

Name: \_\_\_\_\_

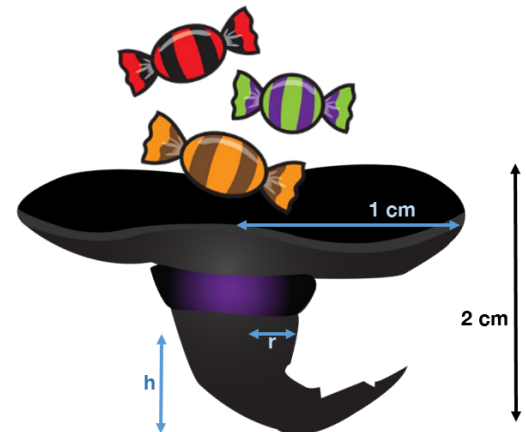
Quiz Score: \_\_\_\_/10

Answer each question completely in the area below. Show all work and explain your reasoning. If the work is at all ambiguous, it is considered incorrect. No phones, calculators, or notes are allowed. Anyone found violating these rules will be asked to leave immediately. Point values are in the square to the left of the question. If there are any other issues, please ask the instructor.

- 6 1. Due to a mysterious bag shortage, a child realizes that a witch's hat is a perfect vessel for candy on Halloween. Roughly speaking, the hat (and therefore the candy filling it) can be modeled as a cylindrical cone with volume described by:

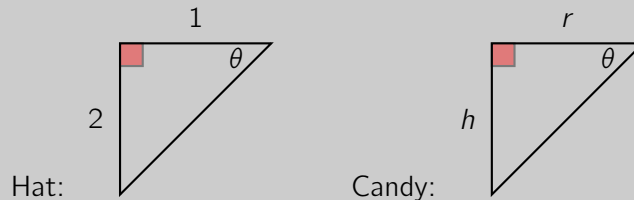
$$V = \frac{1}{3}\pi r^2 h,$$

where  $r$  is the radius of the cone and  $h$  is the height. If the hat has a base radius of 1 m and a height of 2 m, and the child is throwing candy in the hat at a rate of  $1 \text{ m}^3$  per minute, how fast is the level of candy rising when there is currently 1 m of candy in the hat? How much longer can the child trick-or-treat until the hat is completely full of candy?



**Solution:** While this problem may seem difficult, check your notes to notice that we did this **exact** problem with different numbers in class.

We care about the volume of candy which can be described by the provided equation, but the first issue is that we do not necessarily know what  $r$  and  $h$  are for the candy. We know  $h$  and the rate of change of  $h$ , but it turns out that  $r$  is related to  $h$  through similar triangles. Notice we have similar triangles:



From this, we can conclude there is a constant ratio:

$$\frac{1}{2} = \frac{r}{h},$$

which suggests  $h = r/2$ . Meaning our equation becomes:

$$V = \frac{1}{3}\pi \left(\frac{h}{2}\right)^2 h = \frac{\pi}{12}h^3.$$

which we can take derivatives of:

$$\frac{dV}{dt} = 3\frac{\pi}{12}h^2 \frac{dh}{dt} \implies \frac{dh}{dt} = \frac{dV}{dt} \frac{4}{\pi h^2}.$$

We know at the time we care about  $h = 1$ ,  $dV/dt = 1$ , providing us an answer of  $dh/dt = 4/\pi \approx 1.27$  m per minute.

Notice at this time, the volume of the candy is  $V = \frac{\pi}{12}(1)^3 = \pi/12$  and the maximum amount of candy is the volume of the bag:  $V = \frac{\pi}{12}(2)^3 = 8\pi/12$ . Thus:

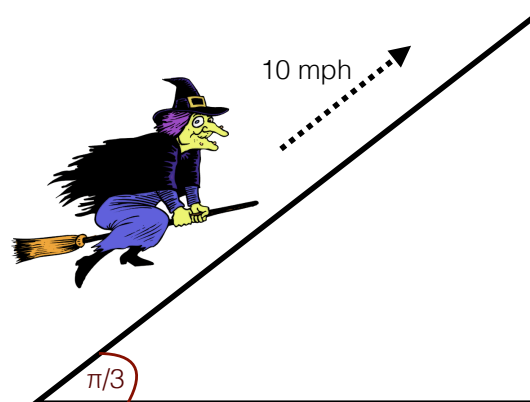
$$V_{\text{remaining}} = V_{\text{max}} - V_{\text{current}} = \frac{2\pi}{3} - \frac{\pi}{12} = \frac{7\pi}{12} \text{ m}^3.$$

Notice, the volume changes at a constant rate, meaning we know the amount of time it takes for the volume to change this amount:

$$\text{time to fill remaining} = \frac{\text{volume remaining}}{\text{rate of change of volume}} = \frac{\frac{7\pi}{12} \text{ m}^3}{1 \text{ m}^3/\text{min}} = \frac{7\pi}{12} \approx 1.83 \text{ min.}$$

Notice, this would **not** have worked by computing the current height and maximum height. Why? **The rate of change of the height is not a constant.** It turns out, that is not a problem if you know more advanced calculus (i.e. integrals), but with our current techniques, we really only know how to deal with a constant rate.

- 4 2. Taking off from the ground, a witch flies on her broom at a constant angle (relative to the ground) of  $\pi/3$  radians. Her broom is capable of flying at a constant rate of 10 miles per hour. Envious humans observe her from earth. At what rate is the witch's altitude (measured from the ground) changing?



**Solution:** This problem was much simpler than it seems. Notice that the angle  $\pi/3$  remains constant, meaning the hypotenuse  $h$  and the altitude  $a$  are always related by:

$$\sin\left(\frac{\pi}{3}\right) = \frac{a}{h},$$

but notice  $\sin(\pi/3)$  is a number, a constant, particularly  $\sin(\pi/3) = \sqrt{3}/2$ . Many students took derivatives and ended up with  $\cos(\pi/3)$ , which does not make sense because it is a constant. Knowing this, we simply have that  $a = \frac{\sqrt{3}}{2}h$  and therefore:

$$\frac{da}{dt} = \frac{\sqrt{3}}{2} \frac{dh}{dt}.$$

Plugging in our one known value,  $dh/dt = 10$ , we get that  $da/dt = 5\sqrt{3}$  miles per hour.