Lebesgue Measure

1. Outer measure: for \( A \subseteq \mathbb{R} \), define \( m^* A = \inf \left\{ \sum_k r_k : A \subseteq \bigcup_k I_k, I_k \text{ intervals} \right\} \) (Outer measure).

2. Inner measure: \( m_* A = \sup \left\{ \sum_k r_k : A \supseteq \bigcup_k I_k, I_k \text{ intervals} \right\} \) (Inner measure).

3. \( m^*(I) = l(I) \) (for intervals) is the outer measure of an interval.

4. Non-measurable set: consider \( f \) if \( f \) is discontinuous, \( f \) is not measurable.

5. Translation invariance: \( m^*(E + a) = m^*(E) \) for all \( a \in \mathbb{R} \) and measurable \( E \).

6. A set is measurable if and only if the outer measure equals the inner measure: \( m^* E = m_* E \).

7. Measurable set: \( E \subseteq \mathbb{R} \), \( m^* E = m_* E \).

8. Measure: \( m^* \) is countably subadditive: \( m^* (\bigcup_n A_n) \leq \sum_n m^* A_n \).

9. Littlewood's three principles: (a) \( v \geq 0 \)

10. Weak Lebesgue measure: \( m^* E \leq 0 \), \( E \) measurable, \( E \rightarrow \mathbb{R} \), given \( E \), \( x \in \mathbb{R} \).

11. Measure: \( m(E) = \sup \left\{ m^* \left( \bigcup_n A_n \right) : A_n \subseteq E, A_n \text{ measurable} \right\} \).

12. Measurable set: \( E \subseteq \mathbb{R} \), \( m(E) = m^*(E) = m_*(E) \).

13. Measurability: \( E \) measurable if and only if \( m^* E = m_* E \).

14. Outer measure: \( m^* E = \inf \left\{ \sum k r_k : A \subseteq E \right\} \).

15. Inner measure: \( m_* E = \sup \left\{ \sum k r_k : A \supseteq E \right\} \).

16. Null set: \( m_*(E) = 0 \).

17. Countable additivity: \( m^* (\bigcup_n E_n) = \sum_n m^* E_n \).

18. Countable additivity: \( m_* (\bigcup_n E_n) = \sum_n m_* E_n \).

19. Complete measure: \( m^* (E) = m_* (E) \) for all \( E \subseteq \mathbb{R} \).

20. Equivalence: \( m^* (E) = m_* (E) \).

21. Generalized Cantor set: \( E \) measurable, \( m^* E = 0 \), \( m_* E = 0 \).

22. Cantor set: \( E \) measurable, \( m^* E = 0 \), \( m_* E = 0 \).

23. Measurability: \( E \subseteq \mathbb{R} \), \( m^* E = m_* E \).

24. Measure: \( m(E) = \sup \left\{ m^* \left( \bigcup_n A_n \right) : A_n \subseteq E, A_n \text{ measurable} \right\} \).

25. Measure: \( m(E) = \inf \left\{ m_* \left( \bigcup_n A_n \right) : A_n \supseteq E, A_n \text{ measurable} \right\} \).

26. Lebesgue measure: \( m^*(E) = \inf \left\{ \sum k r_k : A \subseteq E \right\} \).

27. Lebesgue measure: \( m^*(E) = \sup \left\{ \sum k r_k : A \supseteq E \right\} \).

28. Equivalence: \( m^* (E) = m_* (E) \).

29. Countable additivity: \( m^* (\bigcup_n E_n) = \sum_n m^* E_n \).

30. Countable additivity: \( m_* (\bigcup_n E_n) = \sum_n m_* E_n \).

31. Completeness: \( m^* (E) = m_* (E) \) for all \( E \subseteq \mathbb{R} \).

32. Equivalence: \( m^* (E) = m_* (E) \).

33. Countable additivity: \( m^* (\bigcup_n E_n) = \sum_n m^* E_n \).

34. Countable additivity: \( m_* (\bigcup_n E_n) = \sum_n m_* E_n \).

35. Completeness: \( m^* (E) = m_* (E) \) for all \( E \subseteq \mathbb{R} \).

36. Equivalence: \( m^* (E) = m_* (E) \).

37. Countable additivity: \( m^* (\bigcup_n E_n) = \sum_n m^* E_n \).

38. Countable additivity: \( m_* (\bigcup_n E_n) = \sum_n m_* E_n \).

39. Completeness: \( m^* (E) = m_* (E) \) for all \( E \subseteq \mathbb{R} \).

40. Equivalence: \( m^* (E) = m_* (E) \).


**Lebesgue Integral**

- **Definition:** If \( f \) is *Lebesgue integrable* on \( [a, b] \), then \( \int_a^b f(x) \, dx \) is the least upper bound of \( \{ \sum_{i=1}^n f(x_i) \Delta x_i : \text{finite, \( \Delta x_i \) partition \( [a, b] \) \} \}

- **Properties:**
  1. If \( f \) is integrable on \( [a, b] \), then \( f(x) \) is bounded on \( [a, b] \).
  2. For any partition \( P \) of \( [a, b] \),
     \[ \sum_{i=1}^n f(x_i) \Delta x_i \leq \int_a^b f(x) \, dx \leq \sum_{i=1}^n f(x_i^*) \Delta x_i \]

- **Examples:**
  - \( f(x) = 1 \) on \( [0, 1] \), \( \int_0^1 1 \, dx = 1 \)
  - \( f(x) = x \) on \( [0, 1] \), \( \int_0^1 x \, dx = \frac{1}{2} \)

- **Fundamental Theorem of Calculus:** If \( f \) is continuous on \( [a, b] \), then \( F(x) = \int_a^x f(t) \, dt \) is differentiable on \( (a, b) \) and \( F'(x) = f(x) \) for all \( x \) in \( (a, b) \).

- **Monotone Convergence Theorem:** If \( \{ f_n \} \) is a sequence of non-negative measurable functions such that \( f_n \to f \) almost everywhere on \( [a, b] \), then \( \int_a^b f_n \, dx \to \int_a^b f \, dx \) as \( n \to \infty \).

- **Fatou's Lemma:** If \( f_n \) is a sequence of measurable functions such that \( f_n \to f \) almost everywhere, then \( \int_a^b \liminf f_n \, dx \leq \liminf \int_a^b f_n \, dx \).

- **Lebesgue's Dominated Convergence Theorem:** If \( f_n \to f \) almost everywhere and \( |f_n| \leq g \) for some integrable \( g \), then \( \int f_n \, dx \to \int f \, dx \) as \( n \to \infty \).

- **Lebesgue's Dominated Convergence Theorem:** If \( f_n \to f \) almost everywhere and \( |f_n| \leq g \) for some integrable \( g \), then \( \int f_n \, dx \to \int f \, dx \) as \( n \to \infty \).

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**L^p Spaces.**

*General Holder's: \( \frac{1}{r} + \frac{1}{s} = \frac{1}{p} \) so \( \|fg\|_r \leq \|f\|_s \|g\|_r \). Induction: 3.

- Holder's inequality: \( f \in L^p, g \in L^q \), \( \frac{1}{r} = \frac{1}{p} + \frac{1}{q} \).
- Minkowski's inequality: \( f \in L^p \), \( \|f + g\|_p \leq \|f\|_p + \|g\|_p \).
- Young's inequality: \( ab \leq \frac{a^p}{p} + \frac{b^q}{q} \) for \( p, q \geq 1 \) and integrable (eq: \( f^{\frac{1}{p}} + g^{\frac{1}{q}} \)).

- Banach spaces: \( \|f_n - f\| \to 0 \) implies \( f_n \to f \) in norm.
- Fubini's theorem: \( \int \int f(x)g(y) \, dx \, dy = \int \left( \int f(x)g(y) \, dy \right) \, dx \).

**Riesz-Fischer: L^2 are complete (Bessel's ineq.)**

1. Let \( f \in L^2 \) and \( \int f \overline{g} \, dx = 0 \) for all \( g \in L^2 \).

2. Define \( \langle f, g \rangle = \int f \overline{g} \, dx \) and \( \|f\|_2 = (\int |f|^2 \, dx)^{\frac{1}{2}} \).

3. Important facts:
   - completeness of \( L^2 \).
   - Hölder's inequality: \( \|fg\|_r \leq \|f\|_s \|g\|_r \).

**Inclusions: inclusion for \( L^p \), for \( m(X) < \infty \),**

1. \( L^p(X) \supseteq L^q(X) \) for \( p \leq q \).
2. \( L^p(X) \subsetneq L^q(X) \) for \( p > q \).

**Approximation in \( L^p \)**

1. If \( f \in L^p \), then \( \text{for every } \varepsilon > 0, \text{ there exists } \phi \in C_0^\infty \text{ such that } \|f - \phi\|_p < \varepsilon \).
2. The \( p \)-step and compact \( f \in L^p \) iff \( f \in L^p \) as \( \|f\|_p < \infty \).
3. If \( f \in L^p \), then \( f \in L^q \) iff \( \|f\|_q < \infty \).

**Measure spaces:**

- For \( X \subset \mathbb{R} \), \( m(X) = \lambda(X) \).
- \( \lambda^p(X) \subsetneq \lambda^q(X) \) for \( p < q \).

**Almost everywhere:**

- If \( f \in L^p \), then \( f \in L^q \) iff \( \int |f|^q \, dx < \infty \).

**Generalized functions:**

- \( \delta \) is a distribution that is \( 0 \) outside a compact set.

**Lebesgue spaces:**

- \( L^p(X) \) is the space of all measurable functions \( f : X \to \mathbb{C} \) such that \( \|f\|_p < \infty \).

**Conclusion:**

- \( L^p \) spaces are complete metric spaces.
- \( L^p \) spaces are Banach spaces.
- \( L^p \) spaces are reflexive.
- \( L^p \) spaces are separable.

**Important properties:**

- \( \|f + g\|_p \leq \|f\|_p + \|g\|_p \).
- \( \|f\|_p = 0 \) iff \( f = 0 \) almost everywhere.

**Notes:**

- \( L^p \) spaces are the most important in functional analysis.
- \( L^p \) spaces are used to study functions and their integrability.

**Examples:**

- \( L^1 \) space of integrable functions.
- \( L^2 \) space of square-integrable functions.
- \( L^\infty \) space of essentially bounded functions.

**Properties:**

- \( L^p \) spaces are Banach spaces.
- \( L^p \) spaces are reflexive.
- \( L^p \) spaces are separable.

**Applications:**

- In probability theory.
- In harmonic analysis.
- In partial differential equations.
General measure theory.

1. Measurable space: \((X, \mathcal{B})\), where \(\mathcal{B}\) is a family of subsets of \(X\), \(\sigma\)-algebra of \(X\). A measurable space is a triple \((X, \mathcal{B}, \mu)\), where \(\mu\) is a measure on \((X, \mathcal{B})\).

2. Measure: \(\mu: \mathcal{B} \rightarrow [0, \infty]\), \(\mu(\emptyset) = 0\), countable addition, \((X, \mathcal{B}, \mu)\) measure space.

3. Count: additive, \(\mu(A) = \mu(B)\) if \(A \cap B = \emptyset\).

4. Finite measure: \(\mu(X)<\infty\) \(\Rightarrow \mu(X) = \sum \mu(X_i)\), where \(X_i\) are disjoint sets.

5. \((X, \mathcal{B}, \mu)\) is complete if \(\mathcal{B}\) contains all subsets of sets of measure zero.

6. Lebesgue measure is complete; \(L^1\) not complete.

7. Completion: Given \((X, \mathcal{B}, \mu)\), \(L^1\) complete \((X, \mathcal{B}', \mu)\).

8. \(\mathcal{B} \subseteq \mathcal{B}'\). \(E \in \mathcal{B}' \Rightarrow \exists E_0 \in \mathcal{B}\) \(E_0 \subseteq E, \mu(E_0) = \mu(E).

9. \(E \in \mathcal{B}'\). \(\mu(E) < \infty\) \(\Rightarrow \exists E_0 \in \mathcal{B}\) \(E_0 \subseteq E, \mu(E_0) = \mu(E)\).

10. \(E \in \mathcal{B}'\). \(\mu(E) = \infty\) \(\Rightarrow \exists E_0 \in \mathcal{B}\) \(E_0 \subseteq E, \mu(E_0) = \infty\).

11. \(\mu: \mathcal{B} \rightarrow [0, \infty]\) is a measure. \(\mu(\emptyset) = 0, \mu(E) + \mu(F) = \mu(E \cup F)\) if \(E \cap F = \emptyset\).

12. A measure is countably additive if \(\mathcal{B}\) is countably generated.

13. \((X, \mathcal{B}, \mu)\) a finite measure space if \(\mu(X) < \infty\).

14. \(L^1\) if simple. \(\sum \mu(\mathcal{B}_i)\) if \(\mathcal{B}\) is countable.

15. \(L^p\) if \(p < \infty\).

16. Signed measure: \(\omega: \mathcal{B} \rightarrow \mathbb{R}\) \(\Rightarrow \omega\) is a measure.

17. Positive and negative part of \(\omega\).

18. Jordan decomposition: \(\omega = \omega^+ - \omega^-\) where \(\omega^+\) and \(\omega^-\) are positive and negative.

19. Hahn decomposition: \(A = A_+ \cup A_-\) where \(A_+ \cup A_- = X\).

20. \(\mu(A) = \mu(A_+) - \mu(A_-)\).

21. \(\omega: \mathcal{B} \rightarrow \mathbb{R}\) a signed measure on \((X, \mathcal{B})\). There is a positive \(\mu\) and a negative \(\nu\) (not unique).

22. \((X, \mathcal{B})\) a finite measure space, \(\omega\) integrable \(\Rightarrow \sum \omega(B_i) \leq \mu(B_i)\).
Measure and outer measure

\[ \mu^*: \text{measurable extended real-valued set \( f \)} \mapsto \mu^*(f) = \inf \{ \mu(A) : f \leq A, A \in \mathcal{A} \} \]

Product measures: \( (\mathcal{X}, \mathcal{A}, \mu) \) and \( (\mathcal{Y}, \mathcal{B}, \nu) \), complete measure spaces

\[ \sigma^* \text{-measurable sets } \sigma^* = \{ A \times B : A \in \mathcal{A}, B \in \mathcal{B} \} \]

- \( \mathcal{A} \times \mathcal{B} \) is a \( \sigma^* \text{-algebra} \)
- \( \mu^* \) is a complete measure

\[ \mu^*(A \times B) = \mu(A) \nu(B) \]

Carathéodory: let \( \mu \) be a measure on algebra \( \mathcal{A} \)

\[
\lambda \text{ can be extended to a complete } \sigma \text{-algebra of sets } S \supseteq \mathcal{A}^c
\]

\[ \text{induced by } \mu. \text{ Then } \mu^* \text{ is } \sigma \text{-measurable.} \]

- Fubini: let two complete measure spaces, integrable on \( X \times Y \). Then

\[ \int (x,y) f(x,y) \, d\mu \]

\[ \text{almost all } x, \int_y f(x,y) \, d\nu(y) \]

If \( \mu \) is finite, \( \mu^* \) is the only measure on the smallest \( \sigma \text{-algebra} \)

\[ \text{containing } \mathcal{A} \text{ which is an extension of } \mu. \]

Ex.: \( h \) integrable on \( X \times Y \), \( h(x,y) = 1 \) on \( X \times \{ y \} \). Then \( h \) integrable on \( X \times Y \) and

\[ \int_{X \times Y} h(x,y) \, d\mu = \int_x \int_y h(x,y) \, d\mu(y) \, d\mu(x) \]

- For general \( f(x,y) \)

\[ \int_{X \times Y} f(x,y) \, d\mu = \int_x \left( \int_y f(x,y) \, d\nu(y) \right) \, d\mu(x) \]

- \( \mathcal{X} \times \mathcal{Y} = \mathcal{A} \) \& \( \mathcal{B} \) \& \( \mathcal{A} \times \mathcal{B} \)

\[ \text{measurable is an } (\mathcal{P}_X, \mathcal{P}_Y) \]

\[ \sigma \text{-algebra, } \mathcal{P}_X \cap \mathcal{P}_Y \text{ is a } \sigma \text{-algebra.} \]

\[ \int_X f(x) \, d\mu(x) = \int_X \left( \int_Y f(x,y) \, d\nu(y) \right) \, d\mu(x) \]

1. \( f \) measurable, \( h(x,y) \) integrable on \( X \times Y \):

2. \( f(x,y) = f(y,x) \) integrable on \( X \times Y \):

3. \( f(x,y) = f(x,y) \) integrable on \( X \times Y \):

4. \( f(x,y) = f(x,y) \) integrable on \( X \times Y \):

5. \( f(x,y) = f(x,y) \) integrable on \( X \times Y \):

- Tonelli: let two \( \sigma \text{-finite mesures, } \mu \times \nu \) measurable on \( X \times Y \). Then:

\[ \int (x,y) f(x,y) \, d\mu \times \nu = \int_x \int_y f(x,y) \, d\nu(y) \, d\mu(x) \]

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- Tonelli: let two \( \sigma \text{-finite mesures, } \mu \times \nu \) measurable on \( X \times Y \). Then:

\[ \int (x,y) f(x,y) \, d\mu \times \nu = \int_x \int_y f(x,y) \, d\nu(y) \, d\mu(x) \]

- Tonelli: let two \( \sigma \text{-finite mesures, } \mu \times \nu \) measurable on \( X \times Y \). Then:
Differentiation:

If a set \( E \) is the join of \( \{x \in \mathbb{R}^n : f(x) \leq 0\} \) and \( \{x \in \mathbb{R}^n : f(x) \geq 0\} \), then for each \( x \in \mathbb{R}^n \)

\[
\text{Ex: monotone fn, disc. at \( a \), \( f \) cont. on \( [a, b] \)}
\]

\[
\text{Consider } -\left( f(a) + g(b) \right) < \int_a^b f(t) \, dt < f(a) + g(b)
\]

\[\text{for all } a < b \text{ in } \mathbb{R}^n.\]

**Vitali Lemma:** \( \exists \varepsilon > 0 \). For a Vitali covering \( \{V_0, V_1, \ldots, V_n\} \), define \( f_1, f_2, \ldots, f_n \) as approximations,

\[
\text{cf} \quad m^*(E \setminus V_i) \leq \varepsilon.
\]

**If** \( f \) is continuous on \([a, b]\), \( f' \) is defined \( \forall x \in \mathbb{R}^n \). Then \( f \) is differentiable on \( [a, b] \) if for each \( x \in \mathbb{R}^n \),

\[
\int_a^b f(x \in [a, b]) = \left. \int_a^b f(x) \, dx \right|_{x=a}^{x=b}.
\]

**Then a measurable \( f \) is differentiable if \( f \) is absolutely continuous.**

\[
\text{Ex: } f(x) = x^2 \text{ is continuous, but not absolutely continuous on } [0, 1].
\]

**Lebesgue \( \Rightarrow \) absolute continuous.**

<table>
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<th>Example</th>
<th>Application</th>
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<tr>
<td>( f(x) = x^2 )</td>
<td>Differentiable on ( [0, 1] )</td>
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</table>

**Convex Functions:**

Let \( g(x) = \langle x, v \rangle + f(x) \) and \( h(x) = \langle x, w \rangle + \phi(x) \).

\[
\text{If } f \text{ and } h \text{ are convex, then } g \text{ is convex.}
\]

F, \( f \in [a, b] \) exists for almost all \( x \) in \( [a, b] \).

**Indefinite integral:** \( F(x) = \int_a^x f(t) \, dt \).

\[
\text{If } F \text{ is integrable, then } \frac{d}{dx} F(x) = f(x) \text{ almost everywhere on } [a, b].
\]

If \( F \) is absolutely continuous on \( [a, b] \), then \( F \) is integrable.

\[
\text{then } \int_a^b f(x) \, dx = F(b) - F(a).
\]

If \( f \) is absolutely continuous on \( [a, b] \), then \( f' \) exists a.e. on \( [a, b] \).

**If** \( f \) is absolutely continuous on \( [a, b] \), then \( \int_a^b f(x) \, dx = F(b) - F(a) \), where \( F \) is an antiderivative of \( f \).

\[
\int_a^b f(x) \, dx = F(b) - F(a).
\]

\[
\text{Theorem: } f \text{ is convex on } [a, b], \text{ then } F \text{ is absolutely continuous on } [a, b].
\]

\[
\text{If } \psi \in C^1([a, b]), \text{ then } \psi \text{ is the convex envelope of } \psi''
\]

\[
\text{in general, } \int_a^b \psi(x) \, dx \leq \psi(F(b)) - \psi(F(a)).
\]

\[
\text{Let } \mu \text{ be a measure on } [a, b], \text{ then } \int_a^b \psi(x) \, d\mu \leq \psi(F(b)) - \psi(F(a)).
\]

\[
\text{If } \psi \text{ is convex and } f \text{ is integrable, then } \int_a^b f(x) \psi(x) \, dx = \psi(F(b)) - \psi(F(a)).
\]

\[
\text{Proof: } \int_a^b f(x) \psi(x) \, dx = \psi(F(b)) - \psi(F(a)) - \psi(F(b)) + \psi(F(a)).
\]

\[
\int_a^b f(x) \psi(x) \, dx = \psi(F(b)) - \psi(F(a)).
\]
Modes of convergence.

- \( f_n \to f \) in \( L^p \) if \( \int |f_n - f|^p \to 0 \) as \( n \to \infty \).
- \( f_n \to f \) a.e. if \( f_n \to f \) in measure and \( f_n \to f \) in some subsequence.
- \( f_n \to f \) in measure if \( \int |f_n - f| \to 0 \) as \( n \to \infty \).
- \( f_n \to f \) in measure (a.e.) if \( \mu(\{x : |f_n(x) - f(x)| > \varepsilon \}) \to 0 \) as \( n \to \infty \).

Examples:

- \( f_n \to f \) in \( L^p \) but \( f_n \not\to f \) in measure.
- \( f_n \to f \) in measure but \( f_n \not\to f \) in \( L^p \).
- \( f_n \to f \) a.e. but \( f_n \not\to f \) in measure.
- \( f_n \to f \) in measure and \( f_n \to f \) in \( L^p \), then \( f_n \to f \) in \( L^q \), for any \( p \leq q \).

Ultrafilter Limit:

- \( f_n \to f \) in measure if \( \mu(\{x : |f_n(x) - f(x)| > \varepsilon \}) = 0 \) for some ultrafilter \( \mathcal{U} \) on \( \mathbb{N} \).
- \( f_n \to f \) a.e. if \( \mu(\{x : |f_n(x) - f(x)| > \varepsilon \}) = 0 \) for some ultrafilter \( \mathcal{U} \) on \( \mathbb{N} \).

- \( f_n \to f \) in measure if \( \int |f_n - f| \to 0 \) as \( n \to \infty \).

Examples:

- \( f_n \to f \) in measure but \( f_n \not\to f \) almost everywhere.
- \( f_n \to f \) almost everywhere but \( f_n \not\to f \) in measure.
- \( f_n \to f \) in measure and \( f_n \to f \) almost everywhere.

- \( f_n \to f \) in \( L^p \) if \( f_n \to f \) in measure and \( f_n \to f \) in \( L^p \).
- \( f_n \to f \) in \( L^p \) if \( f_n \to f \) in measure and \( f_n \to f \) a.e.

- \( f_n \to f \) in measure if \( \mu(\{x : |f_n(x) - f(x)| > \varepsilon \}) \to 0 \) as \( n \to \infty \).
- \( f_n \to f \) a.e. if \( \mu(\{x : |f_n(x) - f(x)| > \varepsilon \}) \to 0 \) as \( n \to \infty \).
- \( f_n \to f \) in measure if \( \mu(\{x : |f_n(x) - f(x)| > \varepsilon \}) \to 0 \) as \( n \to \infty \).
- \( f_n \to f \) a.e. if \( \mu(\{x : |f_n(x) - f(x)| > \varepsilon \}) \to 0 \) as \( n \to \infty \).

Generalized Uniform Convergence:

- \( f_n \to f \) in measure if \( |f_n(x) - f(x)| \to 0 \) for a.e. \( x \) as \( n \to \infty \).

Examples:

- \( f_n \to f \) in measure but \( f_n \not\to f \) almost everywhere.
- \( f_n \to f \) almost everywhere but \( f_n \not\to f \) in measure.
- \( f_n \to f \) in measure and \( f_n \to f \) almost everywhere.

- \( f_n \to f \) in \( L^p \) if \( f_n \to f \) in measure and \( f_n \to f \) in \( L^p \).
- \( f_n \to f \) in \( L^p \) if \( f_n \to f \) in measure and \( f_n \to f \) a.e.

- \( f_n \to f \) in measure if \( \mu(\{x : |f_n(x) - f(x)| > \varepsilon \}) \to 0 \) as \( n \to \infty \).
- \( f_n \to f \) a.e. if \( \mu(\{x : |f_n(x) - f(x)| > \varepsilon \}) \to 0 \) as \( n \to \infty \).

- \( f_n \to f \) in measure if \( \mu(\{x : |f_n(x) - f(x)| > \varepsilon \}) \to 0 \) as \( n \to \infty \).
- \( f_n \to f \) a.e. if \( \mu(\{x : |f_n(x) - f(x)| > \varepsilon \}) \to 0 \) as \( n \to \infty \).
1) Riemann-Lebesgue Lemma: For a function $f$ in $L^1$, as $\epsilon \to 0$: 
$$ \lim_{\epsilon \to 0} \int_{-\epsilon}^{\epsilon} f(x) e^{ix} \, dx = 0. $$

2) Holder: $r > 1$, \( \frac{1}{r} + \frac{1}{s} = 1 \rightarrow \|f\|_r \leq \|f\|_p \|f\|_q $  

3) $L^p$ space: \( \{ f \mid \|f\|_p < \infty \} \), \( \|f\|_p = \left( \int |f|^p \, dx \right)^{1/p} \)  

4) Ca. Ha:  

5) Example: For \( f \in L^1 \), \( f \geq 0 \) \( \Rightarrow \int f \leq \|f\|_1 \).