

Name: _____

Quiz Score: _____/10

Answer each question completely in the area below. Show all work and explain your reasoning. If the work is at all ambiguous, it is considered incorrect. No phones, calculators, or notes are allowed. Anyone found violating these rules will be asked to leave immediately. Point values are in the square to the left of the question. If there are any other issues, please ask the instructor.

1. Consider the following limit:

$$\lim_{x \rightarrow \infty} \frac{6x^3 + 3x + 3}{9x^3 + 4x^2 + \pi}. \quad (1)$$

3 (a) Compute this limit if it exists. If it does not, explain why. *Hint:* divide everything by x^3 .

Solution: As mentioned in the hint, we can divide everything by x^3 and examine the behavior:

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{6x^3 + 3x + 3}{9x^3 + 4x^2 + \pi} &= \lim_{x \rightarrow \infty} \frac{6 + \frac{3}{x^2} + \frac{3}{x^3}}{9 + \frac{4}{x} + \frac{\pi}{x^3}} \\ &= \frac{\lim_{x \rightarrow \infty} 6 + \lim_{x \rightarrow \infty} \frac{3}{x^2} + \lim_{x \rightarrow \infty} \frac{3}{x^3}}{\lim_{x \rightarrow \infty} 9 + \lim_{x \rightarrow \infty} \frac{4}{x} + \lim_{x \rightarrow \infty} \frac{\pi}{x^3}} \\ &= \frac{6 + 0 + 0}{9 + 0 + 0} \\ &= \frac{2}{3}. \end{aligned}$$

Note, this means $2/3$ is a horizontal asymptote for this equation.

A number of you just **stated** the limit. When I ask you to **compute** or **calculate** something, please show work! I think this is especially reasonable considering I provided the hint, which was exactly the work I expected to be shown.

2 (b) Using techniques described in class, we are able to effectively read off this limit just by looking at the polynomials in the numerator and the denominator. Why is this so? Explain this technique and how it relates to this problem. *Hint:* your argument should be based around the *leading (largest) order* term of each polynomial.

Solution: As mentioned in class, we can heuristically examine the behavior of this function for large x (which describes the behavior as $x \rightarrow \infty$). Roughly:

$$\lim_{x \rightarrow \infty} \frac{6x^3 + 3x + 3}{9x^3 + 4x^2 + \pi} \approx \frac{6(\text{large number})^3 + \text{negligibly small terms}}{9(\text{large number})^3 + \text{negligibly small terms}}.$$

The terms are small because x^2 , x , and constants grow **much** slower than x^3 , so you can think of the x^3 term as capturing all the behavior for large x , thus we can ignore all the small terms. With this in mind, we can examine the leading order or largest order terms of each polynomial to determine the limit. One way I like to think of this limit is that the top polynomial and the bottom polynomial are racing as $x \rightarrow \infty$. One could be much faster than the other or they could be roughly evenly matched.

In this case, we have:

$$\lim_{x \rightarrow \infty} \frac{6x^3 + 3x + 3}{9x^3 + 4x^2 + \pi} \approx \frac{\cancel{6(\text{large number})^3}}{\cancel{9(\text{large number})^3}} \approx \frac{6}{9} = \frac{2}{3}.$$

Notice we can “cancel” the large terms because they describe growth at roughly the same rate. We can consider two other scenarios where the exponents differ:

$$\frac{6(\text{large number})^2}{9(\text{large number})^3} \rightarrow 0$$

$$\frac{6(\text{large number})^3}{9(\text{large number})^2} \rightarrow \infty.$$

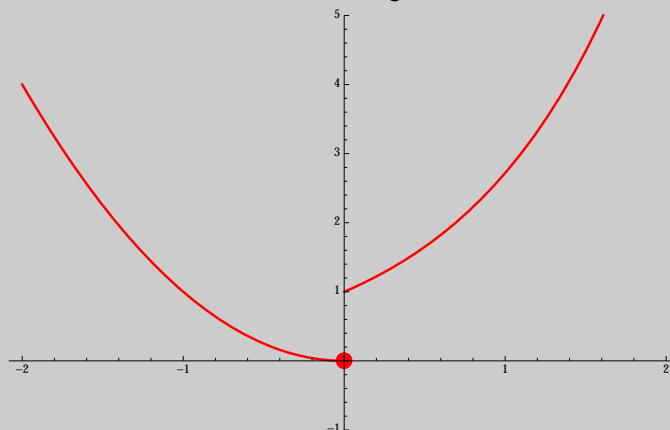
In the first case above, the denominator grows much faster than the numerator, so the limit tends toward 0. In the second case, the opposite is true: the numerator grows much faster than the denominator so it continues to grow to ∞ .

2. Consider the following function:

$$f(x) = \begin{cases} e^x & \text{if } x > 0 \\ x^2 & \text{if } x \leq 0. \end{cases}$$

2 (a) Sketch a graph of the function.

Solution: The graph of the function looks something like:



Notice, $e^0 = 1$ and $f(0) = 0$.

3 (b) Is this function continuous at $x = 0$? Why or why not? Provide concrete justification using the definition of continuity. *Hint:* consider $\lim_{x \rightarrow 0} f(x)$.

Solution: The definition of continuity says that for f to be continuous at $x = 0$, then we need $\lim_{x \rightarrow 0} f(x) = f(0)$. In this case, $f(0) = 0$, thus we need the limit from both sides to equal this value. From the left, it's clear $\lim_{x \rightarrow 0^-} f(x) = 0$, but notice from the right, $e^0 = 1$, meaning the function is approaching 1. That is, $\lim_{x \rightarrow 0^+} f(x) = 1$. Thus, $\lim_{x \rightarrow 0} f(x)$ **does not exist** and therefore cannot equal $f(0)$, meaning the function is **discontinuous** at $x = 0$.