have this calculation done from problem 1(a)!**. Thus:

$$\begin{vmatrix} -1 & 3 & 0 \\ -1 & 2 & 1 \\ -3 & 5 & 4 \end{vmatrix} = 0.$$

What can we conclude from this? The vectors are **linearly dependent**, since the determinant was 0. That is, we can write a linear combination of two of the vectors to obtain the other. Or, equivalently, we can sum these vectors in a linear combination (with non-trivial coefficients) to get the zero vector.

4. Consider the matrix:

$$\begin{bmatrix} 1 & -2 & 0 & 2 & 3 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

5 (a) Define a basis of a vector space.

10 (b) Find a basis for the solution space of the homogeneous system $\mathbf{Ax} = \mathbf{0}$.

5 (c) What is the dimension of the solution space from part (b)?

Solution: This question was asked in a slightly modified form on a quiz and the practice exam.

(a) *Copied from the quiz solution:* A basis, $\mathcal{B}$, is a collection of vectors in V, that have the following two properties:

1. The vectors in $\mathcal{B}$ are linearly independent. That is, say

$$\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\},$$

for convenience. Note that a basis does **not** need to be finite, but I'm just supposing it is for the sake of explanation. The definition of linear independence is that if

$$c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n = \mathbf{0},$$

then this necessarily implies that (all) $c_1 = \ldots = c_n = 0$. That is, the only way to add these vectors to get $\mathbf{0}$ is by taking all the coefficients to be 0.

2. The second condition is that the vectors in $\mathcal{B}$ span the vector space V. In symbols, $\operatorname{span}\mathcal{B} = V$. What this means is that: given any vector $\mathbf{u} \in V$, there exists some $a_1, \ldots, a_n$ such that

$$\mathbf{u} = a_1\mathbf{b}_1 + \cdots a_n\mathbf{b}_n.$$

In other words, we can build anything in our vector space V as a linear combination of our basis elements.

(b) We see that if we associate each column with a variable, say x_1, x_2, x_3, x_4, x_5, we have two leading variables and three free variables. Thus, parameterize the three free variables: $x_2 = r$, $x_4 = s$, and $x_5 = t$. Remembering that our linear system is $\mathbf{Ax} = \mathbf{0}$, the second row corresponds to the equation

$$x_3 + x_4 + x_5 = 0 \qquad x_3 + s + t = 0 \qquad \Longrightarrow \qquad x_3 = -s - t.$$

Similarly, the row equation suggests

$$x_1 - 2x_2 + 2x_4 + 3x_5 = 0 \qquad x_1 - 2s + 2r + 3t = 0 \qquad \Longrightarrow \qquad x_1 = 2r - 2s - 5t.$$

Thus, we write our solution as:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2s - 2r - 3t \\ r \\ -s - t \\ s \\ t \end{bmatrix} = r \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 2 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}.$$

From this, we see that our basis consists of the three vectors comprising the linear combination above. Thus,

$$B = \left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

(c) Three vectors comprise our basis, so the dimension of this subspace is 3, even though these vectors look like they live in $\mathbb{R}^5$, that is effectively irrelevent. Note that the number of free variables is always the dimension of the solution space for the homogeneous linear system.

______/5 pts

5. For each of the following, state the form of the general solution.

5 (a)
$$2y'' - 7y + 3y = 0.$$

5 (b)
$$y^{(4)} + 18y'' + 81y = 0.$$

Solution: These both were taken directly from the homework.

(a) This is a constant coefficient, linear ODE, by guessing a solution of the form e^{rx}, we get the characteristic equation:
$$2r^2 - 7r + 3 = (2r-1)(r-3) = 0.$$
Thus, our two roots are $r = 1/2$ and $r = 3$. These are distinct, real roots, meaning we have our basis functions:
$$B = \left\{ e^{x/2}, e^{3x} \right\},$$
which create general solutions of the form
$$y = c_1 e^{x/2} + c_2 e^{3x}.$$

(b) Again, this is a constant coefficient, linear differential equation, meaning we get the characteristic equation:
$$r^4 + 18r^2 + 81 = (r^2 + 9)^2 = 0.$$
Thus, we have 4 roots, total, $\pm 3i$ repeated twice (multiplicity 2). Our basis of solutions is then:
$$B = \{\sin 3x, \cos 3x, x \sin 3x, \cos 3x\},$$
which provides general solutions of the form
$$y(x) = (c_1 + c_2 x)\cos 3x + (c_3 + c_4 x)\sin 3x.$$
Note we always have the same number of unknowns as total roots (or order of our differential equation). We can't combine the linear terms into a single and distribute it. We need all 4 coefficients.

15 6. Find a particular solution to

$$y''' + y'' = 3e^x + 4x$$

You do not need to solve for **any** of the constants. Just give the appropriate form!

Solution: This is slightly modified (to be easier) from problem was done in class. We have a non-homogeneous, linear (constant coefficient) differential equation. Thus, we know that solutions are of the form

$$y(x) = y_c + y_p,$$

where y_c are complementary solutions, or solutions to the homogeneous problem and y_p is a particular solution to the non-homogeneous problem.

We first try to find y_c, which can be found by solving the homogeneous equation

$$y_c''' + y_c'' = 0,$$

which, since we have a constant coefficient differential equation, yields the characteristic equation

$$r^3 + r^2 = r^2(r+1) = 0.$$

We have roots $r = 0$ (twice) and $r = -1$, meaning the basis for our complementary solutions is

$$B = \left\{1, x, e^{-x}\right\},$$

and we have the form

$$y_c(x) = c_1 + c_2 x + c_3 e^{-x}.$$

We now turn to the particular solution, for which our naive guess would be:

$$(A + Bx + Cx^2) + De^x,$$

however, we see that we have duplication from our homogeneous part (specifically, that a constant and x are repeated), so our actual guess is of the form

$$y_p(x) = x^2(A + Bx) + De^x.$$

When we plug this into the ODE, we can solve for the coefficients (even though this part was not asked), we find

$$y_p''' + y_p'' = (6B + De^x) + (2A + 6Bx + De^x) = 3e^x + 4x^2.$$

Equating powers of x and the exponential, we get:

$$\begin{aligned} 2D = 3 \qquad &\Longrightarrow\ D = 3/2 \\ 6B = 4 \qquad &\Longrightarrow\ B = 2/3 \\ 6B + 2A = 0 \qquad &\Longrightarrow\ A = -2. \end{aligned}$$

Thus, our total solution is

$$y(x) = y_c(x) + y_p(x) = c_1 + c_2 x + c_3 e^{-x} - 2x^2 + \frac{2}{3}x^3 + \frac{3}{2}e^x.$$

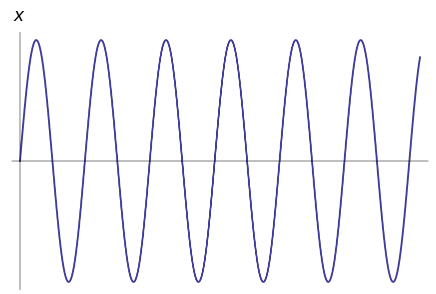
7. Match each of the following differential equations to the appropriate plot of the solution.

   It's recommended that you consider these as spring-mass systems and not actually solve them.

   Explaining your choice may earn you partial credit.

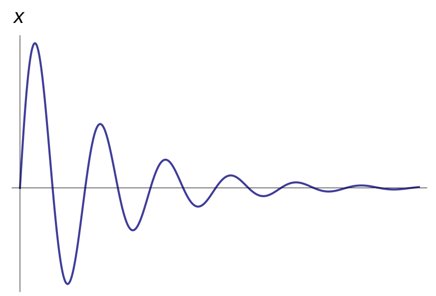
5 (a) $\ddot{x} + \dot{x} + 3x = 0.$

(1)



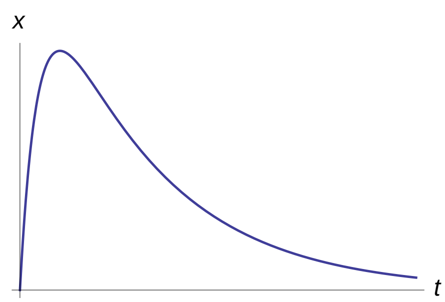
5 (b) $\ddot{x} + 3x = 0.$

(2)



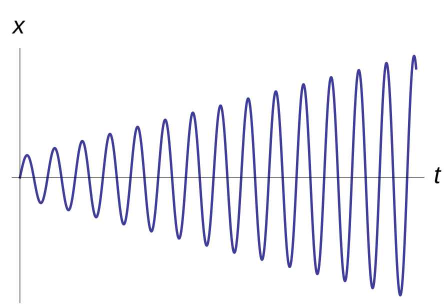
5 (c) $\ddot{x} + 5\dot{x} + 3x = 0.$

(3)



5 (d) $\ddot{x} + 81x = \cos 9t.$

(4)



Solution:

(a) We first classify this equation: it is a damped, unforced oscillator. Thus, it can be characterized by one of three states: underdamped, critically damped, overdamped. We immediately know that all 3 of these die out as $t \to \infty$, so that eliminates our possibilities to just (2) or (3). What's the difference between these three cases? The roots of the characteristic equation, which in this case are:

$$r^2 + r + 3 = 0 \qquad \Longrightarrow \qquad r_\pm = \frac{-1 \pm \sqrt{1-12}}{2}.$$

We see that we have complex roots (with a negative real part), which corresponds to things that look like:

$$x(t) = c_1 e^{-at} \cos bt + c_2 e^{-at} \sin bt.$$

Graphically, this an oscillator with a decaying envelope, or the underdamped case. Thus, it must be (2).

(b) Here, we have an undamped, unforced oscillator. We effectively aren't taking any energy out of the system with damping or putting energy into the system. Thus, the only solution that captures this conservation of energy is (1). We can also see this by noting that the characteristic equation is

$$r^2 + 3 = 0 \qquad \Longrightarrow \quad r_\pm = \pm\sqrt{3}i.$$

That is, we have purely imaginary roots, so our solutions look like

$$x(t) = c_1 \cos\sqrt{3}t + c_2 \sin\sqrt{3}t,$$

pure oscillations.

(c) Here, we have yet another damped, unforced oscillator. However, the characteristic equation in this case is

$$r^2 + 5r + 3 = 0 \qquad \Longrightarrow \qquad r_\pm = \frac{-5 \pm \sqrt{25-12}}{2}.$$

Note that this produces two real (negative) roots. Therefore, our solutions look something of the form

$$x(t) = c_1 e^{-at} + c_2 e^{-bt}.$$

This is the overdamped case. The solutions die out immediately, corresponding to (3).

(d) Here, we have an undamped, force oscillator. We first see that the natural frequency in this case can be seen by solving the homogeneous part:

$$r^2 + 81 = 0 \qquad \Longrightarrow \qquad r_\pm = \pm 9i,$$

which means the homogeneous part has oscillations at the natural frequency of $\omega_0 = 9$. Note that we are forcing the system at its natural frequency, and therefore we have the classic case of resonance. From a mathematical perspective, since the right hand side is a duplication of our homogeneous part, we would guess

$$x_p(t) = c_1 t \cos 9t + c_2 t \sin 9t,$$

which clearly grows in time. Thus, this must be our remaining choice, (4).

Bonus Questions

8. **(Current Events)** Recently, an app named Bark'N'Borrow has gotten widespread attention, including an article in the New York Times. What is the purpose of this app?

Solution: This is an app where you can basically rent a dog to play with, but the app looks like a a dating app, where they list the dogs interests and credentials. It's hilarious.

Check it out. `https://barknborrow.com/`

Source: `http://well.blogs.nytimes.com/2015/11/12/online-matchmaking-but-with-dogs-as-dates/`

9. **(Goats)** *Buzkashi*, translated "goat dragging", is a sport in which horse-mounted players attempt to drag a goat carcass toward a goal. It is the national sport of which country?

Solution: Afghanistan. This question was pretty dark but after Googling "fun goat facts" for a long time, it was the best I could come up with.

10. **(Math)** When first opened, the Millennium Bridge in London wobbled from side to side once a critical number of pedestrians walked on it.

Let $x(t)$ be the displacement of the bridge. The effective force of a single pedestrian was found experimentally to be proportional to $\dot{x}$:

$$F = F_0\dot{x}.$$

Thus, for N pedestrians walking on the bridge, the equation describing the displacement of the bridge is

$$M\ddot{x} + c\dot{x} + kx = F_0 N\dot{x}.$$

Find an expression for the critical number of pedestrians that will cause the bridge to wobble. *Hint:* this can be thought of as the point that *critical damping* occurs.

Solution: Notice this looks forced, but it really isn't, since we can rearrange:

$$m\ddot{x} + (c - F_0 N)\dot{x} + kx = 0,$$

which has characteristic equation

$$mr^2 + (c - F_0 N)r + k = 0,$$

and therefore roots

$$r_\pm = \frac{-c + F_0 N \pm \sqrt{(c - F_0 N)^2 - 4mk}}{2m}.$$

Thus, this is critically damped when the stuff in the square root is 0 and underdamped (oscillations) when it is complex, or

$$(c - F_0 N)^2 - 4mk < 0$$

and we can do some manipulation:

$$(c - F_0 N)^2 < 4mk \implies \qquad c - F_0 N < 2\sqrt{mk} \implies \qquad \frac{c - 2\sqrt{mk}}{F_0} < N.$$