

Final Exam

Math 1310 - Engineering Calculus I

December 17, 2014

Answer each question completely in the area below. Show all work and explain your reasoning. If the work is at all ambiguous, it is considered incorrect. No phones, calculators, or notes are allowed. Anyone found violating these rules will be asked to leave immediately. Point values are in the square to the left of the question. If there are any other issues, please ask the instructor.

By signing below, you are acknowledging that you have read and agree to the above paragraph, as well as agree to abide University Honor Code:

Name: _____
Signature: _____
uID: _____

Solutions

Question	Points	Score
1	15	
2	15	
3	20	
4	20	
5	10	
6	15	
7	15	
8	15	
9	15	
10	15	
11	15	
12	15	
13	15	
Total:	200	

Advice: If a problem is giving you trouble, skip it and return later! Spending too much time on a problem will provide you too little time to do the whole exam. Good luck!

1. Compute the following limits:

5 (a) $\lim_{x \rightarrow 9} \frac{\sqrt{x}-3}{x-9}$

Solution: This problem screams for the use of our conjugate trick, that is:

$$\lim_{x \rightarrow 9} \frac{\sqrt{x}-3}{x-9} \cdot \frac{\sqrt{x}+3}{\sqrt{x}+3} = \lim_{x \rightarrow 9} \frac{\cancel{(x-9)}}{\cancel{(x-9)}(\sqrt{x}+3)} = \lim_{x \rightarrow 9} \frac{1}{\sqrt{x}+3} = \frac{1}{6}.$$

5 (b) $\lim_{x \rightarrow -\infty} \frac{7x^2-x+11}{4-x}$

Solution: As with all of our limits going to $\pm\infty$, we examine the powers of the numerator and denominator. Here, we have that the power of the numerator is larger than that of the denominator, so we can immediately conclude that the limit must be either $+\infty$ or $-\infty$. To determine which it is, we consider the sign: as $x \rightarrow -\infty$, x^2 stays positive in the numerator, but $-x$ becomes positive. Thus, we have a positive number divided by a positive number, meaning we have:

$$\lim_{x \rightarrow -\infty} \frac{7x^2 - x + 11}{4 - x} = +\infty.$$

5 (c) $\lim_{x \rightarrow 0^+} \cos(x)^{1/x}$

Solution: Here, if we plug in $x = 0$, we see that we have the indeterminate form 0^∞ . Thus, we need to apply the ln trick to use L'Hôpital's rule. That is:

$$\lim_{x \rightarrow 0^+} \cos(x)^{1/x} = \lim_{x \rightarrow 0^+} e^{\frac{1}{x} \ln[\cos x]} = e^{\lim_{x \rightarrow 0^+} \frac{1}{x} \ln[\cos x]}.$$

What is this limit in the exponent? It's an indeterminate product, that is $0 \times \infty$, meaning we apply L'Hôpital's rule after getting it into the appropriate form:

$$\lim_{x \rightarrow 0^+} \frac{1}{x} \ln[\cos x] = \lim_{x \rightarrow 0^+} \frac{\ln \cos x}{x}.$$

From here, we take the derivative of the top and bottom, as prescribed by L'Hôpital's rule:

$$\lim_{x \rightarrow 0^+} \frac{\ln[\cos x]}{x} = \lim_{x \rightarrow 0^+} \frac{\frac{\sin x}{\cos x}}{1} = \lim_{x \rightarrow 0^+} \tan x = 0.$$

Thus, our total limit is $e^0 = 1$.

- 15 2. Using the definition of the derivative, compute the derivative of the following function:

$$f(x) = \frac{3}{7}x - \frac{1}{5}.$$

Solution: The definition of the derivative states:

$$f'(x) \stackrel{\text{def}}{=} \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}.$$

Thus, we plug in our particular $f(x)$ into this definition which yields:

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\left[\frac{3}{7}(x+h) - \frac{1}{5}\right] - \left[\frac{3}{7}x - \frac{1}{5}\right]}{h} \\ &= \lim_{h \rightarrow 0} \frac{\left[\cancel{\frac{3}{7}x} + \frac{3}{7}h - \frac{1}{5}\right] - \left[\cancel{\frac{3}{7}x} - \frac{1}{5}\right]}{h} \\ &= \lim_{h \rightarrow 0} \frac{\frac{3}{7}h}{h} \\ &= \frac{3}{7}. \end{aligned}$$

It's always good to double check your work via some other test and this is clearly the right answer via the power rule.

3. Compute the following derivatives:

5 (a) $x^2 e^{3x}$

Solution: This is an application of the product rule. We also know that $d/dx \{e^{u(x)}\} = e^u u'$, yielding:

$$\frac{d}{dx} \{x^2 e^{3x}\} = 2x e^{3x} + x^2 3e^{3x}.$$

5 (b) $\frac{(1+x^2)^{50}}{\ln x}$

Solution: Just as easy as the previous problem, this is simply an application of the quotient rule. Recall that $d/dx \{\ln u\} = u'/u$, yielding:

$$\frac{d}{dx} \left\{ \frac{(1+x^2)^{50}}{\ln x} \right\} = \frac{[50(1+x^2)^{49} 2x] \ln x - (1+x^2)^{50} \frac{1}{x}}{(\ln x)^2}.$$

No need to simplify further. Note that the derivative of the top is an application of the chain rule.

5 (c) $\tan^{-1}(x^2)$

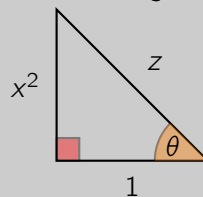
Solution: Here, you can either memorize the derivative of $\tan^{-1}(u)$ or you (like me), would rather just rederive it using implicit differentiation:

$$y = \tan^{-1}(x^2) \implies \tan y = x^2.$$

Taking the derivative of both sides, recalling that $\tan' = \sec^2$:

$$\sec^2(y) y' = 2x \implies y' = \frac{2x}{\sec^2 y}.$$

We're almost done, but what is $\sec y$? We can establish this by considering the triangle described by $\tan y = 2x$, which suggests that we have some angle θ such that we have the following picture:



Notice, we got this picture because \tan describes the opposite edge over the adjacent edge, which we know the ratio is $2x$. We can immediately conclude that $z = \sqrt{x^4 + 1}$. Recall that $\sec y = 1/\cos y$, where $\cos y$ describes the adjacent over the hypotenuse, meaning that \sec describes the hypotenuse over the adjacent. Thus, $\sec^2 y = (x^4 + 1)$. Thus, our total derivative becomes:

$$y' = \frac{2x}{\sec^2 y} = \frac{2x}{x^4 + 1}.$$

5 (d) $\int_1^{\sin x} \cos \sqrt{t} dt$

Solution: For some reason, this problem seemed to confuse many people but was simply an application of the Fundamental Theorem of Calculus. Recall the statement:

$$\frac{d}{dx} \left\{ \int_a^x g(t) dt \right\} = g(x).$$

Here, we have similar but we must use the chain rule with $u = \sin x$ and:

$$g(u) = \int_0^u \cos \sqrt{t} dt.$$

This suggests that:

$$\frac{d}{dx} \{g(\sin x)\} = g'(\sin x) \frac{d}{dx} \{\sin x\}.$$

The first term is exactly the statement of the Fundamental Theorem, which says we can just plug in $\sin x$ for t and the second term is just the derivative of \sin , which yields:

$$\frac{d}{dx} \{g(\sin x)\} = \cos(\sqrt{\sin x}) \cos x.$$

4. Consider the following function:

$$f(x) = \frac{x^2}{(x-2)^2}.$$

- 5 (a) Find the vertical and horizontal asymptotes of $f(x)$.

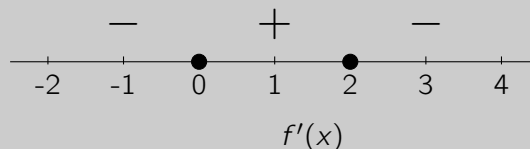
Solution: To find the vertical asymptotes, we consider where the function “blows up”. It’s clear the only point that this occurs is $x = 2$. To find the horizontal asymptotes, we consider $\lim_{x \rightarrow \infty} f(x)$ and $\lim_{x \rightarrow -\infty} f(x)$, which in this case are both the same. We can see the power of the top and bottom are the same and the ratio of their coefficients is 1, thus, $\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow -\infty} f(x) = 1$, and therefore 1 is a horizontal asymptote.

- 5 (b) Find the intervals on which $f(x)$ is increasing or decreasing.

Solution: Increasing or decreasing corresponds to a derivative being positive or negative respectively. Thus, we use the quotient rule to compute this:

$$f'(x) = \frac{2x(x-2)^2 - 2x^2(x-2)}{(x-2)^4} = \frac{2x(x-2) - 2x^2}{(x-2)^3} = \frac{2x^2 - 4x - 2x^2}{(x-2)^3} = -\frac{4x}{(x-2)^3}.$$

Notice we have two distinct critical numbers: $x = 2$, where f' doesn't exist and $x = 0$, where $f' = 0$. Thus, we can create the following number line:



To obtain this number line, we can plug in easy numbers like $x = -1$, $x = 1$, $x = 3$, which correspond to each region to obtain the correct signs. This tells us that the function is only increasing in the middle region and decreasing on the outer two.

- 5 (c) Find the extrema(min/max) of $f(x)$.

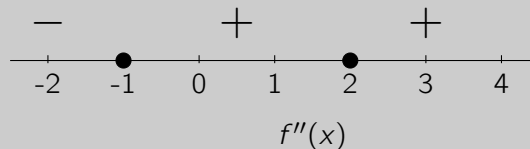
Solution: By the number line above, it’s clear by the first derivative test that if the function is decreasing and then increasing at $x = 0$, it must be a local minimum. The other critical point corresponds to an asymptote so it makes no sense to consider it an extrema.

- 5 (d) Find the intervals of concavity and the inflection points of $f(x)$.

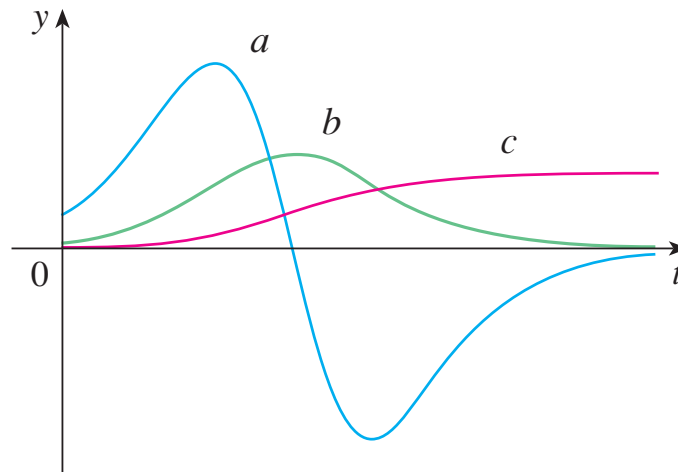
Solution: We take the derivative again to obtain the second derivative, using the quotient rule:

$$f''(x) = \frac{-4(x-2)^3 + 4x(x-2)^2 \cdot 3}{(x-2)^6} = \frac{8(x+1)}{(x-2)^4}.$$

Here, it’s clear our two inflection points are -1 and $x = 2$, again creating a number line:

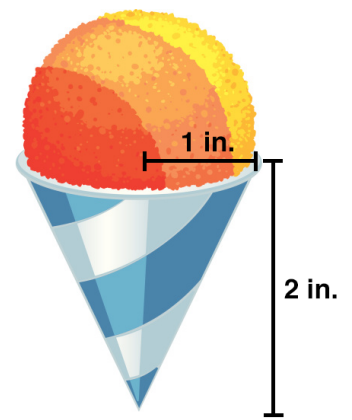


- 10 5. Below contains the graph of a function f , as well as its derivative f' and its second derivative f'' . Identify each and explain your reasoning for full credit.



Solution: With these problems, the easiest technique is to identify the flat spots and match the derivative being zero. Here, we see that a has two flat spots, but does either b or c have a zero at these points? Clearly not. Thus, this provides a as a good candidate for f'' . Also, b has a flat spot in the middle and we can see that a has a zero at this point, meaning we suspect $b' = a$ and therefore $b = f'$. The last curve c is almost never flat, increasing always, meaning its derivative should be roughly always positive, which b is, thus we can conclude $c' = b$. This means that $c = f$, just as we suspected. There are many ways to reason through these. This is just one example.

- 15 6. Shaved ice is one of Chris's favorite summer treats that comes in a perfectly cylindrical paper paper cone. Unfortunately, he eats it way too slowly and it leaks out of the bottom of the paper cone. Assuming that he have eaten enough of the shaved ice so that it is completely in the cone (that is, nothing is sticking out of the top), we measure the paper cone and find it is 2 inches tall with a base radius of 1 inch. If the melted ice is leaking out the bottom at a rate of $1/2$ inch³/min, how fast is the height changing when it is currently at 1 inch high?

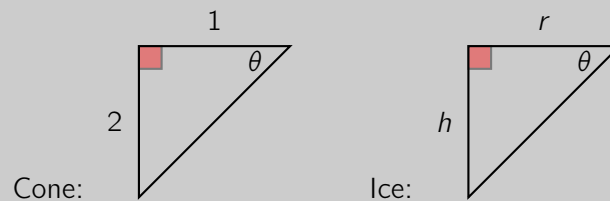


Hint: The volume of a cone is: $V = \frac{1}{3}\pi r^2 h$.

Solution: At this point, we've done many examples nearly identical to this. We know that the volume of shaved ice is described by:

$$V = \frac{1}{3}\pi r^2 h,$$

but we want to eliminate one of the variables on the right hand side. How do we do this? We observe that the radius and height of the ice is described by a similar triangle to the corresponding radius and height of the cone itself, which can be seen from the following triangles:



From this, we can conclude there is a constant ratio:

$$\frac{1}{2} = \frac{r}{h} \implies r = \frac{1}{2}h.$$

Plugging this back into our volume equation, we find:

$$V = \frac{1}{3}\pi \left(\frac{h}{2}\right)^2 h = \frac{\pi}{12}h^3.$$

We can now take the derivative of both sides to yield:

$$\frac{dV}{dt} = \frac{3\pi}{12}h^2 \frac{dh}{dt}.$$

We know all of the values except for $\frac{dh}{dt}$, meaning we can plug them in:

$$-\frac{1}{2} = \frac{3\pi}{12}1^2 \frac{dh}{dt} \implies \frac{dh}{dt} = -\frac{2}{\pi}.$$

Notice, the rate of change of the volume is negative because it is leaking out, that is, the volume is decreasing.

- 15 7. Using implicit differentiation, find the slope of the tangent line to the ellipse given by the following equation and the point:

$$3x^2 + \frac{y^2}{4} = 4, \quad \text{at } (1, 2).$$

Solution: Here, this is a simple implicit differentiation problem. Taking the derivative of both sides yields:

$$6x + \frac{2y y'}{4} = 0 \implies y' = -\frac{12x}{y}.$$

We have the derivative and a point, meaning we have all of the information we need for the tangent line, which is described by, at a point a :

$$y = f'(a)(x - a) + f(a).$$

Here, we know that $f(1) = 2$, so we can compute:

$$f'(1) = -\frac{12(1)}{(2)} = -6.$$

Putting this all together, our tangent line is:

$$y = -6(x - 1) + 2.$$

- 15 8. A rectangular storage container with an open top has a volume of 10 m^3 . The base of the container must be a square. The material for the base costs \$10 per square meter and the material for the sides of the container cost \$6 per square meter. Find the cost of the materials for the cheapest such container.

Solution: This is an optimization problem, meaning we need a constraint and an objective that we want to maximize or minimize. Here, we are told that the constraint is that the box must have a prescribed volume, 10. Assuming the two sides of the square base of the box are of x , and the height is y we know the volume can be described by:

$$V = 10 = x^2y.$$

We need now consider the objective function, which is the cost of the box. We must consider the surface area of the box, excluding the top, yielding:

$$\text{cost} = 1 \times \$10x^2 + 4 \times \$6xy.$$

Notice we have 4 sides of the box that cost \$6 per square meter, and only the square base costs \$10 per square meter. Thus, we need to optimize the cost by eliminating one of the variables. The easiest variable to eliminate, I think, is y from the constraint:

$$10 = x^2y \implies y = \frac{10}{x^2}.$$

Using this in our objective function, we find that:

$$\text{cost} = 10x^2 + 24x \frac{10}{x^2} = 10x^2 + \frac{240}{x}.$$

If we take a derivative we find that:

$$\text{cost}' = 20x - \frac{240}{x^2}.$$

We set the derivative equal to 0 to try to find the minimum cost, yielding:

$$0 = 20x - \frac{240}{x^2} = 20x^3 - 240 \implies x^3 = 240/20 \implies x = \sqrt[3]{12} \approx 2.28.$$

We can convince ourselves this is a minimum because the derivative is negative to the left of this point and positive to the right.

- 15 9. Find two nonnegative numbers whose sum is 9 and so that the product of one number and the square of the other number is a maximum.

Solution: This problem was very easy if read correctly. The first statement says that we have two numbers, which we'll call x, y have a sum 9, meaning:

$$x + y = 9.$$

We have another requirement, that we maximize the product of one of the numbers and the square of the other, that is:

$$\text{maximize: } xy^2.$$

Notice, this is exactly a constrained optimization problem. Our first equation is our constraint and our second equation is our objective. Thus, we can eliminate one of the variables:

$$x = 9 - y,$$

to yield our new objective, which I'll call f :

$$f(y) = (9 - y)y^2.$$

Notice, we also have the constraint that $0 \leq y \leq 9$ since we would have negative numbers otherwise. Thus, to find the maximum, we take a derivative and set it equal to 0:

$$f'(y) = 18y - 3y^2 = y(18 - 3y) = 0.$$

Notice this has two solutions $y = 0, y = 6$. If we plug this into our objective, we see which is bigger. When we have $y = 0$, we see that the product is clearly 0. If $y = 6$, we see the product is $36 \times 3 = 108$, which is clearly larger than 0 and therefore our maximum. Several of you took non-calculus approaches to this problem, which would fail if the answer were not integers. There is no reason the answer had to be integers, so checking all pairs between 0 and 9 is an ineffective and incorrect strategy. For this reason, I can only give minimal credit.

15 10. Compute the following integral:

$$\int_0^1 \frac{dx}{(1 + \sqrt{x})^4}$$

Hint: Try $u = 1 + \sqrt{x}$.

Solution: This was perhaps the trickiest problem on the whole exam, even with the u substitution provided. Consider the u substitution, which also means that $du = dx/2\sqrt{x}$. Notice though, if we rearrange the u substitution, we see that $\sqrt{x} = u - 1$. Therefore:

$$du = \frac{dx}{2\sqrt{x}} \implies 2\sqrt{x}du = dx \implies 2(u - 1)du = dx.$$

With this in mind, let's return to our original integral. Our new bounds become: $x = 0 \implies u = 1$, $x = 1 \implies u = 2$. Therefore:

$$\begin{aligned} \int_0^1 \frac{dx}{(1 + \sqrt{x})^4} &= \int_1^2 \frac{2(u - 1)du}{u^4} \\ &= 2 \int_1^2 \frac{u - 1}{u^4} du \\ &= 2 \int_1^2 \left[\frac{u}{u^4} - \frac{1}{u^4} \right] du \\ &= 2 \left[\frac{u^{-2}}{-2} - \frac{u^{-3}}{-3} \right]_{u=1}^{u=2} \\ &= 2 \left[-\frac{1}{2(2)^2} + \frac{1}{3(2)^3} + \frac{1}{2(1)^2} - \frac{1}{3(1)^3} \right]. \end{aligned}$$

Don't try to simplify this any more than above. It will cause you pain. If you're curious, though, the answer is $1/6$.

- 15 11. Compute the following integral:

$$\int_1^2 \ln x \, dx$$

Solution: We did this example in class. This problem is integration by parts. By LIATE, it's clear the choice for u is $\ln x$. Thus, if we choose $u = \ln x$, this leaves only $dv = dx$. Integrating and differentiating respectively, we find that $u = dx/x$ and $v = x$. Integration by parts says that:

$$\int_1^2 u \, dv = uv \Big|_{x=1}^{x=2} - \int_1^2 v \, du.$$

Plugging in our actual functions yields:

$$\begin{aligned} \int_1^2 u \, dv &= uv \Big|_{x=1}^{x=2} - \int_1^2 v \, du \\ &= x \ln x \Big|_{x=1}^{x=2} - \int_1^2 x \frac{1}{x} \, dx \\ &= [2 \ln 2 - 0] - \int_1^2 1 \, dx \\ &= 2 \ln 2 - 1. \end{aligned}$$

- 15 12. Use partial fraction decomposition to evaluate the following integral:

$$\int \frac{dx}{x^2 - x - 2}.$$

Solution: The first observation is that we can factor $x^2 - x - 2 = (x + 1)(x - 2)$, which allows us to use partial fraction decomposition:

$$\frac{1}{x^2 - x - 2} = \frac{A}{x + 1} + \frac{B}{x - 2}.$$

Finding a common denominator, and matching the numerators of this equation, we find that:

$$1 = A(x - 2) + B(x + 1).$$

If we consider matching the powers of x , we first consider the constant term:

$$1 = -2 \times A + B \times 1.$$

We also consider the x terms:

$$0 = A + B.$$

The solution to this system is $B = 1/3$, $A = -1/3$, thus we have transformed our integral into:

$$\int \frac{dx}{x^2 - x - 2} = \int \left[-\frac{1/3}{x + 1} + \frac{1/3}{x - 2} \right] dx.$$

Recall that the antiderivative of u'/u is $\ln|u|$, meaning we have the following integral result:

$$\int \frac{dx}{x^2 - x - 2} = \int \left[-\frac{1/3}{x + 1} + \frac{1/3}{x - 2} \right] dx = -\frac{1}{3} \ln|x + 1| + \frac{1}{3} \ln|x - 2| + C.$$

- 15 13. By constructing an argument comparing this integral to another, argue whether the following improper integral converges or diverges:

$$\int_1^{\infty} \frac{e^{-x} + 7}{\sqrt{x}} dx$$

Solution: This was the simplest problem I could come up with from this section. Recall what we know about the integral of:

$$\int_1^{\infty} \frac{1}{x^p} dx.$$

We know that if $p \leq 1$ the integral diverges and if $p > 1$, the interval converges. We typically (in this class, always) want to compare with some p . The obvious choice here is $p = 1/2$. Notice that:

$$e^{-x} + 7 \geq 7 \geq 1.$$

Thus, we can conclude that:

$$\frac{e^{-x} + 7}{\sqrt{x}} \geq \frac{1}{\sqrt{x}}.$$

What information do we gain from this? We know the same must be true of the integrals:

$$\int_1^{\infty} \frac{e^{-x} + 7}{\sqrt{x}} dx \geq \int_1^{\infty} \frac{1}{\sqrt{x}} dx.$$

Thus, our desired integral is larger than the second integral, but we know the behavior of the second integral: it diverges. Our integral is therefore larger than an integral that is too large to converge, meaning it is also too large to converge, and therefore our integral also diverges.