Midterm Exam I
Math 2250 - Differential Equations & Linear Algebra
October 2, 2015

Answer each question completely in the area below. Show all work and explain your reasoning. If the work is at all ambiguous, it is considered incorrect. No phones, calculators, or notes are allowed. Anyone found violating these rules or caught cheating will be asked to leave immediately. Point values are in the square to the left of the question. **If there are any other issues, please ask the instructor.**

By signing below, you are acknowledging that you have read and agree to the above paragraph, as well as agree to abide University Honor Code:

Name: ______________________________

Signature: ______________________________

uID: ______________________________

Solutions

Question	Points	Score
1	15	
2	10	
3	15	
4	20	
5	20	
6	30	
Total:	110	

Note: There are 9 questions on the exam with 110 points available, but the exam will be graded out of 100.

1. Consider the differential equation:

$$\frac{dy}{dx} = \pi\sqrt{xy}.$$

5 (a) Which of the following properties describe this differential equation? (there may be more than one answer)

A. linear
B. separable
C. autonomous
D. first-order

Solution: This equation is a **first-order**, **separable** differential equation. It is separable because we can write the right-hand-side as $f(x)g(y)$, but note that we can immediately see this is therefore not autonomous, which would require the right-hand-side to be of the form just of $g(y)$, that is, only depending on the current state. It's first-order because the highest derivative is a first derivative and it's non-linear because it contains $\sqrt{y}$.

10 (b) Solve the differential equation.

Solution: **This problem was directly from a quiz!** If you didn't remember this, the first part of the problem suggests that since the equation is separable, we use separation of variables to separate and integrate:

$$\frac{dy}{\sqrt{y}} = \pi\sqrt{x}dx.$$

From here, we can just integrate both sides, on the left: with respect to y and on the right: with respect to x, yielding

$$\int \frac{dy}{\sqrt{y}} = \int \pi\sqrt{x}dx.$$
$$2\sqrt{y} = \frac{2\pi}{3}x^{3/2} + C,$$

which we can solve explicitly for y to yield

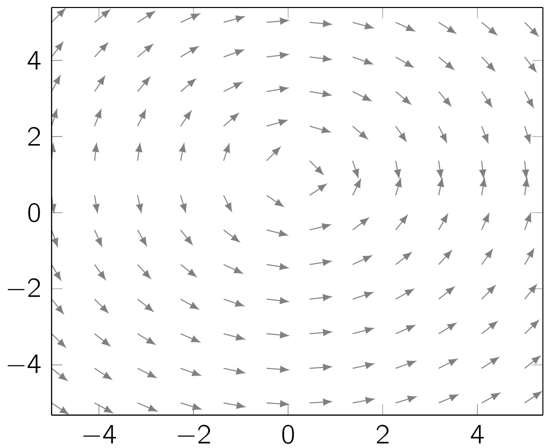
$$y = \left(\frac{\pi}{3}x^{3/2} + C\right)^2.$$

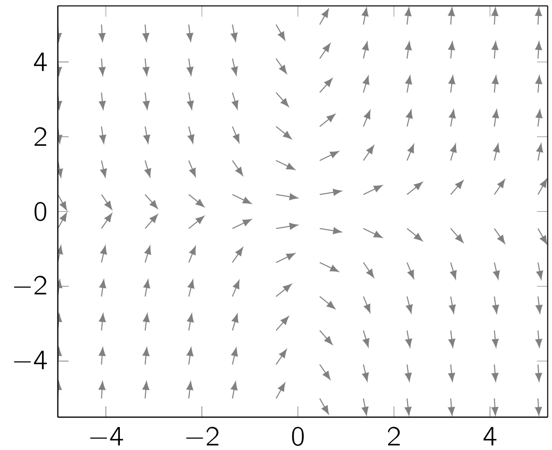
Note, we don't have an initial condition, so we're left with a family of solutions described by some parameter C.

2. Consider the differential equation:

$$\frac{dy}{dx} = \frac{x}{1-y}.$$

5 (a) Identify which of the following is the correct slope field for this differential equation and explain your reasoning.



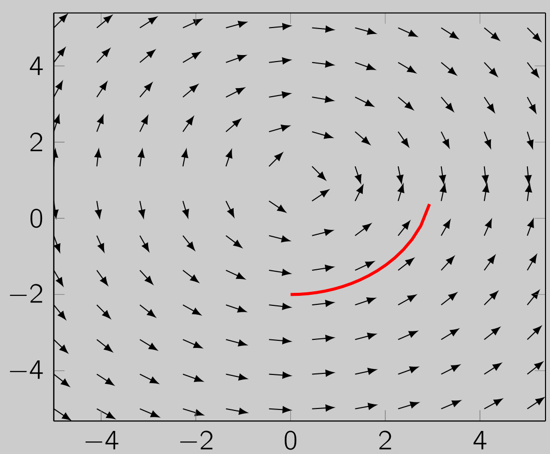


Solution: **This slope field problem was on the lab!** Note that something interesting happens when $y = 1$, specifically, the slope becomes effectively infinite. Also at $x = 0$, the slope should become 0. We see that this is satisfied by the **left**.

5 (b) Using your slope field, draw a rough sketch of the solution to the initial value problem with $y(0) = -2$.

Solution:

For the trajectory, we just start at the point $(0, 2)$ and draw where the arrows take us. Interestingly, we see that the solution stops around $y = 0$, which is perfectly fine. The result looks like the following:



The exact solution to this IVP is actually $y(x) = 1 - \sqrt{9 - x^2}$, which we see agrees: at $x = 3, y = 0$, the solution "stops" and can't extend further as x increases.

3. Consider the initial problem

$$\frac{dx}{dt} = t^2 - x, \qquad x(0) = 1.$$

5 (a) Construct an approximation to the solution $x(t)$ to on the interval $[0, 1]$ using Euler's method with step-size $h = 0.5$.

Solution: This was directly from a quiz! Note that what we normally call x, y here, we actually have t, x. It doesn't matter what we call them, but it's important to keep track of what the independent and dependent variables are for our differential equation. Euler's method, in algorithm says:

$$x_{n+1} = x_n + hk, \qquad k := f(t_n, x_n).$$

Thus, in this problem, we have $h = 1/2$, meaning we have 3 t_j values: $t_0 = 0$, $t_1 = 0.5$, $t_2 = 1$. We know the initial value $x_0 = 1$. Thus, at the first step:

$$x_1 = x_0 + hf(t_0, x_0) = 1 + 0.5\left[0^2 - 1\right] = 0.5.$$

We perform one more step:

$$x_2 = x_1 + hf(t_1, x_1) = 0.5 + 0.5\left[0.5^2 - 0.5\right] = 0.375.$$

Thus, the values x_0, x_1, x_2 approximate the true solution values $x(0), x(0.5), x(1)$.

5 (b) List and describe the 3 types of numerical error discussed. *Hint:* two have to do with the approximation, one has to do with the limitations of computers.

Solution:

1. local error: this is the error we introduce at each step in our approximation
2. cumulative error: after performing a sequence of steps, we get local error at each and the sum of these local errors is the cumulative error. This is the one we have theorems about.
3. round-off error: due to computers being unable to store certain numbers exactly, there is inherently error in doing basic operations

5 (c) What is the trade-off of using a different scheme to perform this numerical approximation? For concreteness, you can compare with the Runge-Kutta method.

Solution: We discussed that the order of the error of Euler's method was h but the order of the error of the Runge-Kutta method was of order h^4. This means that: reducing h causes the maximum possible error to reduce much more significantly for the Runge-Kutta method. In that sense, it is more *accurate*. However, the Runge-Kutta method requires more function evaluations of our $f(x, y)$, and therefore requires more computation (or time).

4. A 120 G (gallon) tank initially contains 90 lb (pound) of salt dissolved in 90 G of water. Brine containing 2 lb/G of salt flows into the tank at a rate of 4 G/minute. The well-stirred mixture in the tank flows out at a rate of 3G/minute.

10 (a) Write down a differential equation and initial condition for the amount of salt in the tank, $x(t)$.

Solution: **This problem was from the review!** As with every tank problem we just need to account for the rate of salt in and rate of salt out. For the rate in, we have:

$$\text{rate in} = c_{\text{in}} r_{\text{in}} = 2\frac{\text{lb}}{\text{G}} \cdot 4\frac{\text{G}}{\text{min}} = 8\frac{\text{lb}}{\text{min}}.$$

Note the units of this are exactly what we want: a rate of salt intake, or a change in salt (pounds) per time (minutes).

The rate out is slightly more tricky, but we did an example almost identical to this in class. The thing to note is: what is the volume of the liquid in the tank? It starts out 90 G, but note that the inflow rate is larger than the outflow rate, meaning we're gaining liquid. In fact, the net change is 1 G/min, so $V(t) = 90 + 1t$. Using this, we know the concentration is just the amount of salt divided by the volume, thus:

$$\text{rate out} = c_{\text{out}} r_{\text{out}} = \frac{x(t)}{V(t)}\frac{\text{lb}}{\text{G}} \cdot 3\frac{\text{G}}{\text{min}} = \frac{3x(t)}{90+t}\frac{\text{lb}}{\text{min}}.$$

Thus, the net rate of change in the amount of salt is just the rate in minus the rate out:

$$\frac{dx}{dt} = \text{rate in} - \text{rate out} = 8 - \frac{3x}{90+t}.$$

We also note that there is 90 lb initially, so our initial condition is $x(0) = 90$.

10 (b) Find an explicit solution to the differential equation.

Solution: We discussed that (almost) every tank problem is a linear first order differential equation, which this one indeed is. Thus, we'll use an integrating factor. Rearranging a little:

$$\frac{dx}{dt} + \frac{3x}{90+t} = 8,$$

we see that we have an equation of the form

$$\frac{dx}{dt} + P(t)x = Q(t), \qquad P(t) = \frac{3}{90+t}, \qquad Q(t) = 8.$$

Thus, our integrating factor becomes

$$\mu(t) = \exp\left\{\int \frac{3}{90+t}\,dt\right\} = \exp\left\{3\ln|90+t|\right\} = (90+t)^3.$$

Our transformed equation is then:

$$\frac{d}{dt}\left\{\mu(t)x(t)\right\} = \mu(t)Q(t),$$

which, using actual forms is

$$\frac{d}{dt}\left\{(90+t)^3 x(t)\right\} = 8(90+t)^3.$$

We integrate both sides to yield

$$(90+t)^3 x(t) = \int 8(90+t)^3\,dt = 2(90+t)^4 + C.$$

Since we have $x(0) = 90$, we have

$$(90+0)^3 90 = 2(90+0)^4 + C \implies C = -90^4.$$

Thus, our explicit solution becomes

$$x(t) = 2(90+t) - \frac{90^4}{(90+t)^3}.$$

5. Consider the differential equation

$$\frac{dx}{dt} = kx - x^3.$$

5 (a) Consider $k > 0$. Find the equilibria of the system in this case.

Solution: **This problem was from the homework!** In the case that $k > 0$, we want to find the values of x such that $f(x) = kx - x^3 = 0$. Notice we can factor this:

$$0 = kx - x^3 = x(k - x^2) = x\left(x - \sqrt{k}\right)\left(x + \sqrt{k}\right).$$

We can only factor this way because $k > 0$, but we see there are three equilibria: $x = 0, \sqrt{k}, -\sqrt{k}$.

5 (b) Again, for $k > 0$, draw the phase diagram for the equilibria you found in part(a) and use this to determine the stability of each of the equilibria.

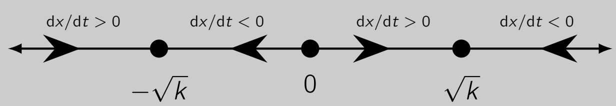
Solution: We know that we have the three equilibria from the previous part, we just need to check whether dx/dt is positive or negative in each of the regions the equilibria partition. If $x < -\sqrt{k}$, we see that we get:

$$\frac{dx}{dt} = \underbrace{x}_{-}\underbrace{(k - x^2)}_{-},$$

so we conclude the arrow should go to the right. In the region $-\sqrt{k} < x < 0$, we see

$$\frac{dx}{dt} = \underbrace{x}_{-}\underbrace{(k - x^2)}_{+},$$

so here $dx/dt < 0$. We can perform another analysis for the other two cases but they're very similar. The result is the following phase diagram:



From this, we can see $x = \sqrt{k}, -\sqrt{k}$ are stable and $x = 0$ is unstable since the arrows are pointing inward (and outward) respectively.

5 (c) Now consider the case where $k \leq 0$. Find the equilibria in this case and state their stability.

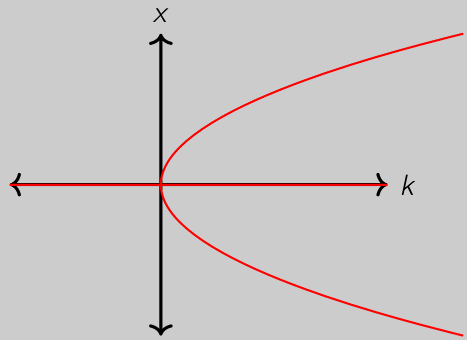
Solution: In the case that $k \leq 0$, we see that we can write the equilibria:

$$0 = kx - x^3 = x(k - x^2),$$

however, now, since $k \leq 0$, the second term *cannot* be factored into real roots, meaning $x = 0$ is the only equilibrium.

5 (d) For this differential equation, we have a *pitchfork bifurcation* at $k = 0$. Draw the corresponding bifurcation diagram.

Solution: The idea of drawing the bifurcation diagram is: we draw the value of the equilibria (x) as a function of the parameter k. From the previous parts, we see that there are three equilibria when $k > 0$ and only one when $k \leq 0$. In the former case, the values are $x = \pm\sqrt{k}$ and squaring both sides yields $x^2 = k$, which is just a parabola. Thus, the plot looks something like the following:



Notice that this indeed looks like a pitchfork, hence the name of the bifurcation. The important thing to note is how we have 3 or 1 equilibria based on the value of$k$.

6. You are conducting an experiment on bacterial growth. Suppose that you suspect the equation that describes bacterial population is of the form

$$p(t) = Ae^t + B\ln t + Ct,$$

where A, B and C are constant coefficients that we hope to determine.

5 (a) Suppose we take the following experimental data about the population of the bacteria:

t	1	2	4
$p(t)$	2	4	9

From the data taken, write down a linear system involving A, B, C.

Solution: We simply plug in each $t, p(t)$ pair and write down the corresponding result. From this, we find the linear system:

$$\begin{aligned} p(1) &= eA + 0 + C = 2, \\ p(2) &= e^2 A + (\ln 2)B + 2C = 4, \\ p(4) &= e^4 A + (\ln 4)B + 4C = 9. \end{aligned}$$

Note, this may look a bit scary, by $\ln 2$ and e^2 are just numbers. This is still exactly the same flavor of linear system we've been studying.

5 (b) Write down the corresponding augmented coefficient matrix for the linear system from part (a).

Solution: To get the system in augmented coefficient form, we take the matrix of coefficients, **A**, which is a 3×3 in this case, and glue on or "augment" the **b** values, which are the right hand sides of the equation. That is, our augmented coefficient matrix is:

$$\left[\ \mathbf{A} \mid \mathbf{b}\ \right] = \left[\begin{array}{ccc|c} e & 0 & 1 & 2 \\ e^2 & \ln 2 & 2 & 4 \\ e^4 & 2\ln 2 & 4 & 9 \end{array}\right].$$

10 (c) Use Gaussian Elimination to get the matrix into echelon form.

Solution: We start with our matrix and perform the procedure of eliminating terms under the current leading row. That is, we'll focus on the first row first column and try to eliminate below that:

$$\left[\begin{array}{ccc|c} e & 0 & 1 & 2 \\ e^2 & \ln 2 & 2 & 4 \\ e^4 & 2\ln 2 & 4 & 9 \end{array}\right] \xrightarrow{(-e)R_1+R_2} \left[\begin{array}{ccc|c} e & 0 & 1 & 2 \\ 0 & \ln 2 & 2-e & 4-2e \\ e^4 & 2\ln 2 & 4 & 9 \end{array}\right] \xrightarrow{(-e^3)R_1+R_3}$$

$$\left[\begin{array}{ccc|c} e & 0 & 1 & 2 \\ 0 & \ln 2 & 2-e & 4-2e \\ 0 & 2\ln 2 & 4-e^3 & 9-2e^3 \end{array}\right] \xrightarrow{(-2)R_2+R_3} \left[\begin{array}{ccc|c} e & 0 & 1 & 2 \\ 0 & \ln 2 & 2-e & 4-2e \\ 0 & 0 & 2e-e^3 & 4e-2e^3+1 \end{array}\right].$$

We see that this is indeed echelon form, all of the leading terms need not be 1 but every leading term is strictly to the right of the one above it.

5 (d) Use the matrix you found in part (c) to determine A, B and C.

Solution: Although we could do Gauss-Jordan elimination to get this into reduced row echelon form, it's a bit of a mess, so let's just back substitute instead. Recall our echelon form:

$$\left[\begin{array}{ccc|c} e & 0 & 1 & 2 \\ 0 & \ln 2 & 2-e & 4-2e \\ 0 & 0 & 2e-e^3 & 4e-2e^3+1 \end{array}\right],$$

which, when we take the bottom equation suggests that

$$(2e-e^3)C = 4e-2e^3+1 \implies C = 2+\frac{1}{2e-e^3}.$$

The next equation reads

$$B\ln 2 + (2-e)C = 4-2e,$$

but we know what C is, so we have:

$$B\ln 2 + (2-e)\left(2+\frac{1}{2e-e^3}\right) = 4-2e \implies B = \frac{e-2}{e^3\log 2 - 2e\log 2}$$

Lastly, the first equation says

$$Ae + C = 2 \implies Ae + 2 + \frac{1}{2e-e^3} = 2 \implies A = \frac{1}{e^3(e-2)}.$$

5 (e) After doing the experiment, you the foolishness of your first equation and now realize the correct model is actually

$$p(t) = Ae^t + B\ln t + Ct + D,$$

again where A, B, C and D are constant coefficients. Is your data set still large enough to determine the coefficients? If not, at least how many more observations do we need to make? Why?

Solution: Note, if we wrote this as a linear system, we would have 3 equations and 4 unknowns, or a 3×4 matrix, which, when we row reduce given the current data, we would have a free variable and therefore not enough information for a unique answer. If we took one more piece of data (assuming it wasn't redundant), we possibly would have enough information for a unique solution.

Bonus Questions

7. **(Math)** A **Bernoulli** differential equation is one of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x)y^n.$$

Note this is *not* a differential equation of a form we have discussed solving, but we can actually transform it into one by taking the substitution $u = y^{1-n}$.

Using this substitution, transform the equation and explain how to solve it.

Solution: It's first useful to divide by y^n, in which case we get

$$y^{-n}\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y^{n-1} = Q(x).$$

Now, take a derivative of the u substitution:

$$\frac{\mathrm{d}u}{\mathrm{d}x} = (1-n)y^{-n}\frac{\mathrm{d}y}{\mathrm{d}x} \qquad \Longrightarrow \qquad \frac{1}{1-n}\frac{\mathrm{d}u}{\mathrm{d}x} = y^{-n}\frac{\mathrm{d}y}{\mathrm{d}x}.$$

Thus, we have

$$\frac{1}{1-n}\frac{\mathrm{d}u}{\mathrm{d}x} + P(x)u = Q(x),$$

and multiplying everything by $(1-n)$ yields

$$\frac{\mathrm{d}u}{\mathrm{d}x} + (1-n)P(x)u = (1-n)Q(x),$$

which is a first-order linear differential equation we can solve with integrating factors.

8. **(Current Events)** This week, NASA announced possible chemical evidence of liquid water on the surface of Mars. Although it seems like the natural next step, NASA is in fact banned from sending the rovers to investigate for a very legitimate reason. Why might this reason be? *Hint:* what might the rover have on it that would be a problem?

Solution: NASA is prohibited from doing this by the 1967 Outer Space International Treaty, which you can read about here:

`http://www.unoosa.org/oosa/en/ourwork/spacelaw/treaties/introouterspacetreaty.html`

The reason is: the rover has germs/microbes on it from Earth and therefore could potentially infect the water supply of Mars if it physically investigated.

9. **(Other)** What is the Latin name for a Yak?

Solution: Bos grunniens.