Fixed point theorems.

Th: (Picard - Lindelof) Sup f \in C([a,b] \times [0,\infty)), u_0 \in \mathbb{R}^n, (t_0, u_0) \in U.

If f is locally Lipschitz continuous in the second argument, uniformly wrt. the first, \exists \exists \exists  \exists local solution x(t) \in \mathbb{C}(\mathbb{R})

K is a contraction if \|K_1 - K_2\| > \epsilon > 0 \forall \epsilon > 0, w.b. B(0,\infty).

K is a contraction \iff \|K_1 - K_2\| > \epsilon > 0 \forall \epsilon > 0, w.b. B(0,\infty). If so, then

K is a contraction \iff \|K_1 - K_2\| > \epsilon > 0 \forall \epsilon > 0, w.b. B(0,\infty).

Lemma: f \in C([a,b] \times [0,\infty)) \Rightarrow \exists x(t) \in \mathbb{C}(\mathbb{R}) \text{ s.t. } x(t) \in \mathbb{C}(\mathbb{R}) \text{ (e.g. } f \text{ is Lipschitz)}

Basic existence and uniqueness

(IVP): \begin{align*}
\frac{dx}{dt} &= f(t, x), \quad x \in \mathbb{R}^n, f \in C([a, b] \times \mathbb{R}^n), x_0 \in \mathbb{R}^n, u_0 \in \mathbb{U}.
\end{align*}

Existence: \exists x(t) < \infty \Rightarrow \exists x(t) < \infty \Rightarrow \exists x(t) < \infty

Feasibility: \exists x(t) < \infty \Rightarrow \exists x(t) < \infty \Rightarrow \exists x(t) < \infty

Conclusion: \exists x(t) < \infty \Rightarrow \exists x(t) < \infty \Rightarrow \exists x(t) < \infty

Basic existence and uniqueness

Basic existence and uniqueness
### Dependence on initial condition

- Gronwall's inequality:
  \[
  \text{If } x(t) \leq e^{t} + \int_{0}^{t} e^{s} \gamma(s) ds, \quad t \in [0,T], \quad x(0) \geq 0, \quad \gamma \in L^1([0,T])
  \]
  Then:
  \[
  x(t) \leq e^{t} \left(1 + \int_{0}^{t} e^{s} \gamma(s) ds\right)
  \]

### Existence of solutions

- If \( \phi \in C_{0}\) is locally Lipschitz and \(a \geq 0\) and \(b \geq 0\), then:
  \[
  x(t) = \phi(t) + \int_{0}^{t} e^{-s} \left(b \phi(s) + a \Phi(s)\right) ds
  \]

### Uniqueness of solutions

- If \( \phi \) is continuous and \( a \geq 0 \), then there exists a unique solution:
  \[
  x(t) = \phi(t) + \int_{0}^{t} e^{-s} \left(b \phi(s) + a \Phi(s)\right) ds
  \]

### Unbounded solutions

- If \( \phi \) is continuous and \( a \geq 0 \), then there exists a solution that remains unbounded.

### Dependence on initial condition

- If \( \phi \) is continuous and \( a \geq 0 \), then:
  \[
  x(t) = \phi(t) + \int_{0}^{t} e^{-s} \left(b \phi(s) + a \Phi(s)\right) ds
  \]

### Proof of existence

- Suppose \( \phi \) is continuous and \( a \geq 0 \).
  \[
  x(t) = \phi(t) + \int_{0}^{t} e^{-s} \left(b \phi(s) + a \Phi(s)\right) ds
  \]

### Proof of uniqueness

- Suppose \( \phi \) is continuous and \( a \geq 0 \).
  \[
  x(t) = \phi(t) + \int_{0}^{t} e^{-s} \left(b \phi(s) + a \Phi(s)\right) ds
  \]

### Unbounded solutions

- If \( \phi \) is continuous and \( a \geq 0 \), then there exists a solution that remains unbounded.

### Dependence on initial condition

- If \( \phi \) is continuous and \( a \geq 0 \), then:
  \[
  x(t) = \phi(t) + \int_{0}^{t} e^{-s} \left(b \phi(s) + a \Phi(s)\right) ds
  \]
Thm: \( \text{Let } U = \mathbb{R}^n, S \subseteq T \in \mathbb{R}^n, L(t) \in L(U, W) \). \( \forall h \in L(U, W) \in L(T, \mathbb{R}^n) \). Then all solutions are defined \( \forall t \in \mathbb{R}^n \).

Proof: \( \forall h \in L(U, W) \in L(T, \mathbb{R}^n) \). Then \( \forall h \in L(U, W) \in L(T, \mathbb{R}^n) \).

\[ w(t) \text{ is a coupled bell.} \]

Thus, \( \exists t \text{ such that } w(t) \in L(U, W) \). Then \( \exists t \text{ such that } w(t) \in L(U, W) \).

\[ A \text{ is continuous on } \mathbb{R}^+ \text{ with } L(U, W) \text{ be integrable.} \]

Then \( x_n \) has a uniformly convergent subsequence.

\[ x_n(t) \to x(t) \text{ as } n \to \infty \text{ if } x_n \text{ is a sequence of } L(U, W) \text{ and bounded.} \]

Then \( x_n(t) \) converges at \( t \) to \( x(t) \). The diagonal sequence \( \tilde{x}(t) = x_n(t) \) converges to \( x(t) \).

Fix \( t \in \mathbb{R}^+ \) and \( t \to h(t) \). The Cylindrical Vector \( h(t) \) is bounded by some positive constant \( C \). Choose \( N \) such that \( |t| < \frac{N}{2} \).

Thus \( |t| < \frac{N}{2} \) and \( h(t) \) is bounded by some positive constant \( C \). Choose \( N \) such that \( |t| < \frac{N}{2} \).

Thus \( |t| < \frac{N}{2} \).

Then \( x(t) = x_n(t) \) for \( n \geq N \).

Thus \( x_n(t) \to x(t) \) as \( n \to \infty \).

Thus \( x_n(t) \to x(t) \) as \( n \to \infty \).

The linear autonomous eq. of order \( n \) is \( \begin{cases} x^{(n)}(t) + a_{n-1}x^{(n-1)}(t) + \cdots + a_1x'(t) + a_0x(t) &= 0 \\ x(t) & \text{on } t \in \mathbb{R} \end{cases} \)

Then \( x(t) = e^{At}x(0) + \int_0^t e^{A(t-s)}f(s)ds \).

Then \( x(t) = e^{At}x(0) + \int_0^t e^{A(t-s)}f(s)ds \).

Where \( x(t) = \begin{cases} x_0 & \text{if } t = 0 \\ \frac{d}{dt}x(t) & \text{if } t > 0 \end{cases} \)

Thus \( x(t) = e^{At}x(0) + \int_0^t e^{A(t-s)}f(s)ds \).

Thus \( x(t) = e^{At}x(0) + \int_0^t e^{A(t-s)}f(s)ds \).

Where \( f(t) = \begin{cases} f_0 & \text{if } t = 0 \\ \frac{d}{dt}f(t) & \text{if } t > 0 \end{cases} \)

Thus \( x(t) = e^{At}x(0) + \int_0^t e^{A(t-s)}f(s)ds \).

Thus \( x(t) = e^{At}x(0) + \int_0^t e^{A(t-s)}f(s)ds \).

Where \( f(t) = \begin{cases} f_0 & \text{if } t = 0 \\ \frac{d}{dt}f(t) & \text{if } t > 0 \end{cases} \)

Thus \( x(t) = e^{At}x(0) + \int_0^t e^{A(t-s)}f(s)ds \).

Thus \( x(t) = e^{At}x(0) + \int_0^t e^{A(t-s)}f(s)ds \).

Where \( f(t) = \begin{cases} f_0 & \text{if } t = 0 \\ \frac{d}{dt}f(t) & \text{if } t > 0 \end{cases} \)

Thus \( x(t) = e^{At}x(0) + \int_0^t e^{A(t-s)}f(s)ds \).

Thus \( x(t) = e^{At}x(0) + \int_0^t e^{A(t-s)}f(s)ds \).

Where \( f(t) = \begin{cases} f_0 & \text{if } t = 0 \\ \frac{d}{dt}f(t) & \text{if } t > 0 \end{cases} \)

Thus \( x(t) = e^{At}x(0) + \int_0^t e^{A(t-s)}f(s)ds \).

Thus \( x(t) = e^{At}x(0) + \int_0^t e^{A(t-s)}f(s)ds \).

Where \( f(t) = \begin{cases} f_0 & \text{if } t = 0 \\ \frac{d}{dt}f(t) & \text{if } t > 0 \end{cases} \)

Thus \( x(t) = e^{At}x(0) + \int_0^t e^{A(t-s)}f(s)ds \).

Thus \( x(t) = e^{At}x(0) + \int_0^t e^{A(t-s)}f(s)ds \).

Where \( f(t) = \begin{cases} f_0 & \text{if } t = 0 \\ \frac{d}{dt}f(t) & \text{if } t > 0 \end{cases} \)

Thus \( x(t) = e^{At}x(0) + \int_0^t e^{A(t-s)}f(s)ds \).

Thus \( x(t) = e^{At}x(0) + \int_0^t e^{A(t-s)}f(s)ds \).

Where \( f(t) = \begin{cases} f_0 & \text{if } t = 0 \\ \frac{d}{dt}f(t) & \text{if } t > 0 \end{cases} \)

Thus \( x(t) = e^{At}x(0) + \int_0^t e^{A(t-s)}f(s)ds \).

Thus \( x(t) = e^{At}x(0) + \int_0^t e^{A(t-s)}f(s)ds \).

Where \( f(t) = \begin{cases} f_0 & \text{if } t = 0 \\ \frac{d}{dt}f(t) & \text{if } t > 0 \end{cases} \)

Thus \( x(t) = e^{At}x(0) + \int_0^t e^{A(t-s)}f(s)ds \).
General linear first order system: \( x'(t) = A(t)x(t) \), \( A \in C(I, \mathbb{R}^{n \times n}) \)

Uniqueness solution: \( \exists x'(t), x(0) \text{ and defined } \forall t \in I \) by \( x(t) = x(0) \exp(-\int_0^t A(s)ds) \)

Principal matrix solution: \( \Phi(t) = \exp(-\int_0^t A(s)ds) \) at \( t = 0 \)

Then \( \Phi(t)x(0) = g(t) \)

Proof: \( \Phi(t) = \exp(-\int_0^t A(s)ds) \) is a fundamental matrix solution.

If \( \Phi(t) = 0 \), \( \Phi(t) \) is a fundamental matrix solution.

From two fundamental solutions \( \Psi(t), \Phi(t) \), we have \( \Psi(t) = \Phi(t) \Psi(0) \)

Then the system is asymptotically stable.

If \( \Phi(t) \) have \( \infty \) real part, and \( \| \Phi(t) \| \leq 0 \),

Then the system is asymptotically stable.

\( \tau(t) = \Phi(t)t \)

(Continuous logarithm) \( \Psi(t) = \Phi(t) \exp(\int_0^t \Phi^{-1}(s)ds) \)

Proof:

\[ \frac{d}{dt} \Phi(t) + A(t) \Phi(t) = 0 \] (variation of constant)

\[ \Phi(t_2) = \Phi(t_1) \exp(-\int_{t_1}^{t_2} A(s)ds) \]

\[ \int_{t_2}^{t_1} A(s)ds = \Phi(t_1) \Phi^{-1}(t_2) \]

\[ \phi(t) = \Phi(t)x(0) = \int_{t_0}^{t} \Phi(s)x(0)ds \]

Lienard's formula: \( \phi(t) = \Phi(t)x(0) \) if trace of \( A \) is continuous.

Reduction of order (D'Alembert): \( A(t) \) is known, then \( \Phi(t) = \exp(\int_0^t A(s)ds) \)

Make \( \phi(t) = \Phi(t)x(0) \Rightarrow \phi(t) = \Phi(t)A(t)x(0) \)

Linear eq. of order \( n+1 \): \( x^{(n+1)} + a_1x^{(n)} + \ldots + a_nx + \frac{b(t)}{n!}x = f(t) \)

\[ x(t) = x_0 + \int_{t_0}^{t} f(s)ds \exp(-\int_{t_0}^{s} A(u)du) \]

Reduction of order: \( \omega(t) = \Phi(t)\psi(t) \)

By Laplace transform:

\[ \int_{t_0}^{t} f(s)ds \exp(-\int_{t_0}^{s} A(u)du) = \mathcal{L}^{-1}\left\{ \Phi(t)\psi(t) \right\} \]
ODE's in the complex domain:

Let \( g(z) : (0 \leq z < \infty) \rightarrow \mathbb{C} \) be analytic and hold:

Then \( w = \mathfrak{p}(z) \), \( w(z) \rightarrow \mathfrak{p} \), has a unique analytic solution defined on \( (0, \infty) \), \( w = \text{max} \{ z, \mathfrak{p}(z) \}, z \geq 0 \).

First order autonomous eq: \( x = \mathfrak{g}(x), \ x(0) = x_0, \ f \in C([0, \infty)) \)

- For \( \mathfrak{g}(x) = 0 \), \( \mathfrak{g}(0) = 0 \Rightarrow \) solution \( \mathfrak{g}(t) \) satisfies \( \mathfrak{g}(\mathfrak{g}(t)) = t \)
- For \( \mathfrak{g}(x) \) strictly monotone nondecreasing, \( \mathfrak{g}^{-1}(t) = \mathfrak{g}(t) \) is the solution.

Maximal interval:
If \( \mathfrak{g}(x) < \varepsilon, \ v > 0 \) in some \( (x, y) \Rightarrow x = 0, \)

\[
T_1 = \max \{ \mathfrak{f}(x) \in (0, \infty), \ t \geq \mathfrak{f}(x) \in C([0, \infty)), \ v \in C(T_1, T_1) \}
\]

\[
\frac{dt}{dx} = \frac{1}{\mathfrak{g}(x)}, \ \frac{dx}{dt} = \mathfrak{g}(x)
\]

\[\mathfrak{g}(x) = x_0, \ \mathfrak{g}(x) = x_0 \]

\[\frac{1}{\mathfrak{g}(x)} = \frac{1}{x_0}, \ \mathfrak{g}(x) = x_0 \]

\[T_2 = \int_0^{\infty} \frac{dx}{\mathfrak{g}(x)} = \infty
\]

Case 1: \( x - x_0 \neq 0, \ \mathfrak{f}(x) \neq x_0, \ \mathfrak{h}(x) \neq x_0 \)

\[\frac{dt}{dx} = \frac{1}{\mathfrak{g}(x)}, \ \mathfrak{g}(x) = x_0 \]

\[T_2 < \infty \quad \text{and} \quad x_0 = \mathfrak{g}(x_0), \ v = v_0 \text{ to } \infty \]

Case 2: \( x - x_0 = 0, \ \mathfrak{f}(x) = x_0, \ \mathfrak{h}(x) = x_0 \)

\[\frac{dt}{dx} = \frac{1}{\mathfrak{g}(x)}, \ \mathfrak{g}(x) = x_0 \]

\[T_2 \to \infty, x_0 \to \infty \quad \text{if} \quad \mathfrak{g}(x) > x_0 \text{ not maximal}
\]

\[x = x_0, \mathfrak{g}(x) = x_0, t = T_1 \quad \text{(not unique continues)}
\]

\[x = x_0, \mathfrak{g}(x) = x_0 \Rightarrow \mathfrak{f}(x) = \mathfrak{g}(x), \ T_2 = \infty
\]

Sup. \( w = \mathfrak{g}(z) \) is linear, \( A : z \to e^{inz}, \ b : z \to e^{inz} \)

\[\mathfrak{g}(z) = z, \mathfrak{f}(z) = z, \mathfrak{h}(z) = z, \ T_2 = \infty \]

are analytic in a simply connected domain \( \mathbb{C} \). Then \( \mathfrak{f} \in C(\mathbb{C}) \)

the IVP has a unique solution defined on all \( x \). The space

\[x > 0 \to \mathfrak{f}(x) > 0, \ (x, x_0) = (0, \infty), \ T_2 = \infty \]

will \( \mathfrak{g}(z) = z, \mathfrak{h}(z) = z, \ T_2 = \infty \)

\[\mathfrak{f}(z) = \mathfrak{g}(z), \mathfrak{f}(z) = \mathfrak{g}(z), \mathfrak{h}(z) = \mathfrak{g}(z), \mathfrak{h}(z) = \mathfrak{g}(z)
\]

\[x = x_0, \mathfrak{g}(x) = x_0 \Rightarrow \mathfrak{f}(x) = \mathfrak{g}(x), \ T_2 = \infty
\]

\[\mathfrak{g}(z) = z, \mathfrak{f}(z) = z, \mathfrak{h}(z) = z, \ T_2 = \infty
\]

\[\mathfrak{g}(z) = z, \mathfrak{f}(z) = z, \mathfrak{h}(z) = z, \ T_2 = \infty
\]

\[\mathfrak{g}(z) = z, \mathfrak{f}(z) = z, \mathfrak{h}(z) = z, \ T_2 = \infty
\]

Qualitative analysis:

\[\begin{cases}
E_{\lambda} : \lambda(t) = (1 - \lambda)x - \lambda x, \\
\lambda(t) = (1 - \lambda)x - \lambda x
\end{cases}
\]

\[E_{\lambda} : \lambda(t) = (1 - \lambda)x - \lambda x
\]

Vector fields: \( F_1(x, y) = -x, F_2(x, y) = -y \). The ODE \( x'(t) = F(x(t)) \)

\[\frac{dt}{dx} = \frac{1}{\mathfrak{g}(x)}, \ \mathfrak{g}(x) = x_0 \]

\[x(t) = \mathfrak{g}(x(t)), \ \mathfrak{g}(x(t)) = x_0 \]

say that the tangent vector to the curve at \( x(t) \) is \( F(x(t)) \),

\[\text{(i.e., } x(t) \text{ is tangent at each point of } F)\]
Laplace transform:

- \( \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st}f(t)\,dt \), \( s \in \mathbb{C} \), \( f \in L^1[0, \infty) \), analytic for \( Re(s) > a \).

- \( \mathcal{L}\{e^{at}f(t)\} = \mathcal{L}\{f(t)\} e^{-as} \)

- For \( x \in C^1([0, \infty)), x(0) + x(1) \leq \varepsilon \) we have \( \mathcal{L}\{x(t)\} \approx \mathcal{L}\{x(t) - x(0)\} \)

- For \( Re(s) > 0 \),

Dynamical Systems:

\[
\begin{align*}
& \dot{x} = f(x, t) \quad x \in \mathbb{R}^n, \quad t \in \mathbb{R}, \\
& f \in C^1(\mathbb{R}^n \times \mathbb{R}), \quad k \geq 1, \quad \text{finite valued}.
\end{align*}
\]

A solution \( x(t) \) is called a limit cycle, a periodic orbit, if there is a positive \( T \) such that \( x(t + T) = x(t) \).

- A periodic orbit \( x(t) \) is called a closed orbit, if \( x(t) = x(Tt) \) for all \( t \).

- If \( x(t) \) is a periodic orbit, then \( x(t) \) is complete if \( T \to \infty \).

- If \( x(t) \) is a periodic orbit, then \( x(t) \) is compact if \( T \to \infty \).

- If \( x(t) \) is a periodic orbit, then \( x(t) \) is closed if \( T \to \infty \).

- If \( x(t) \) is a periodic orbit, then \( x(t) \) is compact and complete if \( T \to \infty \).

- If \( x(t) \) is a periodic orbit, then \( x(t) \) is compact and closed if \( T \to \infty \).

- If \( x(t) \) is a periodic orbit, then \( x(t) \) is compact and closed if \( T \to \infty \).

- If \( x(t) \) is a periodic orbit, then \( x(t) \) is compact and closed if \( T \to \infty \).

- If \( x(t) \) is a periodic orbit, then \( x(t) \) is compact and closed if \( T \to \infty \).

- If \( x(t) \) is a periodic orbit, then \( x(t) \) is compact and closed if \( T \to \infty \).

- If \( x(t) \) is a periodic orbit, then \( x(t) \) is compact and closed if \( T \to \infty \).

Every \( \phi \) compact \( L^1 \)-valued set \( \subset C^1(\mathbb{R}, \mathbb{R}) \), contains a maximal compact subset.

The Straightening-out of vector fields: \( \phi = \phi(t, x) = x(0) + \int_0^t \frac{\partial}{\partial x} f(x, s)\,ds \), \( f \in C^1(\mathbb{R}^n \times \mathbb{R}) \), \( x = x(t) \) is a compact, \( L^1 \)-valued set.

Every \( \phi \) compact \( L^1 \)-valued set \( \subset C^1(\mathbb{R}, \mathbb{R}) \), contains a maximal compact subset.

Theorem: \( \phi = \phi(t, x) = x(0) + \int_0^t \frac{\partial}{\partial x} f(x, s)\,ds \), \( f \in C^1(\mathbb{R}^n \times \mathbb{R}) \), \( x = x(t) \) is a compact, \( L^1 \)-valued set.

Every \( \phi \) compact \( L^1 \)-valued set \( \subset C^1(\mathbb{R}, \mathbb{R}) \), contains a maximal compact subset.

Theorem: \( \phi = \phi(t, x) = x(0) + \int_0^t \frac{\partial}{\partial x} f(x, s)\,ds \), \( f \in C^1(\mathbb{R}^n \times \mathbb{R}) \), \( x = x(t) \) is a compact, \( L^1 \)-valued set.

Every \( \phi \) compact \( L^1 \)-valued set \( \subset C^1(\mathbb{R}, \mathbb{R}) \), contains a maximal compact subset.

Theorem: \( \phi = \phi(t, x) = x(0) + \int_0^t \frac{\partial}{\partial x} f(x, s)\,ds \), \( f \in C^1(\mathbb{R}^n \times \mathbb{R}) \), \( x = x(t) \) is a compact, \( L^1 \)-valued set.

Every \( \phi \) compact \( L^1 \)-valued set \( \subset C^1(\mathbb{R}, \mathbb{R}) \), contains a maximal compact subset.

Theorem: \( \phi = \phi(t, x) = x(0) + \int_0^t \frac{\partial}{\partial x} f(x, s)\,ds \), \( f \in C^1(\mathbb{R}^n \times \mathbb{R}) \), \( x = x(t) \) is a compact, \( L^1 \)-valued set.

Every \( \phi \) compact \( L^1 \)-valued set \( \subset C^1(\mathbb{R}, \mathbb{R}) \), contains a maximal compact subset.
Stability of fixed points.

- \( x_0 \) is Lyapunov-stable if \( \forall \epsilon > 0 \), \( \exists \delta > 0 \) such that if \( \| x(0) - x_0 \| < \delta \), then \( \| x(t) - x_0 \| < \epsilon \) for all \( t \geq 0 \).

- \( x_0 \) is asymptotically stable if it is stable and \( \lim_{t \to \infty} \| x(t) - x_0 \| = 0 \).

- \( x_0 \) is exponentially stable if \( \| x(t) \| \leq C e^{-\lambda t} \| x(0) \| \) for some \( C, \lambda > 0 \).

- Example: \( x_0 = 0 \), \( x' = x \). Then, \( x(t) = C e^t \). If \( C = 0 \), \( x_0 = 0 \) is marginal. If \( C \neq 0 \), \( x_0 \) is unstable.

Newton's equation (1.9): \( x' = F(x) \).

- Energy: \( E = \frac{1}{2} \| x \|^2 + V(x) \).

- \( \frac{dx}{dt} = 0 \) along solutions.

- Solutions with \( x(0) > 0, x(0) = 0 \), is given by \( x(t) = x_0 e^{-t} \), with \( E = \frac{1}{2} \| x \|^2 + V(x) \).

- Fixed points: \( x = 0 \), \( V(x) = 0 \). \( x = 0 \) is an equilibrium point.

- \( (4.1) \) has min. at \( x_0 \), \( E(x) \). A Lyapunov function, \( F(x) \), is stable.

- A fixed point cannot be exponentially stable.

Poincaré-Bendixson Th.

If \( \text{Exp. stab. via linearization) Sup. } \exists C, \text{for } t \to \infty, \text{as } \exists \text{fixed Jordan curve.} \)

- \( \mathbb{R}^2 \): Jordan curve, the every Jordan curve (homeomorphic image of \( S^1 \)) has \( \exists \mathbb{R} \). Then, \( x_0 \) is stable.

- \( \mathbb{R}^2 \) is connected.

Ex. \( f(x) = x \), \( f(t) = 0 \). Then \( x(t) = \frac{1}{2} t \), \( \frac{dx}{dt} \), \( \frac{d^2 x}{dt^2} \).

- Linearization fun.: \( \text{Let } f(x) = 0, \text{a neighborhood.} \)

**Lemma:** If \( f(x) > 0 \), \( f(x) > 0 \), \( f(x) = 0 \), \( f(x) < 0 \), then \( f(x) \) is a connected function.

**Lemma:** \( f(x) = 0 \), \( f(x) > 0 \), \( f(x) = 0 \), \( f(x) < 0 \), ordered by \( t \). Then, \( x(t) = x(t) \).

- Poincaré-Bendixson Th.: If \( \exists x \), \( \exists x \). \( \exists x \).

- A minimal compact, connected set is a periodic orbit.

Rek: If \( A \) is affine, is enough to prove

\[ \frac{d}{dt} (\phi(t)) = \frac{d}{dt} (\phi(t)) = \frac{d}{dt} (\phi(t)) = \frac{d}{dt} (\phi(t)) \]

- Bounded: \( \exists \phi(t) \), \( \forall t \), \( \forall \theta \).

- Generalized: \( \exists \phi(t) \), \( \forall t \), \( \forall \theta \).

- Case 1: \( \phi(t) \) is a fixed orbit on.

- Case 2: \( \phi(t) \) is a regular periodic orbit on.

- Case 3: \( \phi(t) \) consists of (finite many) fixed points \( \phi(t) \) and non-
  - closed orbits. \( \phi(t) \), \( \forall \theta \).
More on Newton's eq:

Writing \( p = \dot{x}, q = x, H(p, q) = \frac{p^2}{2} + U(q) \) is called Hamiltonian.

and the eq. are:
\[
\frac{\dot{q}}{\dot{p}} = \frac{\partial H}{\partial p}, \quad \frac{\dot{p}}{\dot{q}} = -\frac{\partial H}{\partial q}.
\]

Example: (pendulum) \( \ddot{x} = -\sin x \) (x displacement from \( x = 0 \)).

\( U(x) = 1 - \cos x \). Consider \( x \in (-\pi, \pi) \); \( x = 0 \) is a fixed point.

\( x = 0 \) is stable (\( U \) has min).

\( E(x) = \frac{\dot{x}^2}{2} + U(x) \); level sets of \( E(x) = ct \) are invariant.

i) \( E = 0 \): \( (x, \dot{x}) = (0, 0) \)

ii) \( 0 < E < 2 \): \( E = ct \) is homeomorphic to a circle; no fixed pt. outside \( \Omega \).

a) \( E = 2 \): fixed point \( x = \pm x \); level set has two orbits between \( -\pi, 0 \).

b) \( E > 2 \): closed orbits

\( x = 0, \dot{x} = x, E = \frac{\dot{x}^2}{2} + \frac{x^2}{2} \)

Lepsius's hint: ex:

i) \( s(t) \), any \( s(t) \) with bounded derivative.

ii) \( x^2 \) is loc. (not global) lip.

iii) \( \dot{x} \) is not zero.

iv) \( x^{1/2} \) on \( [0, 1] \) is diff. not lip.

v) An everywhere diff. \( s(t) \) is Lipschitz if \( \| \dot{s} \| \leq M \).