CLASSIFYING $VII_0$ SURFACES WITH $b_2 = 0$ VIA GROUP THEORY

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Abstract. We give a new proof of Bogomolov’s theorem, that the only $VII_0$ surfaces with $b_2 = 0$ are those constructed by Hopf and Inoue. The proof follows the strategy of the original one, but it is of purely group-theoretic nature.

Our aim is to provide a new proof of the well known:

Theorem 1. [Bo76, Bogomolov] Every $VII_0$ surface with $b_2 = 0$ carries a holomorphic foliation, and therefore it is either a Hopf, or a Inoue surface.

The first part of the proof in op.cit., which is the content of our Section I, aims at constructing an affine structure on such surfaces, as well as establishing its uniqueness. Both the existence and uniqueness statements follow from the vanishing of certain cohomology groups, which could be naturally though of as obstruction spaces. This structure provides a representation of the fundamental group of the surface into $\text{GL}_2(\mathbb{C})$. On the other hand, the automorphism group $\text{Gal}(\mathbb{C}/\mathbb{Q})$ acts naturally on the set of flat vector bundles, and it can be shown that indeed the tangent bundle is stable under this action. This follows from the fact that these surfaces have very few vector bundles with at least one non-vanishing cohomology group, and that cohomology with locally constant coefficients is not altered by the Galois action. It follows that the Galois acts, on the linear representation arising from the affine structure, by conjugation. This observation imposes very strong arithmetic restrictions on such representation. We can state:

Proposition 1. Let $X$ be a $VII_0$ surface with $b_2 = 0$ and no holomorphic foliation. Then $X$ carries an affine structure, defined by a representation $\rho : \pi_1(X) \to \text{Aff}(2, \mathbb{C})$. Furthermore, $\text{Gal}(\mathbb{C}/\mathbb{Q})$ acts on its linearization $l\rho : \pi_1(X) \to \text{GL}_2(\mathbb{C})$ by conjugation. In particular, the image of $l\rho$ can be conjugated in $\text{GL}_2(\mathbb{Q})$, or in the units $H_2(F)$ of a quaternion algebra over $F$, a real quadratic extension of $\mathbb{Q}$. 
After this was achieved, the original strategy, first sketched in [Bo76] and then expanded on in [Bo82], aimed at deriving a contradiction profiting from the strong arithmetic constraints we have on the representation. Unfortunately this derivation is rather long, and contains complicated topological arguments. Our contribution is to replace it by a simpler group-theoretic proof. In section II we analyze the geometry behind the affine structure just constructed. This is all very elementary and explicit. A key role is played by the existence of a real subspace $R \subset \mathbb{C}^2$, which is invariant under the affine transformations in the image of $\rho$. Its existence is a consequence of the results of Section I, specifically that the tangent bundle of $X$ has one-dimensional cohomology in degree 1. Such $R$ provides us with a contractible, two-dimensional real Lie group of affine transformations that fix $R$ pointwise, and so that the quotient of $\mathbb{C}^2 \setminus R$ by its action, is a Moebius band. The Lie group’s action lifts, via the developing map attached to the affine structure, to the universal cover of $X$, and it commutes with that of the fundamental group. It follows that the quotient of an open dense subset of the universal cover, modulo our Lie group, is acted upon by the fundamental group of $X$. This quotient is an open differentiable surface, with an unramified map to the above-mentioned Moebius band. The existence of this surface is of crucial importance, since we can now use the geometry of Riemann surfaces to discover strong properties of the fundamental group of $X$. Specifically, it turns out that

Proposition 2. The kernel of the natural projection $\pi_1(X) \to \text{SL}_2(\mathbb{R})$ is free, unless the developing map of the affine structure $X \to \mathbb{C}^2$ is a, possibly infinite, cyclic covering over its image.

This is the content of Corollary II.9 which concludes Section II.

In section III we complete the proof of Theorem I. We start by excluding the possibility that the linearized representation lands in $\text{SL}_2(\mathbb{Z})$ or $H_2(\partial F)$. Indeed, such groups are, up to finite index, fundamental groups of hyperbolic Riemann surfaces, and this would provide $X$ with a profusion of holomorphic one-forms, which is impossible. The very same idea is not so easy to carry over when denominators appear in the representation, even though they provide us with a decomposition of the fundamental group into an amalgamated product of free groups. It is probably true that factorization theorems, in the spirit of [GS92], still hold, in a suitable form, for non-Kahler surfaces. However, we preferred to gather informations on the fundamental groups of these surfaces, by more elementary methods. The linear space $R$, whose existence is briefly discussed above, might be responsible for
the existence of some real 2-dimensional tori or Klein bottles in $X$. Those have the special property that their complement, in $X$, is an Eilenberg-Maclane space. If such submanifolds are not present, our informations on the fundamental group are enough to deduce that its cohomological dimension cannot exceed 3, and hence derive a contradiction. This is done in Theorem III.4. If such submanifolds are present, the same method might not work. However, a careful analysis of the interplay between our affine structure and those submanifolds, reveals a strong rigidity of our developing map $\tilde{X} \to \mathbb{C}^2$ - it must be an embedding. From here it is easy to conclude, and this is the content of Theorem III.5. We remark that our arguments provide the existence of a holomorphic foliation on $X$. That $X$ is either a Hopf or a Inoue surface, is a theorem of Inoue, [Io74].

It is interesting to remark that the key player, in the story, is our two dimensional Lie group acting on $\tilde{X}$. In fact, its orbits are holomorphic curves, but the resulting foliation is not holomorphic.

We remark that an analytic approach to this theorem appeared in [LYZ90], and was further completed in [Te94] and [LYZ94].

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I. Affine structures on VII$_0$ surfaces with $b_2 = 0$ and no holomorphic foliations

In this section we prove most of Proposition 1, namely all the statements but the fact that $F$ is real. We will be able to exclude imaginary quadratic fields in Section II. We recall that a VII$_0$ surface is a compact, complex surface with $H^0(X, K_X^n) = 0$ for every $n \in \mathbb{Z}$, $H^0(X, \Omega^1_X) = 0$, and $H^1(X, \mathcal{C})$ is 1-dimensional. Among those with $b_2 = 0$, examples include Hopf and Inoue surfaces. For a description of these examples, and more informations about such surfaces, we refer to the treatment in [Bo76] and [Io74]. In this section, we follow the original scheme of [Bo76] rather closely.

From now on, $X$ will be a VII$_0$ surface with $b_2 = 0$ and no holomorphic foliations, i.e. $H^0(X, \mathcal{L} \otimes \Omega^1) = 0$ for every rank 1 coherent sheaf $\mathcal{L}$. 
Proposition I.1. The tangent bundle of $X$ admits a unique flat, torsion-free connection.

Proof. Let us consider the obstructions to the existence of such an object:

- The obstruction to the existence of a connection is in $H^1(\text{End}(TX) \otimes \Omega^1)$
- The obstruction to the flatness of a connection is in $H^0(K_X \otimes \text{End}(TX))$
- The obstruction to the uniqueness of a connection is in $H^0(\text{End}(TX) \otimes \Omega^1)$

We shall prove that all these cohomology groups vanish.

Lemma I.2. For every $n \in \mathbb{Z}$, $n \neq 0$, we have $H^0(X, K^n_X \otimes \text{End}(TX)) = 0$.

Proof. If $s$ is a section, it defines a morphism $s : TX \to TX \otimes K^n_X$. Its determinant is a section of $K_X^{4n}$, which must be zero. Hence, if $s$ is non-trivial, its kernel is generically of rank 1 and defines a holomorphic foliation, which is absurd. □

Lemma I.3. $H^0(\Omega^1 \otimes \text{End}(TX)) = H^0(K_X \otimes TX \otimes \text{End}(TX)) = 0$

Proof. Let us prove the vanishing of the first group. Let $s : TX \to \text{End}(TX)$ be a section. Composing with the natural trace map $\text{tr} : \text{End}(TX) \to \mathcal{O}_X$, we obtain a morphism $\text{tr} \circ s : TX \to \mathcal{O}_X$, which must be trivial since otherwise its kernel would define a foliation. Hence each endomorphism in the image of $s$ can be diagonalized. Next, consider the composition of $s \wedge s : K^{-1}_X = TX \wedge TX \to \text{End}(TX) \wedge \text{End}(TX)$ with the commutator $c : \text{End}(TX) \wedge \text{End}(TX) \to \text{End}(TX)$. By Lemma I.2, it must be zero. It follows that the image of $s$ generates a commutative sub-algebra of $\text{End}(TX)$, and hence all its elements are simultaneously diagonalizable with one-dimensional eigenspaces. But then, maybe after an étale double cover of $X$, the tangent bundle of $X$ splits, thus deriving a contradiction.

The second group vanishes for the exact same reason. □

At this point, we proved the vanishing of the last two obstructions mentioned above. On the other hand, since $\chi(X) = 0$, if $h^1(\text{End}(TX) \otimes \Omega^1) \neq 0$ then by Riemann-Roch and Serre duality, at least one among $h^0(\text{End}(TX) \otimes \Omega^1)$ and $h^0(K_X \otimes \text{End}(TX) \otimes TX)$ is non-zero, which is impossible. □

The connection just constructed defines an affine structure on $X$, meaning a morphism $\hat{X} \to \mathbb{C}^2$, which is equivariant for a representation $\rho : \Gamma := \pi_1(X) \to \text{Aff}(2, \mathbb{C})$, with corresponding linearization $l_\rho : \Gamma \to \text{GL}_2(\mathbb{C})$. We will extract informations by studying the orbit of $l_\rho$ under the natural action of $\text{Gal}(\mathbb{C}/\mathbb{Q})$. By way of notation, if a bundle $E$
is defined by a representation $\eta : \Gamma \to \text{GL}_n(\mathbb{C})$, we will denote by $E^\sigma$ the one defined by $\sigma \circ \eta$, for $\sigma \in \text{Gal}(\mathbb{C}/\mathbb{Q})$. As a warm-up we have:

**Proposition I.4** (Bombieri). $K^\sigma_X \simto K_X$ for every $\sigma$.

*Proof.* We will be using the exact sequences

(1) $0 \to C(\mathcal{F}) \to \mathcal{O}(\mathcal{F}) \to d\mathcal{O}(\mathcal{F}) \to 0$

(2) $0 \to d\mathcal{O}(\mathcal{F}) \to \Omega^1(\mathcal{F}) \to K_X \otimes \mathcal{F} \to 0$

**Lemma I.5.** If $\mathcal{F} \neq \mathcal{O}_X$, $K_X$ is of rank 1, then $h^i(X, \mathcal{O}(\mathcal{F})) = 0$ for every $i$.

*Proof.* This is the absence of compact curves, and Riemann-Roch. □

**Lemma I.6.** If $\mathcal{F}$ has rank 1, then $h^0(d\mathcal{O}(\mathcal{F})) = 0$.

*Proof.* This follows by taking global sections in the sequence (2). □

**Lemma I.7.** If $\mathcal{F} \neq \mathcal{O}_X$, $K_X^{-1}$ has rank 1 then $h^i(d\mathcal{O}(\mathcal{F})) = 0$ for every $i$.

*Proof.* We can apply sequence (2) together with Lemma I.5 for $K_X \otimes \mathcal{F}$, to deduce $h^i(d\mathcal{O}(\mathcal{F})) = h^i(\Omega^1(\mathcal{F}))$. If, for some $i$ we have $h^i(\Omega^1(\mathcal{F})) \neq 0$, then by Riemann-Roch it holds for $i = 0$ or $i = 2$. But then we get a foliation, which is absurd. □

**Lemma I.8.** $h^1(C(K_X)) = 1$

*Proof.* By Lemma I.7 we know $h^i(d\mathcal{O}(\mathcal{F})) = 0$ for all $i$. By the sequence (1) $h^1(C(K_X)) = h^1(\mathcal{O}(K_X)) = 1$. □

The above lemmas now merge to conclude the proof: if $K^\sigma_X \neq K_X$, by Lemma I.5 $h^i(\mathcal{O}(K^\sigma_X)) = 0$. Then by the sequence (1) we get

$h^0(d\mathcal{O}(K^\sigma_X)) = h^1(C(K^\sigma_X)) = h^1(C(K_X)) = 1$

which is absurd by lemma I.6. □

Now we turn our attention to the Galois action on $TX$. As one imagines,

**Proposition I.9.** $TX^\sigma \simto TX$. In particular, $\text{Gal}(\mathbb{C}/\mathbb{Q})$ acts on $l_p$ by conjugation.
Proof. Let us start by observing that, for any locally free sheaf \( \mathcal{F} \), the existence of homotheties gives \( h^0(\mathcal{F} \otimes \mathcal{F}^\vee) = h^0(\text{End}(\mathcal{F})) \geq 1 \). We have, again, a series of lemmas.

**Lemma I.10.** \( h^1(\mathcal{C}(TX)) = h^0(\mathcal{O}(TX)) = 1 \)

Proof. The first equality follows from sequence 1 and the vanishing of \( h^i(\mathcal{O}(TX)) \) due to the absence of foliations and Riemann-Roch. Observe that, since \( TX \) is simple, \( h^0(\text{End}(TX)) = 1 \), and since \( h^0(K_X \otimes TX) = 0 \), we conclude via sequence 2. Therefore: \( h^1(\mathcal{C}(TX^\sigma)) = 1 \) for every \( \sigma \in \text{Gal}(\mathbb{C}/\mathbb{Q}) \). By the sequence 1, at least one of the following happens:

- \( h^0(\mathcal{O}(TX^\sigma)) \geq 1 \)
- \( h^1(\mathcal{O}(TX^\sigma)) \geq 1 \)

The first option implies that, by the sequence 2, \( h^0(TX^\sigma \otimes \mathcal{O}) \geq 1 \) so then there exists a morphism \( TX \to TX^\sigma \) which must be an isomorphism.

The second option, along with Riemann-Roch, leads to a further alternative:

- \( h^0(\mathcal{O}(TX^\sigma)) \geq 1 \)
- \( h^0(\mathcal{C}(TX^\sigma)) = h^0(\mathcal{C}(K_X \otimes (\Omega^1)^\sigma)) \geq 1 \)

In the first case, since \( h^0(\mathcal{C}(TX^\sigma)) = 0 \), the sequence 1 implies \( h^0(\mathcal{O}(TX^\sigma)) \geq 1 \), while the sequence 2 gives an isomorphism \( TX \to TX^\sigma \).

In the second case, by the sequence 1 and the identity \( (TX^\sigma)^\vee = (\Omega^1)^\sigma \), we have one more alternative:

- \( h^0(\mathcal{C}(K_X \otimes (\Omega^1)^\sigma)) \geq 1 \)
- \( h^0(\mathcal{O}(K_X \otimes (\Omega^1)^\sigma)) \geq 1 \)

In the first case Proposition I.4 gives \( h^0(\mathcal{C}(K_X \otimes (\Omega^1)^\sigma)) = h^0(\mathcal{C}(K_X^\sigma \otimes \Omega^1)) = h^0(\mathcal{C}(K_X^\sigma \otimes \Omega^1)) \geq 1 \) and hence a foliation.

In the second case, the sequence 2 gives \( h^0(K_X \otimes \mathcal{O}^1 \otimes (\omega_1)^\sigma) \geq 1 \), which defines an isomorphism \( TX \xrightarrow{\sim} K_X \otimes (\Omega^1)^\sigma \). Taking determinants, however, gives \( K_X^\sigma \xrightarrow{\sim} K_X \otimes K_X^\sigma \xrightarrow{\sim} \mathcal{O} \), which is again absurd.

This proves that \( TX \xrightarrow{\sim} TX^\sigma \) for every \( \sigma \). Since \( TX \) is built from \( \rho \), the representations \( l\rho^\sigma \) must be conjugated to each other. \( \square \)

We wish to employ Proposition I.9 to derive concrete arithmetic restrictions on \( l\rho \). As such we start by computing, in full generality, what it means, for a representation, to be acted upon by Galois via conjugation:
Proposition I.11. Let \( \eta : \Gamma \to \text{GL}_2(\mathbb{C}) \) be a representation, such that \( \eta^\sigma \) is conjugated to \( \eta \) for every \( \sigma \in \text{Gal}(\mathbb{C}/\mathbb{Q}) \). Then the image of \( \eta \) is either conjugated to a subgroup of \( \text{GL}_2(\mathbb{Q}) \), or to a subgroup of the units in a quaternion algebra \( H_2(\mathbb{Q}(\sqrt{d})) \).

Proof. The strategy is to compute the Galois action on a matrix in its Jordan form, and see what happens. This is possible, since our condition on the action of Galois is invariant under conjugation.

Lemma I.12. Either the image of \( \eta \) can be conjugated into \( \text{GL}_2(\mathbb{Q}) \), or the image of \( \eta \) contains a diagonalizable matrix which is not a multiple of the identity.

Proof. Assume that every matrix in the image of \( \eta \), has two coincident eigenvalues. It is easy to see, for example by looking at the trace and determinant of the product of two distinct such matrices, that there exists a basis such that every element in the image of \( \eta \) is in its Jordan form. Let us represent such a matrix, in its Jordan form, as:

\[
M = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}
\]

where \( a, b \neq 0 \). Let \( \sigma \) act, in this basis, via a matrix \( \Lambda = (\lambda_{ij}) \). Comparing the coefficients in the identity

\[
\Lambda M^\sigma = M \Lambda
\]

we get

\[
a^\sigma = a; \quad \lambda_{21} = 0; \quad b^\sigma = \lambda_{22} \lambda_{11}^{-1} b
\]

It follows that, for any two matrices \( M, M' \), we have \( b/b' \in \mathbb{Q} \). In particular, after conjugating \( \eta \) with

\[
\begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix}
\]

the image is in \( \text{GL}_2(\mathbb{Q}) \).

We can now analyze what happens to a diagonal matrix which is not a multiple of the identity:

Lemma I.13. Let \( M \), belonging to the image of \( \eta \), have eigenvalues \( x_1 \neq x_2 \). Let \( \sigma \) act, in a basis where \( M \) is diagonal, via a matrix \( \Lambda \). Then we have two cases:

- \( M \) is invariant under \( \sigma \), and \( \Lambda \) is diagonal, or
• $M$ is not invariant, and $\Lambda$ is anti-diagonal, so then $\sigma^2 = 1$.

In particular, the Galois stabilizer of $M$ is a subgroup of a torus $\mathbb{C}^* \times \mathbb{C}^*$, while the orbit of $M$ has two elements.

Proof. As before, we can analyze the coefficients in the identity $3$. □

We proceed, therefore, to analyze the action of Galois on the whole $\Gamma$.

**Lemma I.14.** Assume the image of $\eta$ cannot be conjugated into a subgroup of $\text{GL}_2(\mathbb{Q})$. Then there exists a quadratic extension $F$ of $\mathbb{Q}$, such that the image of $\eta$ lies in $\text{GL}_2(F) \cap H_2(F)$, where $H_2(F) \subset M_2(F)$ is a quaternion algebra over $F$.

Proof. With the notation of Lemma I.13 we have two cases:

• $\sigma$ is in the stabilizer of $M$, and
• $\sigma$ is not.

Let $N$ be an element in the image of $\eta$, represented in the basis where $M$ is diagonal by a matrix

$$N = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and we consider the equation

$$\Lambda N^\sigma = N \Lambda$$

Assume $\sigma$ stabilizes $M$, then $\Lambda$ is diagonal and we have

$$a^\sigma = a; d^\sigma = d; \lambda_1 b^\sigma = \lambda_2 b; \lambda_2 c^\sigma = \lambda_1 c$$

In particular, for any other

$$N' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$$

in the image of $\eta$, both $b/b'$ and $c/c'$ are $\sigma$-invariant. As such, after composing $\eta$ with conjugation by

$$\begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix}$$

the action of $\text{Stab}(M) < \text{Gal}(\mathbb{C}/\mathbb{Q})$ is trivial on $\eta$.

In case $\sigma$ does not stabilize $M$, then $\Lambda$ is anti-diagonal and

$$\sigma^2 = 1; a = d^\sigma; d = a^\sigma; \lambda_2 b^\sigma = \lambda_1 c; \lambda_1 c^\sigma = \lambda_2 b$$
Observe that the last two identities imply \( q := bc \) is \( \sigma \)-invariant. Let \( r := \lambda_{21}/\lambda_{12} \). We have

\[
(8) \quad c = c^{\sigma^2} = r^{\sigma}r^{-1}c
\]

which implies \( r^\sigma = r \), and \( c = rb^\sigma \). It follows that the image of \( \eta \) is defined over a quadratic extension \( F \) of \( \mathbb{Q} \), with Galois group generated by \( \sigma \), and that its image, consisting of matrices of the form

\[
\begin{pmatrix}
a & b \\
rb^\sigma & a^\sigma
\end{pmatrix}
\]

lies in a quaternion algebra \( H_2(F) \).

This concludes the proof of Proposition I.11. \( \square \)

By Proposition I.11, our linearization \( l\rho \) has values in \( \text{GL}_2(\mathbb{Q}) \) or in the units of a quaternion algebra \( H_2(F) \) over a quadratic extension \( F \) of \( \mathbb{Q} \). We can now conclude the proof of Proposition 1. Indeed we are left with the possibility that \( F \) is an imaginary quadratic field, which we proceed now to exclude.

**Lemma I.15.** \( l\rho \) has values in \( \text{GL}_2(\mathbb{Q}) \) or in the units of a quaternion algebra \( H_2(F) \) over a real quadratic extension \( F \) of \( \mathbb{Q} \).

**Proof.** Consider the composite \( \Gamma \to \text{GL}_4(\mathbb{R}) \), of \( l\rho \) followed by the canonical inclusion \( \text{GL}_2(\mathbb{C}) \to \text{GL}_4(\mathbb{R}) \). We keep denoting its image by \( G \), and we let \( \mathbb{R}[G] \subset M_4(\mathbb{R}) \) be the real sub-algebra it generates. Quite generally, for a \( \mathbb{R} \)-algebra \( R \) and a \( \mathbb{R} \)-module \( M \), we have maps

\[
(9) \quad R \to \text{End}_R(M) = C_{\text{End}_R(M)}(R) \subset \text{End}_\mathbb{R}(M)
\]

where \( C \) means centralizer. On the other hand, the centralizer of \( \mathbb{R}[G] \subset M_4(\mathbb{R}) \) acts on \( H^1(\Gamma, \mathbb{C}^2_{\rho}) \) non-trivially by definition, and therefore \( \mathbb{R}[G] \) itself acts on \( H^1(\Gamma, \mathbb{C}^2_{\rho}) \) non-trivially. For a quaternion algebra over a quadratic field \( F \), we have \( H_2(F) \otimes_\mathbb{Q} \mathbb{R} = M_2(\mathbb{R}) \) iff \( F \) is real, while \( H_2(F) \otimes_\mathbb{Q} \mathbb{R} = H_4(\mathbb{R}) \) iff \( F \) is imaginary. Since the real quaternions \( H_4(\mathbb{R}) \) have no non-trivial 2-dimensional representation, we deduce that \( F \) cannot be imaginary.

This concludes the proof of Proposition I.11. \( \square \)
II. Geometry of affine structures

In order to reach a contradiction using our affine structure, it will be important to study in detail the geometry that governs it. This is the aim of the present section. A substantial part of the presentation is taken from [Bo82, chapter 3]. Let us start by setting up the notation. Recall that the affine structure on $X$ is defined by a morphism $V := \tilde{X} \to \mathbb{C}^2$, equivariant with respect to a representation $\rho : \Gamma \to \text{Aff}(2, \mathbb{C})$. This representation is uniquely defined by its linearization $l\rho$, whose image we denote by $G := \text{Im}(l\rho)$, along with a cocycle $\omega \in H^1(\Gamma, \mathbb{C}^2 l\rho)$. We already know the structure of $l\rho$, but something more can be said about $\rho$:

**Lemma II.1.** There exists a real 2-dimensional subspace $R \subset \mathbb{C}^2$, invariant under $\rho(\Gamma)$.

*Proof.* There is a natural embedding $e : \text{Aff}(2, \mathbb{C}) \to \text{Aff}(4, \mathbb{R})$ obtained by considering $\mathbb{C}$ as a real vector space. The composite $e \circ l\rho$ is a reducible real representation, which splits as a direct sum $\rho_1 \oplus \rho_2$ of isomorphic representations $\rho_i : \Gamma \to \text{GL}_2(\mathbb{R})$. A trivial inspection reveals that the centralizer of $e(\text{GL}_2(\mathbb{R}))$ is naturally isomorphic to $\text{GL}_2(\mathbb{R})$, and will be referred to as $\text{GL}_2(\mathbb{R})_K$ in the sequel. Indeed, we can represent the elements of $e(\text{GL}_2(\mathbb{R}))$, in some basis, as matrices

$$
\begin{pmatrix}
aI & bI \\
cI & dI
\end{pmatrix}
$$

where $I$ is the $2 \times 2$ identity, and $a, b, c, d \in \mathbb{R}$, while those elements of $\text{GL}_2(\mathbb{R})_K$ can be written, in the same basis, as

$$
\begin{pmatrix}
a & 0 \\
0 & a
\end{pmatrix}
$$

with $A \in \text{GL}_2(\mathbb{R})$. The intersection $\text{GL}_2(\mathbb{R})_K \cap e(\text{GL}_2(\mathbb{C}))$ is the center of $\text{GL}_2(\mathbb{C})$. $\text{GL}_2(\mathbb{R})_K$ acts transitively on the set of splittings of $\mathbb{R}^4$ into a direct sum of two 2-dimensional subspaces. If we fix an $e(\text{GL}_2(\mathbb{R}))$-invariant splitting $\mathbb{R}^4 = V_1 \oplus V_2$ then we have a real vector spaces decomposition

$$
H^1(\Gamma, \mathbb{C}^2 l\rho) \xrightarrow{\sim} H^1(\Gamma, V_1) \oplus H^1(\Gamma, V_2)
$$

where, by Lemma I.10, each space on the right is real 1-dimensional; correspondingly we obtain that the cocycle $\omega$, defining $\rho$, splits in a pair $\omega_1 \oplus \omega_2$ of real-valued cocycles. The center of $\text{GL}_2(\mathbb{C})$ acts by scalar multiplication on $H^1(\Gamma, \mathbb{C}^2 l\rho)$, and hence the whole $\text{GL}_2(\mathbb{R})_K$ acts transitively on $H^1(\Gamma, V_1) \oplus H^1(\Gamma, V_2) \setminus \{(0,0)\}$. In particular, for some
splitting of $\mathbb{R}^4$ we have $\omega_1 = 1$ and $\omega_2 = 0$, so then the action of $\Gamma$ on $\mathbb{R}^4/V_1$ is linear. The existence of $R$ follows.

We continue by inspecting those elements commuting with the image of $\rho$, in the whole affine group. From the existence of $R$ it follows that the image of $\rho$ lies in $\text{Aff}(2, \mathbb{R})$. We denote by $e : \text{Aff}(2, \mathbb{C}) \to \text{Aff}(4, \mathbb{R})$, and by $G_K$ the centralizer of $e(\text{Aff}(2, \mathbb{R}))$.

**Lemma II.2.** $G_K$ is the subgroup of $\text{GL}_2(\mathbb{R})_K$ acting trivially on $R$. It is isomorphic to $\text{Aff}(1, \mathbb{R})$.

**Proof.** $G_K$ acts on $R$, and since $\text{Aff}(2, \mathbb{R})$ has no center, this action must be trivial. This implies that $G_K < \text{GL}_2(\mathbb{R})_K$. Moreover, any $g \in \text{GL}_2(\mathbb{R})_K$ commutes with the linear part of any $h \in e(\text{Aff}(2, \mathbb{R}))$ by definition. Hence $t := ghg^{-1}h^{-1}$ is a translation. If moreover $g$ acts trivially on $R$, also $t$ does, so then $t = 0$ and $g \in G_K$. That $G_K$ is isomorphic to $\text{Aff}(1, \mathbb{R})$ is clear at this point, since it consists of matrices that can be written, in the above basis, as

$$\begin{pmatrix} 1 & a \\ 0 & b \end{pmatrix}$$

and of course $R$ is spanned by $(1, 0, 0, 0)$ and $(0, 0, 1, 0)$.

We switch our attention to the action of $G_K$ on $\mathbb{C}^2$. We fix a splitting $\mathbb{C}^2 = R \oplus iR$. For a point $x \in \mathbb{C}^2 \setminus R$, there exists a unique line $L_x$, which is a translation of a line contained in $iR$, it contains $x$ and intersects $R$. Such line has a complexification $L^C_x$, which intersects $R$ along a line $R_x$.

**Lemma II.3.** The orbit of $x$ under $G_K$ is $L^C_x \setminus R_x$.

**Proof.** This is particularly clear if we use the matrix representation of $G_K$ given above.

The previous lemma can be rephrased by saying that the quotient space of $\mathbb{C}^2 \setminus R$ under the action of $G_K$ is canonically the Grassmannian manifold, $G(1, 2)$, of affine lines in $R$. There is a fiber bundle structure $s : G(1, 2) \to S^1$, where each fiber parametrizes a family of parallel lines in $R$. $G(1, 2)$ is non-orientable, hence homeomorphic to a Moebius band. The quotient of $\mathbb{C}^2 \setminus R$ by $G_K^+$, a connected component of $G_K$, is a cylinder mapping to $G(1, 2)$ as its canonical orientable double covering. Trying to lift this setup on $V$ under
the unramified developing map, introduced at the beginning of this section, \( V \to \mathbb{C}^2 \), it is natural to introduce the closed submanifold \( V_R := V \times_{\mathbb{C}^2} R \). A crucial observation is that, since \( \rho(\Gamma) \) and \( G_K \) commute, the action of \( \rho(\Gamma) \) descends naturally to the orbit space \( G(1, 2) \). Moreover, since \( G_2^+ \) is connected and simply connected, it action lifts to \( V \setminus V_R \), and commutes with \( \Gamma \). We obtain a commutative diagram

\[
\begin{array}{ccc}
V \setminus V_R & \xrightarrow{G_K^+} & S \\
\downarrow & & \downarrow \\
\mathbb{C}^2 \setminus R & \xrightarrow{G_K^+} & G(1, 2)
\end{array}
\]

where \( S \) is the leaf space for the induced foliation on \( V \). The projection \( S \to G(1, 2) \) is an unramified map, and \( S \) is an open real surface. Denoting by \( X_R \) the compact, real surface which is the image of \( V_R \) under the covering map \( V \to X \), we deduce that \( X \setminus X_R \) is an Eilenberg-MacLane space, and the fibration sequence yields that \( S \) is simply connected.

Recall that we defined \( G \) to be the image of \( l\rho \). We have a commutative diagram of groups with exact rows and columns:
where of course $A$ is the subgroup of translations inside $G_A$, which the image of $\rho$. The previous diagram of continuous maps can be slightly enlarged:

\[
\begin{array}{cccc}
V \setminus V_R & \xrightarrow{G_K^+} & S & \longrightarrow S/K \\
\downarrow & & \downarrow & q \\
\mathbb{C}^2 \setminus R & \xrightarrow{G_K^+} & G(1,2) & \longrightarrow G(1,2)
\end{array}
\]

Observe that $K$ necessarily acts discontinuously on $S$. It is therefore the fundamental group of $S/K$, which being an open surface implies that $K$ is a free group.

Let us denote by $G'$ the image of $G$ under the natural projection $GL_2(\mathbb{R}) \to SL_2(\mathbb{R})$. The following sequence defines a subgroup $K_{Ab} < \Gamma$:

\[
(11) \quad 1 \to K_{Ab} \to \Gamma \to G' \to 1
\]

i.e. $K_{Ab}$ is the extension of $K_A$ by the cyclic group generated by homotheties. A choice of a generating homothety yields a semi-direct product decomposition

\[
(12) \quad K_{Ab} \cong K_A \rtimes H
\]

We proceed to analyze closely our surface $S$ and its map to $G(1,2)$. Denote by $s : \tilde{G}(1,2) \to \mathbb{R}$ the map induced on universal coverings. The main technical result is:

**Theorem II.4.** Assume there exists a connected open subset $U \subset S$ mapping homeomorphically onto $s^*I \subset \tilde{G}(1,2)$, for some open interval $I \subset S^1$. Then $S \to \tilde{G}(1,2)$ is a homeomorphism.

**Proof.** First, denote by $\Delta$ the stabilizer of $U$ inside $\rho(\Gamma)$. Observe that if $I = \mathbb{R}$ then $U$ is closed and hence equal to $S$. Thus we can assume there exists a point $x \in \bar{I} \setminus I$. Denote by $U_x$ the boundary of $U$ over $x$, i.e. the closed subset of $S$ mapping to $x$ and in the closure of $U$. It consists of a disjoint union of open segments, and it is $\Delta$-invariant. Moreover, being a boundary of an open set, its pre-image $G_K^+ \times U_x \subset V$ maps to a compact 3-manifold $T_x \subset X$. Since $U \to \tilde{G}(1,2)$ is an embedding and $S/K$ is Hausdorff, $U_x \to \tilde{G}(1,2)$ is an embedding as well. There is an exact sequence

\[
(13) \quad 1 \to \Delta \cap \mathbb{R}^2 \to \Delta \to \Delta_t \to 1
\]

where $\mathbb{R}^2$ denotes the subgroup of translations, and $\Delta_t$ the image in $GL_2(\mathbb{R})$. If we pick a basis of $\mathbb{R}^2$ such that the image of $U_x \to G(1,2)$ parametrizes lines parallel to the vector
Then $\Delta$ is a group of lower triangular matrices.

**Lemma II.5.** The intersection $\Delta \cap \mathbb{R}^2$ is either trivial or cyclic.

**Proof.** If rank $\Delta \cap \mathbb{R}^2 \geq 2$, then there is at least one translation, $t$, that acts non-trivially on $U_x$. But then necessarily $U_x = \mathbb{R}$, and it maps homeomorphically onto the corresponding line in $G(1, 2)$. It follows that a neighborhood of $U_x$, acted upon by $t$, is contained in $U$, so then $x \in I$, a contradiction. \qed

At this point we distinguish two cases:
- $\Delta$ contains no non-trivial translation;
- The translation subgroup $\Delta$ is generated by $\tau = (0, 1)$.

**Step I:** We deal with the first case, that $\Delta$ contains no translations.

Certainly, the above sequence (13) gives

\[
1 \to U \cap \Delta = [\Delta, \Delta] \to \Delta \to D \cap \Delta = \Delta^{ab} \to 1
\]

where $j : U \xrightarrow{\sim} \mathbb{R}$ is the abelian group of unipotent, lower triangular matrices, and $D$ is the abelian group of diagonal matrices.

**Claim II.6.** $D \cap \Delta$ is contained in the subgroup of homotheties.

**Proof.** Let $d \in D \cap \Delta$ have diagonal entries $a, b$. If $c \in \mathbb{R}$, then then an easy computation gives $dj^{-1}(c)d = j(cba^{-1})$. Thus, if for some $d$, we have $a \neq b$, then there exists an action of $\mathbb{Z}[\frac{1}{n}]$, for some $n \neq 1, -1$, on $[\Delta, \Delta]$. Remember, however, that $\Delta$ is the fundamental group of the compact 3-manifold $T_x$, which is an Eilenberg-MacLane space by default. In particular, its abelianization is a free abelian group on finitely many generators, hence of cohomological dimension 1, while $H^3(T_x, \mathbb{Z}) = \mathbb{Z}$. Looking at the group cohomology spectral sequence, with $\mathbb{Z}/n\mathbb{Z}$ coefficients, attached to our sequence (14), we see that

\[
E_2^{p,q} = H^p(\Delta^{ab}, H^q(D \cap \Delta, \mathbb{Z}/n\mathbb{Z})) = 0, \quad p \geq 2, q \geq 0
\]

which contradicts $E_\infty^3 = \mathbb{Z}/n\mathbb{Z}$, \qed

It follows that $\Delta < \mathbb{R}^2$. Since it acts discretely on $G^+_K \times U_x$, it must be a lattice in $\mathbb{R}^2$, but then its action cannot be co-compact, contradicting the compactness of $T_x$. 
Step 2: We deal with the second case, that $\Delta$ contains a cyclic group of translations. Let

$$g = \begin{pmatrix} a & 0 \\ c & d \end{pmatrix}$$

be an element of $\Delta_l$. Then $g\tau g^{-1} = d\tau$, and hence $\mathbb{Z}[d, d^{-1}]$ acts on the subgroup of translations, which is cyclic so then $d = 1$ and $\tau$ is in the center. But then it is easy to check that sequence $[13]$ splits, and $\Delta$ is a subgroup of $\mathbb{R} \times \text{Aff}(1, \mathbb{R})$, which is discrete and acts co-compactly on $G_{K}^+ \times U_x$. We proceed to show that this is impossible, by looking at possible images $\Delta' < \text{Aff}(1, \mathbb{R})$ of $\Delta$.

Claim II.7. $\Delta'$ cannot be abelian.

Proof. Every abelian subgroup of $\text{Aff}(1, \mathbb{R})$ is contained in a 1-dimensional abelian Lie subgroup $L$. Hence $\Delta$ is contained in the 2-dimensional abelian Lie group $\mathbb{R} \times L$, and if it were discrete it would have rank 2. But then it cannot act co-compactly on $G_{K}^+ \times U_x$. $\square$

Claim II.8. $\Delta'$ must be abelian.

Proof. If it is non-abelian, then it has a non-trivial commutator, hence $\Delta'$ intersects the subgroup of translations. If the image of $\Delta' \to \mathbb{R}^*$ is $\{1, -1\}$, then up to a double covering, it is abelian. If such image contains $a \neq 1, -1$, then its action on the subgroup of commutators is by multiplication by $a$. Hence $\mathbb{Z}[a]$ acts on $[\Delta', \Delta']$, which is therefore infinitely generated. Hence the same holds for $[\Delta, \Delta]$, and $\Delta$ cannot be discrete. $\square$

This contradiction concludes the proof of Theorem II.4

Corollary II.9. If the action of $K_{Ab}$ on $S$ is not free and discrete, then $S/K \to G(1, 2)$ is a cyclic covering map.

Proof. Since $K$ acts freely and discretely on $S$, this means that the group $K_{Ab}/K$, see equation $[12]$ generated by its subgroup of translations $A$, and that of homotheties $H$ is non-discrete on $S/K$. Hence there exists either a translation, or a homothety, acting non-discretely on $S/K$. But then there exists a line $L \subset S/K$ mapping homeomorphically to a line in $G(1, 2)$. Moreover, since the action of $K_{Ab}/K$ is by translations and homotheties, it must non-discrete in a neighborhood $U$ of $L$. Such $U$ therefore satisfies the assumptions of Theorem II.3

$\square$
The previous statement is visibly equivalent to Proposition 2 stated in the introduction.

III. Proof of the main Theorem

Having gathered enough informations about the affine structure on $X$, we are ready to prove the main theorem. We exclude the possibility that $l\rho$ has integral values:

**Proposition III.1.** The image of $l\rho$ cannot be conjugated to a subgroup of $GL_2(\mathbb{Z})$ or $H_2(\mathcal{O}_F)$.

**Proof.** Let us assume $l\rho$ has integral image, which therefore lies inside $SL_2(\mathbb{Z})$, or inside the integral units of $H_2(F) \cap GL_2(\mathcal{O}_F)$. Observe that $SL_2(\mathbb{Z})$ has a free subgroup of finite index, while units in a maximal order of a quaternion algebra carry natural free, co-compact actions on the unit disk, giving rise to Shimura curves, [Sh67]. Either way, up to finite-index subgroups, $\Gamma$ has abelianization with $\text{rank}(\Gamma_{ab}) \geq 2$. By the standard inequality $2h_{1,0}(X) \geq b_1(X) - 1 \geq 1$, an étale cover of $X$ carries a holomorphic foliation, which contradicts our assumptions. \[\square\]

We deduce that $l\rho$ cannot be integral, and therefore denominators must appear non-trivially in the coefficients of the image of $l\rho$. Dealing with denominators is way more delicate, and occupies the rest of the manuscript. From now on, we assume that the representation $l\rho$ has image $G := \text{Im}(l\rho)$ inside $GL_2(\mathbb{Z}[\frac{1}{n}])$ or $H_2(\mathcal{O}_F[\frac{1}{n}])$, which further cannot be conjugated into an integral representation. Moreover, it will be convenient to assume that $G$ is torsion-free. This can be achieved by passing to a finite-index subgroup, thanks to Selberg’s lemma, with the negligible price of replacing $X$ by an étale cover. From this assumption, we will derive a contradiction. A fundamental remark we will need in the sequel is

**Lemma III.2.** $A$ is naturally a $\mathbb{Z}[\frac{1}{n}]$-module.

**Proof.** Recall that $G$ lies in $GL_2(\mathbb{Z}[\frac{1}{n}])$ or $H_2(\mathcal{O}_F[\frac{1}{n}])$. There is a natural action of the group algebra $\mathbb{Z}[G]$ on $A$. We deal with the case $G \subset H_2(\mathcal{O}_F[\frac{1}{n}])$ first. Let $M \in G$ be any matrix with at least one non-integral coefficient. It is easy to see that a $\mathbb{Z}$-linear combination of $M$ and $M^2$ is diagonal with non-integral entries, so then such diagonal matrix provides an action of $\mathbb{Z}[\frac{1}{n}]$ on $A$.

In case $G \subset GL_2(\mathbb{Z}[\frac{1}{n}])$, the reasoning is analogous. It is easy to see that:

- Either $\mathbb{Z}[G]$ contains a diagonal matrix, with non-integral diagonal entries, or
• \( G \) lies in the subgroup of \( GL_2(\mathbb{Z}[\frac{1}{n}]) \) consisting of upper-triangular matrices with integral diagonal entries, and non-diagonal entries with bounded \( p \)-valuation.

In the first case, such diagonal matrix provides an action of \( \mathbb{Z}[\frac{1}{n}] \) on \( A \), while in the second case \( G \) can be conjugated into \( GL_2(\mathbb{Z}) \), which is absurd given our setup described at the beginning of Section III. □

We distinguish two cases:

• \( V_R \) is empty

• \( V_R \) is not empty.

We proceed to exclude both cases. The main idea is to use some paradoxical properties of \( \Gamma \) to deduce that it has the wrong cohomological dimension, and therefore reach a contradiction. We start with:

**Lemma III.3.** The cohomological dimension of \( G' \) is at most 2.

**Proof.** Recall that \( G' \) was defined as the image of \( G = \text{Im}(\rho) \) in \( SL_2(\mathbb{R}) \). Let us treat the case \( G' \subset PSL_2(\mathbb{Z}[\frac{1}{n}]) \) first, which is the most complicated because of torsion in \( PSL_2 \).

The group \( PSL_2(\mathbb{Z}) \) admits a free product decomposition as \( \mathbb{Z}/3\mathbb{Z} * \mathbb{Z}/4\mathbb{Z} \), and hence its classifying space satisfies \( BPSL_2(\mathbb{Z}) = B\mathbb{Z}/3\mathbb{Z} \vee B\mathbb{Z}/4\mathbb{Z} \). By [Se80], \( SL_2(\mathbb{Z}[\frac{1}{n}]) \) can be decomposed as a free product of copies of \( SL_2(\mathbb{Z}) \), amalgamated along modular subgroups \( \Gamma_0(p) \), for primes \( p|n \). This gives a decomposition

\[
BPSL_2(\mathbb{Z}[\frac{1}{n}]) = S_1 \vee T_1 \vee S_2 \vee T_2 \cdots \vee T_k \vee S_{k+1}
\]

where each \( S_i \) is a classifying space for \( PSL_2(\mathbb{Z}) \), which are glued, via the \( T_i \)'s, along the classifying space of a modular group. Let \( f : BG' \to BPSL_2(\mathbb{Z}) \) denote the natural étale cover. Then, since \( G' \) is torsion-free, the \( f^*S_i \) become classifying spaces of free groups, i.e. bouquet of circles. These are glued along 2-cells corresponding to \( f^*T_i \), so then \( BG' \) has cohomological dimension 2.

The case \( G' \subset H_2(\mathcal{O}_F[\frac{1}{n}]) \) can be dealt with similarly: again there is a decomposition of \( H_2(\mathcal{O}_F[\frac{1}{n}]) \) as a free product of copies of \( H_2(\mathcal{O}_F) \), amalgamated along their intersections with modular subgroups of \( PSL_2(\mathcal{O}_F) \). Such intersections are, as in the previous case, fundamental groups of open Riemann surfaces, hence free. Therefore we obtain a decomposition

\[
BH_2(\mathcal{O}_F[\frac{1}{n}]) = S_1 \vee T_1 \vee S_2 \vee T_2 \cdots \vee T_k \vee S_{k+1}
\]
where each $S_i$ is a hyperbolic Shimura curve, \cite{Shimura67}, which are glued via the $T_i$’s along bouquets of circles. Therefore, the cohomological dimension of $BH_2(\mathcal{O}_F[\frac{1}{n}])$ is 2, and the same holds for its étale cover $BG'$.

We can now state the first half of the main theorem:

**Theorem III.4.** $V_R$ cannot be empty.

*Proof.* The crucial tool is Corollary \ref{corollary:V_is_empty}. If $S/K \rightarrow G(1, 2)$ is a cyclic covering map, then the same is true for $V/K \rightarrow C^2 \setminus R$. Observe that the action of $A$ on $C^2 \setminus R$ is non-discrete, precisely every point is an accumulation point in its orbit, by Lemma \ref{lemma:non_discrete_action}. The same holds, therefore, for its action on $V/K$. But $A$ acts discretely on $V/K$, so necessarily $A$ is trivial. Therefore, $X$ is the quotient of $V/K$ by the action of $G$. However, the radial vector field on $C^2$ defines a foliation which is invariant under $GL_2$. This implies that such foliation descends to $X$, which is absurd.

If $K_{Ab}$ acts freely and discretely on $S$, then it is the fundamental group of a Riemann surface, so then its cohomological dimension is 1. Let $p \mid n$ a prime. The cohomology spectral sequence, with $\mathbb{Z}/p\mathbb{Z}$ coefficients, attached to the sequence \ref{sequence:cohomology} satisfies:

\begin{equation}
E_2^{p,q} = H^p(G', H^q(K_{Ab}, \mathbb{Z}/p\mathbb{Z})) = 0, \quad q \geq 1, p \geq 2
\end{equation}

This is a consequence of the fact that $K_{Ab}$ is a surface group, and of Lemma \ref{lemma:cohomology_k_ab}. It follows that $H^4(\Gamma, \mathbb{Z}/p\mathbb{Z}) = 0$, which is absurd since $X$ is both a $K(\Gamma, 1)$ and a compact complex surface.

Not so surprisingly, the second half is:

**Theorem III.5.** $V_R$ cannot be non-empty.

Before getting into the proof, let us recall that the only real surfaces carrying an affine structure are the torus, and the Klein bottle, which is covered by a torus. There are exactly four distinct affine structures on 2-dimensional real tori:

(i) A lattice in $\mathbb{R}^2$
(ii) A cyclic group in $\mathbb{R}^*$
(iii) A lattice in $\mathbb{R} \times \mathbb{R}^*$
(iv) A lattice in $\mathbb{R}^* \times \mathbb{R}^*$

We can now begin the proof of Theorem \ref{theorem:V_R_non_empty}.
Proof. Denote by \( V_R \) the pre-image of \( R \) inside \( V \), and by \( X_R \) its image in \( X \). \( X_R \) is a disjoint union of real, compact surfaces that inherit an affine structure from \( X \). We can assume therefore that every component of \( X_R \) is a torus. Correspondingly, each connected component of \( V_R \) maps homeomorphically to \( R \), and the image can be \( R, R \setminus 0 \), an open half-space \( H \), or a quadrant \( Q \) - by the above remark on affine structures on tori.

Denote by \( V^i_R \) the connected components of \( V_R \), and for each \( i \) let \( U_i = \{ x \in V \text{ s.t. } G_K \cdot x \cap V^i_R \neq \emptyset \} \). This is a \( G_K \)-invariant open neighborhood of \( V^i_R \). Observe that, if \( U_i \cap U_j \neq \emptyset \) for \( i \neq j \), necessarily \( V^i_R \) and \( V^j_R \) map to adjacent domains in \( R \) - the two half-spaces, or two of the four quadrants. If this is the case, let us re-define \( U_i := \bigcup_{j \neq i} U_j \cap U_i \) as the union of those \( U_j \) intersecting \( U_i \). \( \Gamma \) acts on the collection of the \( U_i \)'s, and the stabilizer \( \text{Stab}_i < \Gamma \) of \( U_i \) is isomorphic to a subgroup of an extension of \( \mathbb{Z} \oplus \mathbb{Z} \) by a subgroup of \( S_4 \) - which subgroup depends on which adjacent domains appear. In order to conlude the proof, we fix a component \( U_0 \), and

Claim III.6. \( U_0 = V \).

Proof. Denote by \( W_0 \) the union of those \( V^i_R \) contained in \( U_0 \). We divide into two cases:

- The image of \( W_0 \to R \) is all of \( R, R \setminus 0 \), the union of two half-spaces or the union of at least three quadrants;
- The image of \( W_0 \to R \) is a half-space or a union of at most two quadrants.

In the first case, the complement of \( U_0 \subset \mathbb{C}^2 \) is of real codimension at least 2. It follows that the complement \( V \setminus U_0 \) must have empty interior, since otherwise \( \partial U_0 \) would be a 3-dimensional manifold mapping to a 3-dimensional submanifold of \( \mathbb{C}^2 \). Hence \( U_0 = V \).

In the second case, if \( U_0 \neq V \), its boundary \( \partial U_0 \) is a 3-dimensional \( G_K \)-invariant submanifold. Its image, under projection \( V \setminus V_R \to G(1,2) \), parametrizes those lines that intersect the closure of \( W_0 \), but not \( W_0 \) itself. If \( W_0 \) were a half-space, then \( \partial U_0 = L \times G_K \), with \( L \) a half-line in a fiber of \( G(1,2) \to S^1 \). If \( W_0 \) were the union of at most two quadrants, then \( \partial U_0 = S \times G_K \) were \( S \) is a proper, closed interval contained in a section of \( G(1,2) \to S^1 \). Either way \( \text{Stab}_0 \), having rank at most 2, cannot have a co-compact action on \( \partial U_0 \), contradiction.

Consequently, \( V \to \mathbb{C}^2 \) is an embedding. But then we can conclude as in Theorem III.4 that the subgroup of translations must be trivial and \( X \) carries a holomorphic foliation, which is a contradiction. Theorem III.5 is proven, and this finishes Theorem I as well. \( \Box \)
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