ON THE INFINITESIMAL STANDARD CONJECTURE

Abstract. The aim of the present is to discuss an infinitesimal version of Grothendieck’s weak Lefschetz standard conjecture, and to conjecture an approach to the global one.

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INTRODUCTION

It is a known principle, in the study of the topology of smooth complex projective varieties, that hyperplane sections carry a great amount of topological informations. Such principle can be best understood in terms of the fundamental Lefschetz isomorphisms, stating for $H \subset X$ a smooth ample divisor inside a smooth complex projective variety, that $\pi_p(H) \cong \pi_p(X)$ and $H^p(X, \mathbb{Z}) \cong H^p(H, \mathbb{Z})$ as long as $p \leq \dim \mathbb{C} X - 2$. These isomorphisms depend heavily on the CW-complex structure, particularly on how we can reconstruct the cellular structure of the variety starting with an hyperplane section. However at the price of forgetting torsion and working with rational coefficients, the cohomological Lefschetz isomorphism is an exercise in combining Hodge decomposition and Kodaira-Nakano vanishing. This last result admits a purely algebraic proof, [3], suggesting that it might have
a role in explaining the conjectured algebraic analogue of the Lefschetz isomorphism, 

**Weak Lefschetz-type Conjecture**

Let $X$ be a smooth projective variety over a field of characteristic zero, of dimension $n$. Let $H$ be a smooth hyperplane section, then the natural morphism between Chow groups $CH^p(X)_\mathbb{Q} \to CH^p(H)_\mathbb{Q}$ is an isomorphism if $2p < n - 1$ and injective if $2p \leq n - 1$.

This formulation of the problem is extremely difficult to handle for $p > 1$. Indeed a more reasonable question concerns a comparison between the category of vector bundles on $X$ and $H$ respectively, and there is no loss of generality in taking this point of view by way of

**Fact** [1]

There exists a $\gamma$-filtration on the Grothendieck group of vector bundles on $X$, such that the Chern character induces isomorphisms, on the associated graded components

$$\text{ch} : \text{Gr}_\gamma^{p} K^0(X)_\mathbb{Q} \cong CH^p(X)_\mathbb{Q}$$

Grothendieck’s approach to the case $p = 1$ can be summarized as follows:

1) Denoting by $\hat{X}$ the formal completion of $X$ along $H$, understand the morphism $\text{Pic}(\hat{X}) \to \text{Pic}(H)$. This is a cohomological problem, and Kodaira vanishing easily implies $\text{Pic}(\hat{X}) \cong \text{Pic}(H)$ whenever $\dim(X) \geq 4$.

2) Understand $\text{Pic}(X) \to \text{Pic}(\hat{X})$. This is sensibly more delicate than the previous problem, but surprisingly the dimension constraint $\dim(X) \geq 4$ is enough to imply that any vector bundle over $\hat{X}$ extends to a Zariski neighborhood of $H$, thus proving that $\text{Pic}(X) \cong \text{Pic}(\hat{X})$.

The higher codimension case can be understood along the same lines, namely there is a diagram

$$CH^p(X)_\mathbb{Q} \cong \text{Gr}_\gamma^{p} K^0(X)_\mathbb{Q} \to \text{Gr}_\gamma^{p} K^0(U)_\mathbb{Q} \to \text{Gr}_\gamma^{p} K^0(\hat{X})_\mathbb{Q} \to \text{Gr}_\gamma^{p} K^0(H)_\mathbb{Q} \cong CH^p(H)_\mathbb{Q}$$

where $U$ is any Zariski open neighborhood of $H$. The second map is an isomorphism if $p \leq \dim(X) - 1$, the third one is an isomorphism by way of Grothendieck’s remarkable theorem, I.1.iii on the effective Lefschetz condition, (see [8] for a discussion with applications). The mysterious point is the restriction $\text{Gr}_\gamma^{p} K^0(\hat{X})_\mathbb{Q} \to \text{Gr}_\gamma^{p} K^0(H)_\mathbb{Q}$. In casting light on this obscure morphism, the starting point is that it admits a cohomological interpretation, by way of the fundamental

**Bloch’s formula** [10]
Let $K_M^p$ denote the $p$-th Milnor $K$-theory sheaf. Then for $X$ a smooth algebraic variety we have $CH^p(X)_\mathbb{Q} \simeq H^p(X, K_M^p)_\mathbb{Q}$.

The smoothness assumption is necessary, indeed by II.7 it seems to be extremely unlikely that Milnor $K$-theory sheaves have any interesting relation to vector bundles on non-reduced schemes. In any case, the first result of this work points in the right direction:

**Milnor’s Infinitesimal Lefschetz, II.1**

Let $X$ be a smooth projective variety of dimension $n$, defined over a field of characteristic zero, and $H$ a smooth hyperplane section. Then

$$H^q(\hat{X}, \lim_{\leftarrow} K_M^p(O_{H_n}))_\mathbb{Q} \to H^q(H, K_M^p(O_H))_\mathbb{Q}$$

is an isomorphism if $p + q < n - 1$ and is injective if $p + q \leq n - 1$.

However Thomason and Trobaugh’s monumental work [13] tells that the K-theory spectra deduced from the category of perfect complexes (or what is the same in our context, Quillen’s K-theory of vector bundles) will fix this lack of flexibility proper of Milnor K-theory. One of their many important results is:

**Local-to-global spectral sequence [13]**

Let $X$ be a noetherian scheme. There exists a strongly convergent spectral sequence $E_2^{p,q} = H^p(X, K^q(O_X)) \Rightarrow K^{q-p}(X)$. If $X$ is a projective scheme then the $K$-theory sheaves can be assumed to be Quillen’s $K$-theory of vector bundles.

The existence of such a spectral sequence suggests that an Infinitesimal Lefschetz for Quillen’s $K$-theory would give informations on the infinitesimal extension of vector bundles. Accordingly, the second result of this work is:

**Quillen’s Infinitesimal Lefschetz, III.4**

Let $X$ be a smooth projective variety over a field of characteristic zero, of dimension $n$, and $H$ a smooth hyperplane section. Then

$$H^q(\hat{X}, \lim_{\leftarrow} K^p(O_{H_n}))_\mathbb{Q} \to H^q(H, K^p(O_H))_\mathbb{Q}$$

is an isomorphism if $p + q < n - 1$ and is injective if $p + q \leq n - 1$.

Once again this is not quite satisfactory, since it is not known, and probably false, whether Thomason and Trobaugh’s spectral sequence degenerates at the second page on non-reduced schemes. This is not a big deal, since what we are really interested in are vector bundles on the formal completion, which is by itself a smooth formal scheme. This gives hope for the following to be true:
Conjecture "Formal Bloch's Formula"

Let $\mathcal{X}$ be a smooth formal scheme, with ideal of definition $\mathcal{J}$. If $K^p$ is the Milnor and/or Quillen's $K$-theory sheaf, then there exists an isomorphism

$$\text{Gr}_p K^0(\mathcal{X})_\mathbb{Q} \to H^p(\mathcal{X}, \lim_{\leftarrow n} K^p(\mathcal{O}/\mathcal{J}^n))_\mathbb{Q}$$

where $K^0$ is the Grothendieck group of vector bundles on $\mathcal{X}$ and $2p < \dim(\mathcal{X})$.

The formal Gersten conjecture would immediately imply Grothendieck's weak Lefschetz conjecture. From the point of view of the (possible) failure of the Thomason and Trobaugh's spectral sequence to degenerate at the second page in the non-smooth case, smoothness of the formal scheme should play a key role. The difficulty of the problem is that we are comparing completely different categories. This diversity lies at the heart of the notion of rings that are complete in a linear topology, and cannot be appreciated at the level of the category of modules, since we have in full generality:

**Fact [11]**

Let $A$ be a ring that is complete in a linear topology, i.e. $A = \lim_{\leftarrow i} A_i$. Then the category of modules over $A$ and that of inverse systems of modules over $\{A_i\}$ are equivalent.

What is not true anymore is that such categories up to homotopy are equivalent. Indeed the length of the class of isomorphism of a complex of projective modules can't increase arbitrarily as we run over the index set $I$, for example by Nakayama's lemma. This doesn't hold if we replace isomorphism classes with homotopy classes, whence the derived category of bounded perfect complexes over a formal scheme is, in general, not equivalent to the inverse limit of derived categories of bounded perfect complexes over the sequence of schemes that converge to the formal scheme. This dichotomy can be appreciated at the level of $K$-groups, indeed we have:

**Remark, II.6**

Let $A$ be a smooth local $\mathbb{Q}$-algebra of finite type, and $x \in A$. Then the natural morphism

$$K^p_M(\lim_{\leftarrow n} A/x^n)_\mathbb{Q} \to \lim_{\leftarrow n} K^p_M(A/x^n)_\mathbb{Q}$$

is never injective, for $p \geq 2$.

This suggests that a finer understanding of inverse limits of derived categories can lead to a proof of Grothendieck's Lefschetz Conjecture.

The paper is organized as follows: field means field of characteristic zero.
In the first chapter we recall the Grothendieck-Lefschetz conditions and some fundamental results; then we move onto the notions of Milnor and Quillen’s K-theory, and their relation to algebraic cycles and vector bundles.

In the second chapter we prove the Infinitesimal Lefschetz for Milnor K-theory.

In the third chapter we prove the Infinitesimal Lefschetz for Quillen K-theory.

Remark We have discovered that Patel and Ravindra obtained Theorem III.4 - Quillen’s Infinitesimal Lefschetz using similar techniques, and for the same scope of casting light on the weak Lefschetz conjecture. Their results are in [14].

1. GENERALITIES

1.1. Grothendieck-Lefschetz conditions

Here we recall the main definitions and results concerning the algebraicity of vector bundles on smooth formal subschemes of algebraic varieties in characteristic zero.

1.1.i Definition Let $X$ be an algebraic variety and $Y$ a closed subvariety. Then the pair $(X, Y)$ is said to satisfy the Lefschetz condition $\text{Lef}(X, Y)$ if for any locally free sheaf $F$ defined in a neighborhood $U$ of $Y$, there is a possibly smaller open neighborhood $V$ of $Y$ such that the canonical map induced by completion of $X$ along $Y$

$$H^0(V, F) \to H^0(\hat{X}, \hat{F})$$

is an isomorphism.

The pair satisfies the Effective Lefschetz condition $\text{Eff}(X, Y)$ if $\text{Lef}(X, Y)$ and moreover, for any locally free sheaf $E$ on $\hat{X}$, there is locally free sheaf $F$ defined around $Y$ such that $\hat{F} \simeq E$.

The condition $\text{Lef}(X, Y)$ means that the completion functor from the category of germs of locally free sheaves on $X$ around $H$ to that of locally free sheaves on $\hat{X}$ is fully faithful. Indeed a morphism between locally free sheaves $E, F$ is a global section of the sheaf $\mathcal{H}om(E, F)$.

It is interesting to observe that $\text{Lef}$ can be expressed in terms of cohomological properties of the open subscheme $X – Y$. We have:

1.1.ii Theorem [5] If the cohomological dimension of $X – Y$ is at most $\dim(X) – 2$, then $\text{Lef}(X, Y)$.
Our interest is clearly in the effective Lefschetz condition. It seems a miracle that such algebraization holds almost unconditionally:

**I.1.iii Theorem** If $X$ is a smooth algebraic variety over a field of characteristic zero, $H$ a smooth hyperplane section and $\dim(H) \geq 2$ then $\text{Lef}f(X,H)$. Equivalently, every vector bundle on $\hat{X}$ is the formal completion of a vector bundle defined over a Zariski neighborhood of $H$ in $X$.

### 1.2. Milnor K-theory and algebraic cycles

. For a unitary commutative ring $A$, the Milnor $K$-theory groups are defined as $K^p_M(A)^0 = A^* \otimes_\mathbb{Z} A^* \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} A^*/R$, the product over $p$ copies of $A^*$ and $R$ is the subgroup of Steinberg relations, generated by tensors $x_1 \otimes \cdots \otimes x_p$ such that $x_i + x_j = 1$ for some distinct $i$ and $j$.

The Steinberg relations have some immediate consequences that will be used later on: from the identity $(1 - x)^{-1} = ((1 - x)x^{-1})^{-1}x^{-1}$, since $(x, 1 - x) = 0$ we find $(x, 1 - x)_1^{-1} = x^{-1}(1 - x)^{-1} = 0$. Plugging $1 - x = -x^{-1}(1 - x)$ in, we end up with $(x, -x) = 0$. Multiplying this by 2 we find $2(x, x) = 0$. Moreover $2(xy, xy) = 0$ yields $2(x, y) = -2(y, x)$. We have proved

**I.2.i Fact** The following relations hold in $K^2_M(A)^0 \otimes_\mathbb{Z} \mathbb{Z}[1/2]$, whenever $x, 1 - x, 1 - xy \in A^*$:

$$(x, x) = 0$$

$$(x, y) = -(y, x)$$

An important corollary of the above identities is the following: given an ideal $I \subset A$ let $\pi : K^p_M(A)^0 \otimes_\mathbb{Z} \mathbb{Z}[1/2] \to K^p_M(A/I)^0 \otimes_\mathbb{Z} \mathbb{Z}[1/2]$ be the induced morphism. then ker $\pi$ is generated by symbols of the form $(1 + j, u_1, \ldots, u_{p-1})$ for $j \in I$ and $u_i \in A^*$. In fact, symbols $(x_1, \ldots, x_p)$ such that $x_i + x_k = 1 + j$, for some $j \in I$ clearly generate the kernel. Then, the identity $(x, y) = (x + y, xy^{-1})$, which is true for $x, y, x + y \in A^*$, implies $(\ldots, x_i, \ldots, x_k, \ldots) = (1 + j, x_kx_i^{-1}, \ldots)$ up to a sign.

**I.2.ii Notation** In the sequel it will be convenient to understand all Milnor $K$-groups up to torsion. Thus $K^p_M(A)$ will always mean $K^2_M(A)^0 \otimes_\mathbb{Z} \mathbb{Q}$.

The role of Milnor $K$-theory is now a direct consequence of the following fundamental
"localization sequence":

I.2.iii **Theorem (Kerz, [10])**

If $A$ is a regular local ring with characteristic zero residue field, $g$ an non-unit in $A$ then

$$0 \to K_M^p(A) \to K_M^p(A_g) \to K_M^{p-1}(A/g) \to 0$$

The maps can be described explicitly, the first injection induced by the inclusion $A \to A_g$, the surjection given on generators by $(u_1, ..., u_p) \to 0$ if $u_i \neq g^n$ for all $i = 1, ..., p$ and all positive integers $n$, and $(g, u_2, ..., u_n) \to (u_2, ..., u_n)$, extended by linearity. If $A$ is a local DVR and $g$ is a uniformizing parameter the sequence reads

$$0 \to K_M^p(A) \to K_M^p(Q(A)) \to K_M^{p-1}(k) \to 0$$

$Q(A)$ being the quotient field and $k$ the residue field. If $X$ is a regular scheme with characteristic zero residue fields, and $V$ is a codimension one subvariety of $X$, denote by $O_{X,V}$ the local ring at the generic point of $V$. This is a local DVR and denoting by $K_X$ the constant sheaf of rational functions on the variety $X$, we have

$$0 \to K_M^p(O_{X,V}) \to K_M^p(K_X) \to K_M^{p-1}(K_V) \to 0$$

We can collect all exact sequences of this form as the codimension one subvariety $V$ varies, and continuing in higher codimension

$$0 \to \bigoplus_{\text{codim}V=1} K_M^{p-1}(K_V) \to \bigoplus_{\text{codim}V=2} K_M^{p-2}(K_V) \to \cdots \to \bigoplus_{\text{codim}V=p-1} K_M^1(K_V) \to \bigoplus_{\text{codim}V=p} K^0(K_V) \to 0.$$  

This is a flasque resolution of $K_M^p(O_X)$ because $K_M^{p-i}(K_V)$ is a constant sheaf supported on the generic point of $V$. We can compute the $p$-th cohomology of the global section complex easily: indeed the Grothendieck $K^0$ of a field is $\mathbb{Z}$, while $K^1_M(R) = R^\ast$ by definition. Particularly we deduce

$$H^p(X, K_M^p(O_X)) \simeq \text{Coker} \left\{ \bigoplus_{\text{codim}V=p-1} K^\ast_V \to \bigoplus_{\text{codim}V=p} \mathbb{Z} \right\} = CH^p(X)$$

by the way the boundary maps in the localization sequence are defined.

What we have proved is,

I.2.iv **Theorem**
For $X$ a noetherian regular scheme with characteristic zero residue fields, there is a canonical isomorphism

$$H^p(X, \mathcal{K}_M^p(\mathcal{O}_X)) \simeq CH^p(X)_\mathbb{Q}$$

1.3. Quillen K-theory and algebraic cycles

For a given CW-complex $X$ with perfect fundamental group we can construct a new complex $X^+$ and a canonical map $X \to X^+$ such that $\pi_1(X^+) = 0$ and $H_*(X, \mathbb{Z}) \simeq H_*(X^+, \mathbb{Z})$. The map is visibly functorial and the plus-construction satisfies the obvious universal property, that is if $X \to T$ is any map such that the image of the fundamental group is trivial, then this map factors as $X \to X^+ \to T$.

A similar definition occurs when the fundamental group of $X$ is not perfect, in that case we pretend to construct the space $X^+$ by killing a normal perfect subgroup of the fundamental group. More precisely, fixed any normal perfect subgroup $P \subset \pi_1(X)$ then there is a CW-complex with a map $p : X \to X^+_P$ such that $\pi_1(X^+_P) = \pi_1(X)/P$ and for any $\pi_1(X^+_P)$-module $G$ we have isomorphisms $H_*(X, p^*G) \simeq H_*(X^+_P, G)$.

I.3.i Remark We will be interested in the case of spaces such that $\pi_1 = GL(R)$ with perfect subgroup $SL(R)$, for a local ring $R$.

I.3.ii Theorem The plus-construction exists

Proof: We start with a CW-complex $X$ with perfect fundamental group, so we can attach 2-cells $e_j$ to kill the entire fundamental group. Now we have $X \to X_1 = X \cup_j e_j$, and $\pi_1(X_1) = 0$. This procedure of attaching cells could obviously have modified the homology of $X$, so we proceed to recover its homology. We must observe:

1) by the Hurewicz theorem, $\pi_2(X_1) \simeq H_2(X_1, \mathbb{Z})$

2) There is a relative exact sequence $H_2(X_1) \to H_2(X_1, X) \to H_1(X) = \pi_1(X)_{\text{ab}} = 0$

Since $H_2(X_1, X)$ is generated by the homology classes of the $e_j$, combining 1) and 2) we can find classes $x_j \in \pi_2(X_1)$ that represent $e_j$. Thus we can attach 3-cells $f_k$ to kill these $x_j$. This gives a new space $X_2$, and

I.3.ii.1 Claim $X_2 = X^+$

Indeed we have to compute the relative homology $H_*(X_2, X)$ and the complex of relative
chains is

\[ \cdots \to 0 \to C_3(X_2, X) \to C_2(X_2, X) \to 0 \]

So it’s enough to prove that the boundary of each \( f_j \) is \( e_j \). But this is obvious by the definition of attaching a cell.

In the general case, namely when \( \pi_1 \) is not necessarily perfect, let \( P \) be a perfect normal subgroup and let \( \overline{X} \to X \) be the covering space of \( X \) such that \( \pi_1(\overline{X}) = P \). We know how to construct \( \overline{X} \to \overline{X}^+ \), thus we define \( X^+_P \) as the amalgamated sum \( X \cup_{\overline{X}} \overline{X}^+ \).

I.3.iii Definition

Given a commutative ring \( R \) let \( BGL(R) \) be the classifying space of the discrete group \( GL(R) \) (that is, a space with the homotopy type of a \( K(GL(R), 1) \)). Then the Quillen K-theory of the ring \( R \) is defined as

\[ K^p(R) = \pi_{p}(BGL(R)^+) \]

Quillen’s original definition [12] is much more general, indeed he defined a K-theory of an exact category. He observed that starting with an exact category \( \mathcal{C} \) he could functorially construct a new category \( QQ\mathcal{C} \) such that its classifying space \( BQ\mathcal{C} \) has the remarkable property

\[ \pi_1(BQ\mathcal{C}) = K^0(\mathcal{C}) \]

where \( K^0 \) is the classical Grothendieck K-group of a category where the notion of exact sequence makes sense. Thus the natural definition of Quillen K-theory is

I.3.iv Definition The Quillen p-th K-theory of an exact category \( \mathcal{C} \) is defined as

\[ K^p(\mathcal{C}) = \pi_{p+1}(BQ\mathcal{C}) \]

In the case of rings, the natural category to work with is that of projective modules, so that

\[ K^p(R) = \pi_{p+1}(Vect/R) \]

where \( Vect/R \) is the category of projective modules over \( R \) and this definition agrees with that given by the plus-construction as long as \( p \neq 0 \).

Instead of giving a precise definition of the \( Q \)-construction, we proceed by stating the
main properties of Quillen K-theory and draw an important application to the study of algebraic cycles on regular schemes. These properties are two fundamental Devissage and Localization theorems, explained in [12]:

**I.3.v Quillen’s Devissage**

Let \( \mathcal{A} \) be an abelian category, and \( \mathcal{B} \) a full subcategory closed under taking products, subobjects and quotient objects. Assume that every \( M \in \text{Ob}(\mathcal{A}) \) has a finite filtration \( 0 \subset M_p \subset ... \subset M_0 = M \) with \( M_i/M_{i+1} \in \text{Ob}(\mathcal{B}) \). Then \( QB \to QA \) is a homotopy equivalence.

**I.3.vi Quillen’s Localization**

Let \( \mathcal{A} \) be an abelian category, \( \mathcal{B} \) a Serre subcategory and \( \mathcal{A}/\mathcal{B} \) the categorical quotient, then there is a fibration

\[
QB \to QA \to QA/B
\]

which induces a long homotopy sequence

\[
... \to K^p(\mathcal{A}) \to K^p(\mathcal{A}/\mathcal{B}) \to K^{p-1}(\mathcal{B}) \to ...
\]

We are now in position to prove a ”Bloch formula” for Quillen K-theory, as already discussed in the realm of Milnor K-theory. We have

**I.3.vii Theorem ([12], Gersten Conjecture)**

*If \( X \) is a regular scheme, we have the canonical isomorphism*

\[
H^p(X, K^p(O_X)) \cong CH^p(X)
\]

**Proof:** The idea is to filter the category of coherent sheaves by codimension of support, and then run a spectral sequence from the so filtered localization sequence. Indeed let \( \mathcal{C}_p \) be the Serre subcategory of \( \text{Coh}/X \) generated by coherent sheaves whose support is of pure codimension at least \( p \). By the regularity hypothesis, every irreducible subscheme \( V \) is determined by its generic point \( e_V \), so we get the equivalence of categories...
(Cor. 1 to Thm 4, [12])

\[ \mathcal{C}_p / \mathcal{C}_{p+1} \simeq \bigoplus_{\operatorname{cod}(V) = p} \mathcal{V}ect / \mathcal{K}_V \]

which implies, by the localization sequence, that there is a long exact sequence

\[ \cdots \rightarrow K^q(\mathcal{C}_{p+1}) \rightarrow K^q(\mathcal{C}_p) \rightarrow \bigoplus_{\operatorname{cod}(V) = p} K^q(\mathcal{K}_V) \rightarrow \cdots \]

where the union runs over the q-th K group of the residue field along the subscheme V. This filtered long exact sequence induces naturally a spectral sequence, whose first term is

\[ E_1^{p,q} = \bigoplus_{\operatorname{cod}(V) = p} K^{q-p}(\mathcal{K}_V) \]

converging to \( K^*(X) \), with differentials \( d_1 : E_1^{p,q} \rightarrow E_1^{p+1,q} \).

We are now ready to conclude the proof of the Gersten conjecture:

The differentials induce a complex of groups

\[ (*) 0 \rightarrow K^n(X) \rightarrow \bigoplus_{\operatorname{cod}(V) = 0} K^n(\mathcal{K}_V) \rightarrow \bigoplus_{\operatorname{cod}(V) = 1} K^{n-1}(\mathcal{K}_V) \rightarrow \cdots \rightarrow \bigoplus_{\operatorname{cod}(V) = n} K^0(\mathcal{K}_V) \rightarrow 0 \]

which we can perfectly regard as a complex of sheaves on \( X \), by letting \( K^*(\mathcal{K}_V) \) be the skyscraper sheaf supported on the generic point of \( V \) and \( K^n(X) \) be the sheafification of the functor \( U \rightarrow K^n(\mathcal{O}_X(U)) \).

The key point is that under the regularity assumption, this sequence of sheaves is exact on \( X \). More precisely, Quillen proves I.3.vii.bis Theorem

Let \( A \) be a local regular ring, then the sequence

\[ 0 \rightarrow K^n(A) \rightarrow \bigoplus_{htp = \dim A - 1} K^{n-1}(A/p) \rightarrow \cdots \rightarrow \bigoplus_{htp = \dim A - n} K^0(A/p) \rightarrow 0 \]

is exact.

This implies the Gersten conjecture, for the following two reasons:

1) The above \( (*) \) gives a flasque resolution of the sheaf \( \mathcal{K}^n \) and thus

\[ H^p(X, \mathcal{K}^n(\mathcal{O}_X)) = E_2^{p,n} \]

2) The last map

\[ \bigoplus_{\operatorname{cod}(V) = n-1} K^1(\mathcal{K}_V) \rightarrow \bigoplus_{\operatorname{cod}(V) = n} K^0(\mathcal{K}_V) \]
of the sequence (\ast) is equal to
\[ \bigoplus_{\text{cod}(V)=n} K^*_V \rightarrow \bigoplus_{\text{cod}(V)=n} \mathbb{Z} \]
and this has cokernel equal to the Chow group of codimension p cycles.

\[ \square \]

2. MILNOR’S INFINITESIMAL LEFSCHETZ

In this section we aim to prove the following:

II.1. **Theorem - Milnor’s Infinitesimal Lefschetz**

Let \( X \) be an algebraic variety of dimension at least four over a characteristic zero field, \( H \) an ample divisor. Let \( K^p_M \) be the Milnor K-sheaf, namely the sheaf associated to the presheaf \( U \rightarrow K^p_M(\mathcal{O}_X(U)) \), where \( K^p_M(R) \) denotes the Milnor K-theory of the ring \( R \).

Then the natural map
\[ H^q(\hat{X}, \lim_{\leftarrow} K^p_M(\mathcal{O}_{H_n})) \rightarrow H^q(H, K^p_M(\mathcal{O}_H)) \]
is an isomorphism if \( p + q \leq \dim H - 1 \) and is injective if \( p + q = \dim H \)

The strategy is to compute the kernel
\[ 0 \rightarrow N^p_n \rightarrow K^p_M(\mathcal{O}_{H_{n+1}}) \rightarrow K^p_M(\mathcal{O}_{H_n}) \rightarrow 0 \]
and use Kodaira-Nakano vanishing to draw conclusions on its cohomology. Being interested exclusively at the stalks of the above sequence, after a localization to a suitable neighborhood of a given point \( q \) of \( H \), if will suffice to study
\[ 0 \rightarrow N^p_n \rightarrow K^p_M(\mathcal{O}_{H_{n+1}}(f^n)) \rightarrow K^p_M(\mathcal{O}_{H_n}(f^n)) \rightarrow 0 \]
where \( A \) is a smooth, local \( k \)-algebra and \( f \) generates the ideal of \( H \).

We will first consider the case of a truncated polynomial algebra, \( A[f]/f^n \), where \( f \) has a priori no relations inside \( A \). The starting point to understand the structure of the kernel is a theorem by Van der Kallen:

II.2 **Theorem (Van der Kallen, Bloch [2])** let \( A \) be a local ring containing \( 1/2 \), \( f \) an indeterminate. Then the tangent space \( TK^p_M(A) \) to Milnor K-theory, namely
\[ \text{Ker}\{K^p_M(\mathcal{O}[f]/f^2) \rightarrow K^p_M(A)\} \]
has a natural \( A \)-module structure and is \( A \)-isomorphic to \( \Omega^{p-1}_{A/Q} \).
In the case $p = 2$ the map is defined, on symbols of the form $(1 + af, b)$ for $b \in A^*$ and $a \in A$, by

$$(1 + af, b) \rightarrow adb/b$$

The fact that these symbols generate all the tangent space follows from the "vanishing" theorem of Van Der Kallen, stating that in the above hypotheses, namely $A$ a local ring containing $1/2$, all the symbols of the form $(1 + af, 1 + bf)$ and $(1 + af, q)$, $a, b \in A$ and $q \in Q^*$, vanish in $K^2_M(A[f]/f^2)$, and from the fact that generators for the kernel are of the form $(1 + af, c + ef)$, $a, e \in A$ and $c \in A^*$. All of this can obviously be extended to the case $p \geq 3$, as remarked in [6], pg. 78, with absolutely no sorrow, namely the tangent space to $K^p_M(A)$ is generated by symbols of the form $(1 + af, u_1, ..., u_{p-1})$ for $u_i \in A^*$, has an $A$-module structure and is therefore $A$-isomorphic to $\Omega^{p-1}_{A/Q}$. We report a proof, following [6], of Van der Kallen’s result.

**Proof of II.2**

II.2.i **Lemma** the map $(1 + af, b) \rightarrow adb/b$ is well defined

**Proof:** More precisely we are going to prove the equality

$$(1 + fa, 1 + fb) = (1 + f(a + b), a + b) - (1 + fa, a) - (1 + fb)$$

Indeed we have

$$(1 + fa, 1 + fb) = (1 + fa, 1/e) + (1 + fa, e + fbe) = (1 + fa, 1/e) + (1 + fa, 1 - (1 - e - fbe)) = (1 + fa, 1/e) + ((1 + fa)(1 - e - fbe), 1 - (1 - e - abe))$$

Thus using the relation in the Milnor K-group we deduce

$$(1 + fa, 1 + fb) = (1 + fa, 1/e) + (1 - e + f(a(1 - e) - be), e + fbc)$$

Setting $e = a/a + b$ we can simplify, deducing

$$(1 + fa, 1 + fb) = (1 + fa, (a + b)/a) + (-b/a + b, 1 + fb) = (1 + f(a + b), a + b) - (1 + fa, a) - (1 + fb, b) + (-1, 1 + fb)$$

that is exactly what we claimed.  

II.2.ii **Lemma** for any $a, b \in A$ the relation $(1 + fa, 1 + fb) = 0$ holds in $K^2_M(A[f]/f^2)$

**Proof:** We know by the above lemma and the torsion relations,

$$(1 + fa, 1 + fb) = (1 + f(a - b), 1 + fb)$$
Applying the above lemma to both sides of the equality yields
\[(1 + fa, a) - (1 + f(a - b), a - b) - (1 + fb, b) = (1 + f(a + b), a + b) - (1 + fa, a) - (a + fb, b)\]
so that
\[2(1 + fa, a) = (1 + f(a + b), a + b) + (1 + f(a - b), a - b)\]
This is clearly equivalent to
\[2(1 + f(a + b)/2, (a + b)/2) = (1 + fa, a) + (1 + fb, b)\]
where the left hand side nothing is but \((1 + f(a + b), a + b)\). \(\square\)

II.2.iii **Lemma** Elements of the form \((1 + fa, q)\) with \(q \in \mathbb{Q}^*\) vanish in Milnor K-theory

**Proof:** It is clearly enough to prove that \((1 + fa, p)\) is a torsion element for \(p\) a prime number. As remarked earlier, computations will be completed modulo torsion. Let \(u, v \in \mathbb{Q}^*\) and consider the relation
\[n(1 + af, u) = -n(v, u) + n(v + fva, u) = n(v + fva, u(1 - v - fva)) = (v + fva, u^n(1 - v)^n - nu^{n-1}(1 - v)^{n-1}fva)\]
Choosing \(v = 1/1 - n\) so that \(1 - v = -nv\) we find out
\[n(1 + fa, u) = (v + fva, u^n(-n)^n v^n - nu^{n-1}(-n)^{n-1}v^n fa) = (v, u^n(-n)^n v^n) + (v, 1 + fa) + (1 + fa, u^n(-n)^n v^n)\]
Since \(K^2_M(\mathbb{Q})^0\) is torsion, \(K^2_M(\mathbb{Q}) = 0\). Thus the last expression equals, recalling \(v = 1/1 - n,\)
\[(1 + fa, u^n) + (1 + fa, v^{n-1}) + (1 + fn, (-n)^n) = (1 + fa, u^n) + (n - 1)(1 + fa, (1/n - 1)) + n(1 + fa, n)\]
Finally this implies the recursive formula
\[n(1 + fa, n) = (n - 1)(1 + fa, n - 1)\]
which proves the desired elements to be torsion by reducing to the case \(n = 1\) where the symbols are trivial by definition. \(\square\)

The three lemmas together easily imply Van der Kallen’s isomorphism. \(\square\)
The key point that makes the above isomorphism work, as also suggested by what the lemmas are proving, is the computation of the explicit set of generators. Such a description remains unchanged for $n > 2$ as outlined in proposition 1.3 at pg. 228 of [2]. Thus we can state

**II.3 Corollary** $N^n_p = \text{Ker}\{K^2_M(A[f]/f^{n+1}) \to K^2_M(A[f]/f^n)\}$ is generated by symbols of the form $(1 + cf^n, a)$ and $(1 + ef^n, 1 - f)$ for $a, e \in A^*$ and $c \in A$.

**Proof:** The proof reduces to Van der Kallen’s theorem as follows. Observe first that the map $p : TK^2_M(A[f]/f^{(n+1)}) \to N^n_p$ obtained by sending $\epsilon \to f^n$ is surjective. This is clear by way of the explicit set of generators for the kernel given at pg. 2.

Since $\Omega^1_{A[f]/f^{n+1}} \simeq \Omega^1_A \otimes Q[f]/f^{n+1} + A \otimes \Omega^1_{Q[f]/f^{n+1}}$

the theorem of Van der Kallen shows that $TK^2_M(A[f]/f^{n+1})$ is generated by $(1 + c\epsilon, a)$ and $(1 + c\epsilon, 1 - f)$ for $c, e \in A[f]/f^{n+1}$, $a \in A^*$. By the surjectivity of $p$ we are done. \(\square\)

To be honest, [3] requires also $e \in A$, not only invertible. Anyway, in our situation $A$ is local, so one among $e$ and $e + 1$ is invertible. Then the identity

$$(1 + (1 + e)f^n, 1 - f) = (1 + ef^n, 1 - f) + (1 + f^n, 1 - f)$$

casts any doubt away. Again, this is a result that remains true for all $p \geq 3$, namely $\text{Ker}(K^p_M(A[f]/f^{n+1}) \to K^p_M(A[f]/f^n))$ is generated by symbols of the form $(1 + cf^n, u_1, ..., u_{p-1})$ and $(1 + ef^n, 1 - f, u_2, ..., u_{p-1})$ for $u_i, e \in A^*$ and $c \in A$, since obviously

$\Omega^p_{A[f]/f^n} \simeq \Omega_A^p \otimes Q[f]/f^n + \Omega_{A}^{p-1} \otimes \Omega^1_{Q[f]/f^n}$

Now that it is clear what $N^n_p$ will be, we can state

**II.4 Theorem - The structure of the kernel**

*Notations as above, the map*

$N^n_p \to \Omega^{p-1}_{A/Q} \otimes_A (f^n/f^{n+1})$
given by \((1 + cf^n, u_1, ..., u_{p-1}) \rightarrow du_1/u_1 \wedge ... \wedge du_{p-1}/u_{p-1} \otimes cf^n\) is an isomorphism of \(A\)-modules. The module structure on \(N^p_n\) is exactly the same as that discussed above.

**Proof:** The guide line for the proof is clear. It will be enough to prove that II.4.i

**Claim Symbols** \((1 + cf^n, 1 - f)\) with \(e \in A\) vanish.

**Proof:** In the group \(K^2_M(A[[f]][1/f])\), the identity \((n + 1)(1 + cf^{n+1}, f) = -(1 + cf^{n+1}, c)\) arising from \((1 + cf^{n+1}, cf^{n+1}) = 0\) makes sense. Moreover \((1 + cf^{n+1}, f) = (1 - f(1 + cf^{n+1}), 1 + cf^{n+1} - cf^n)\) so that

\[
(1 + cf^{n+1}, c) = (n + 1)((1 - f)(1 + cf^{n+1}), 1 + cf^{n+1} - cf^n)
\]

(The above calculation is from p.229 of \([3]\)).

By way of theorem 3.6.1 from \([10]\), for \(R\) a local ring with characteristic zero residue field and \(g \in R\), the canonical map \(K^p_M(R) \rightarrow K^p_M(R_g)\) is injective. Applying this to the case \(R = A[[f]]\) and \(g = f\), the above formula (*) survives in \(K^2_M(A[[f]])\).

We can project (*) under the surjective map induced in Milnor K-theory by the canonical projection \(A[[f]] \rightarrow A[f]/f^{n+1}\), obtaining the sequence of identities in \(K^2_M(A[f]/f^{n+1})\)

\[
0 = (1, c) = (n + 1)(1 - f, 1 - cf^n) = (1 - f, (1 - cf^n)^{n+1}) = (1 - f, 1 - (n + 1)cf^n)
\]

This concludes the proof of the claim, and consequently of II.4.

We are now in position to prove Theorem II.1

**Proof of II.1**

**step1:** There is an exact sequence

\[
0 \rightarrow \bigoplus_{r+s=p-1} \Omega^r_k/Q \otimes \Omega^s_{O_H/k} \rightarrow \Omega^{p-1}_{O_H/Q} \rightarrow \Omega^{p-1}_{O_H/k} \rightarrow 0
\]

We can tensor this sequence by \(T^n/T^{n+1} = O_H(-nH)\).

Since \(O_H(-nH)\) is dual to an ample divisor on \(H\), by Kodaira vanishing and Serre duality we find

\[
H^q(H, \Omega^{p-1}_{O_H/k} \otimes O_H(-nH)) = 0
\]

for \(q + r \leq \dim H - 1\). Moreover since \(\Omega^r_k/Q\) is a constant sheaf, \(H^q(H, \Omega^r_k/Q \otimes \mathcal{W}) = \Omega^r_k/Q \otimes H^q(H, \mathcal{W})\) for any sheaf \(\mathcal{W}\). Taking the long cohomology sequence from above we get

\[
H^q(H, \Omega^{p-1}_{O_H/Q} \otimes O_H(-nH)) = 0
\]
as long as \( p + q \leq \dim H \).

**step 2:** form the long cohomology sequence induced by
\[
0 \to N_n^p \to K^p_M(\mathcal{O}_{H_n+1}) \to K^p_M(\mathcal{O}_{H_n}) \to 0
\]
and from the \( \mathcal{O}_H \)-modules isomorphism \( N_n^p \simeq \mathcal{O}_{\mathcal{O}_H} \otimes \mathcal{O}_H(-nH) \), step 1 implies that
\[
H^q(\mathcal{H}, K^p_M(\mathcal{O}_{H_n+1})) \to H^q(\mathcal{H}, K^p_M(\mathcal{O}_{H_n}))
\]
is an isomorphism for \( p + q \leq \dim H - 1 \) and is injective if \( p + q \leq \dim H \).

**step 3:** The natural map
\[
H^q(\hat{X}, \lim \leftarrow K^p_M(\mathcal{O}_{H_n})) \to \lim \leftarrow H^q(\mathcal{H}, K^p_M(\mathcal{O}_{H_n}))
\]
is an isomorphism for \( p + q \leq \dim H \). We show that the conditions of [7], theorem 13.3 of chapter 0 are satisfied. Indeed both the systems \( (K^p_M(\mathcal{O}_{H_n}))_n \) and \( H^r(\mathcal{H}, K^p_M(\mathcal{O}_{H_n})) \), for \( r + p \leq \dim H - 1 \) are surjective. Moreover a basis of the topology consisting of open affine sets trivializing the locally free sheaf \( \mathcal{O}_{\mathcal{O}_H/k}^{-1} \otimes \mathcal{O}_H(-nH) \) for all \( n \) clearly exists, and if \( U \) is such open set then \( H^i(U, \mathcal{O}_{\mathcal{O}_H/Q}^{-1} \otimes \mathcal{O}_H(-nH)|U) = 0 \) for \( i \geq 1 \) and \( n \geq 1 \). This concludes the proof, being then \( (H^q(U, K^p_M(\mathcal{O}_{H_n}))_n \) a system of isomorphic groups.

**step 4:** By step 4 we know \( \lim \leftarrow H^q(\mathcal{H}, K^p_M(\mathcal{O}_{H_n})) \simeq H^q(\hat{X}, \lim \leftarrow K^p_M(\mathcal{O}_{H_n})) \) for \( p + q \leq \dim H \) and by step 2 we know \( H^q(\mathcal{H}, K^p_M(\mathcal{O}_{H_n})) \simeq H^q(\mathcal{H}, K^p_M(\mathcal{O}_{H})) \) for \( p + q \leq \dim H - 1 \). Thus taking the inverse limit we find
\[
H^q(\hat{X}, \lim \leftarrow K^p_M(\mathcal{O}_{H_n})) \simeq H^q(\mathcal{H}, K^p_M(\mathcal{O}_{H}))
\]
if \( p + q \leq \dim H - 1 \).

Similarly if \( p + q = \dim H \) then we have an injection \( H^q(\mathcal{H}, K^p_M(\mathcal{O}_{H_n+1})) \hookrightarrow H^q(\mathcal{H}, K^p_M(\mathcal{O}_{H_n})) \) for every \( n \). Consequently
\[
\lim \leftarrow H^q(\mathcal{H}, K^p_M(\mathcal{O}_{H_n})) \hookrightarrow H^q(\mathcal{H}, K^p_M(\mathcal{O}_{H}))
\]
and thus by step 4 we find the required injectivity.

A couple of interesting consequences of the structure of the kernel are the following remarks, which highlight the bad behaviour of Milnor K theory with respect to inverse
II.6 Remark Given any smooth $\mathbb{Q}$-algebra of finite type $A$ and $x \in A$, the natural morphism

$$K^p_M(\lim_{\leftarrow n} A/x^n) \to \lim_{\leftarrow n} K^p_M(A/x^n)$$

is never injective for $p \geq 2$

**Proof:** First we prove it for $p = 2$ and $A = \mathbb{Q}[x]$. We have an exact sequence

$$0 \to \Omega^1_{\mathbb{Q} / \mathbb{Q}} \to K^2(\mathbb{Q}[x]/x^{n+1}) \to K^2((\mathbb{Q}[x]/x^n)) \to 0$$

meaning that $K^2(\mathbb{Q}[x]/x^n))$ does not depend on $n$. Whence

$$\lim_{\leftarrow n} K^2_M(\mathbb{Q}[x]/x^n) = K^2(\mathbb{Q}) = 0$$

On the other hand $\mathbb{Q}((x))$ is generated over $\mathbb{Q}$ by an uncountable set $x_a$ of algebraically independent elements, i.e. $\mathbb{Q}((x)) \mathbb{Q}(x_a)$. Considering the valuation associated to one of them, say $x_0$, we deduce that $K^2(\mathbb{Q}((x)))^0$ surjects onto the group of invertible elements of an uncountable field, namely $\mathbb{Q}(x_a)_{a \neq 0}$ whence $K^2(\mathbb{Q}((x)))$ surjects onto the same modulo the subgroup generated by the roots of unity. The roots of unity generate a countable subgroup, whence $K^2(\mathbb{Q}((x)))$ is still uncountable and the localization sequence

$$0 \to K^2(\mathbb{Q}[[x]]) \to K^2(\mathbb{Q}((x))) \to \mathbb{Q}^*/(\pm 1) \to 0$$

implies that the same is true for $K^2(\mathbb{Q}[[x]])$. This concludes the proof. The general case $p \geq 3$, and $A$ smooth of finite type follows verbatim.

II.7 Remark Let $A$ be a local, smooth noetherian $k$-algebra. Then Kerz’ localization sequence I.2.iii is never exact for the ring $A[T]/T^n$, $n \geq 2$

**Proof:** We start with $n = 2$. We have a diagram
with exact columns, and exact bottom row. The exactness of the middle row is equivalent to that of the first row, so it is enough to prove that the first row is never exact. To prove this, observe that there exists a unique inclusion \(Q[x]/Q\) \(\to A\) sending \(x \to g\). Exactness of the top row would give, by restriction, exactness of

\[
0 \to \Omega^p_{A/Q} \to \Omega^p_{A_q/Q} \to \Omega^{p-1}_{(A/g)/Q} \to 0
\]

which is visibly impossible. The case \(n > 2\) follows verbatim.

### 3. QUILLEN’S INFINITESIMAL LEFSCHETZ

In light of the proof of the infinitesimal Lefschetz isomorphism for Milnor K-theory, our task is to understand, given a nilpotent extension \(A \to A/I\) of a smooth local algebra \(A/I\), the homotopy type of the homotopy fiber of the map \(p : BGL(A)^+ \to BGL(A/I)^+\). Such homotopy type can be understood completely when the nilpotent extension is of the form \(A[f]/f^n \to A\). Denoting by \(K^p(A[f]/f^n, (f))\) the \(p\)-th relative homotopy group of \(BGL(A[f]/f^n)^+ \to BGL(A)^+\) we have:

**III.1 Theorem (Goodwillie-Hesselholt [9])**

There is an isomorphism of groups

\[
K^p(A[f]/f^n, (f))_Q \cong \oplus_{m \geq 1} \{\Omega^{p+1-2m}_{A/Q}\}^{n-1}
\]

As in the the case of Milnor K-theory, this isomorphism can be written as an isomorphism of sheaves on \(H\):
III.1.bis **Corollary**

Let $X$ be a smooth algebraic variety in char zero, $H$ a smooth ample divisor. Then there exists a canonical isomorphism

$$K^p(\mathcal{O}_{H_{n+1}}, \mathcal{O}_{H_n})_\mathbb{Q} \simeq \oplus_{m \geq 1} \Omega^{p+1-2m}_{\mathcal{O}_H/\mathbb{Q}}(−nH)$$

Moreover the long homotopy sequence of a fibration gives a long exact sequence of sheaves

$$\ldots \to K^{p+1}(\mathcal{O}_{H_n})_\mathbb{Q} \to \oplus_{m \geq 1} \Omega^{p+1-2m}_{\mathcal{O}_{H}^*/\mathbb{Q}}(−nH) \to K^p(\mathcal{O}_{H_{n+1}})_\mathbb{Q} \to K^p(\mathcal{O}_{H_n})_\mathbb{Q} \to \ldots$$

We see at once that the order of the differentials appearing in this long exact sequence fits perfectly with the Kodaira vanishing theorem, indeed repeating the argument in the first step of the proof of the infinitesimal Lefschetz theorem in Milnor K-theory, we obtain

$$H^q(H, \oplus_{m \geq 1} \Omega^{p+1-2m}_{\mathcal{O}_H/\mathbb{Q}}(−nH)) = 0$$

if $p + q \leq \dim H$.

The best possible we could hope at this point is that the long exact sequence splits into short exact sequences. This is indeed the case, by way of the smoothness hypothesis. Indeed, we recall the precise definition in order to make the argument more transparent.

**Definition** Let $k$ be a field. Then a $k$-algebra $A$ is said to be formally smooth if for any $k$-algebra $B$ and a nilpotent ideal $I \subset B$, every $k$-linear map $A \to B/I$ lifts to a map $A \to B$.

It follows that the natural projections $\mathcal{O}_{H_n} \to \mathcal{O}_H$ admits local sections, and in particular localizing to a point, we have the following

III.2 **Tautology**

If $A$ is the regular local ring of the smooth variety $X$ at a point of $H$, where $H$ is defined by $f = 0$, the map $A/f^n \to A/f$ admits a section.

it follows, by the functoriality of the K-theory, that the natural projection

$$K^p(\mathcal{O}_{H_n}) \to K^p(\mathcal{O}_H)$$

is surjective. Thus we can proceed by induction on $n$: 
The vertical map between differentials is surjective, and the map on the right is the identity. An easy diagram chase shows that also the map in the middle is surjective. We obtain,

III.3 Theorem
The long exact sequence for Quillen K-theory of infinitesimal neighborhoods splits into short exact sequences

$$0 \rightarrow \bigoplus_{m \geq 1} \Omega_{A/Q}^{p+1-2m} \rightarrow K^p(A[f]/f^{n+1})_Q \rightarrow K^p(A)_Q \rightarrow 0$$

The situation is then completely analogous to that of Milnor K-theory, and having noticed the compatibility of the rank of the differential forms appearing with the Kodaira vanishing the theorem, we finally copy-and-paste the proof of the infinitesimal Lefschetz in Milnor K-theory to get

III.4 Theorem - Quillen’s Infinitesimal Lefschetz
Let $X$ be a smooth algebraic variety of dimension $n$ in char zero, $H$ an ample divisor and $\hat{X}$ the formal completion along $H$. Then the map

$$H^q(\hat{X}, \lim_{\leftarrow} K^p(O_{H_n})_Q) \rightarrow H^q(H, K^p(O_H)_Q)$$

is an isomorphism if $p + q \leq n - 2$ and is injective if $p + q = n - 1$.

An interesting consequence is that the description of Goodwillie-Hesselholt permits an easy description of the ”continuous” K-theory of a smooth, complete, local algebra over a field of characteristic zero:

III.5 Theorem
Let $A$ be a local smooth $Q$-algebra, $f \in A$ and $A_n = A/f^n$. Then

$$\lim_{\leftarrow} K^p(A_n) \simeq \pi_p \lim_{\leftarrow} (BGL(A_n)^+)$$
The idea is that the plus-construction is well defined up to homotopy, and we can change the source of any morphism preserving its homotopy class, to turn the morphism into a fibration. Consequently we will define $\lim(BGL(A_n)^{+})$ as the inverse limit in the homotopy category, that is the homotopy class of the inverse limit of a sequence of fibrations homotopic to our original sequence.

More precisely, we have

III.5.i Claim Given any map $f : X \to Y$ there is a space $Z_f$ mapping to $Y$, endowed with an homotopy equivalence $X \sim Z_f$ such that $Z_f \to Y$ is a fibration. Moreover if we have a sequence of maps $X_{n+1} \to X_n$ we can turn them into a sequence $Z_{j_{p_n}} \to Z_{j_{p_{n-1}}}$ of fibrations with compatible homotopies $X_n \sim Z_{j_{p_{n-1}}}$.

III.5.ii Remark
In the sequel $\lim X_n$ will be understood as the homotopy class of the space $\lim Z_{j_{p_{n-1}}}$.

Proof of III.5.i: Start with any map $f : X \to Y$. Consider the space $Z = Hom(I, Y)$ naturally endowed with two maps $s, t : Z \to Y$, given by $s(h) = h(0)$ and $t(h) = h(1)$. Now we consider the fiber-square

$$
\begin{array}{ccc}
Z \times_Y X & \longrightarrow & Z \\
p_2 \downarrow & & s \downarrow \\
X & \longrightarrow & Y \\
t & & \\
\end{array}
$$

and let $Z_f$ denote the fiber product $Z \times_Y X$. Of course we can compute explicitly

$$
Z_f = \{(x, h) | f(x) = h(0)\}
$$

and deduce that there is a map $j : X \to Z_f$ sending $x$ to the pair $j(x) = (x, \text{constant path at } f(x))$. The crucial fact, that is easy to see, is that $p_2$ and $j$ are homotopy inverse to each other, even better $p_2$ deformation retracts $Z_f$ on $X$.

By construction it is clear that the endpoint morphism $t : Z_f \to Y$ is a fibration: explicitly to give a map $G : S \times I \to Y$ with a starting lift $\bar{g} : S \times \{0\} \to Z_f$ means that we have a
family of paths in $Y$ parametrized by $S$ and starting in $S$ (corresponding to $G$), and then another family of paths, parametrized by $S$ as well, ending in $S$ (corresponding to $\bar{y}$). With this in mind it is clear that we can lift the map $G$ to $3_f$, simply by first running in the second family of paths, then continuing along the first family progressively with respect to the time.

Now let $p_n : X_{n+1} \to X_n$ be a sequence of maps. Then we can turn them into a sequence of fibrations in the obvious manner: notations as above consider the diagram

$$
\begin{array}{c}
\cdots \longrightarrow 3_{j p_3} \xrightarrow{t} 3_{j p_2} \xrightarrow{t} 3_{p_1} \xrightarrow{t} X_1 \\
\downarrow j \quad \downarrow j \quad \downarrow j \quad \uparrow = \\
\cdots \longrightarrow X_4 \xrightarrow{p_3} X_3 \xrightarrow{p_2} X_2 \xrightarrow{p_1} X_1
\end{array}
$$

and observe that the up horizontal arrow is a sequence of fibration, with vertical arrows homotopy equivalences. We now construct homotopy equivalences $3_n \sim X_{n+1}$ which are compatible with the maps $t$ and $p_n$. This can be done by induction, the first step being $X_1 \to X_1$ the identity.

Assume we have a map $F_n : 3_n \times I \to 3_n$ such that $F_n(0)$ is the identity and $F_n(1)$ the canonical projection $3_n \to X_{n+1}$.

Consider the diagram

$$
\begin{array}{c}
3_{j p_{n+1}} \times \{0\} \xrightarrow{id} 3_{j p_{n+1}} \times \{0\} \xrightarrow{id} 3_{j p_{n+1}} \\
\downarrow \text{inclusion} \quad \downarrow \text{id} \quad \downarrow t \\
3_{j p_{n+1}} \times I \xrightarrow{t \times id} 3_{j p_n} \times I \xrightarrow{F_n} 3_{j p_n}
\end{array}
$$

The fibration property implies that there is a lift of $F_n$ to a map $G_{n+1} : 3_{j p_{n+1}} \times I \to 3_{j p_{n+1}}$ that has the property that the image of $t(G_{n+1}(1))$ is equal to $j(X_{n+1})$, by the above diagram. Thus $G_{n+1}$ is a retraction of $3_{j p_{n+1}}$ onto the pre-image $S_{n+1}$ of $j(X_{n+1})$ under $t$. Of course we can find a retraction $H_{n+1}$ that deformation retracts $S_{n+1}$ onto $j(X_{n+2})$. The
composition \( F_{n+1} = H_{n+1} G_{n+1} \) will be the required homotopy, that is it restricts to \( F_n \) on \( \mathfrak{Z}_{j F_n} \). This concludes the proof of the claim. \( \square \)

**Proof of III.5:** The above claim reduces the proof of III.5 to the following result:

**III.5.ii Fact 10** Let \( X_{n+1} \to X_n \) be a sequence of fibrations. Then for any space \( T \) (if \( \lim \leftarrow \) denotes the first left derived functor of \( \lim \)), the sequence of (pointed) sets

\[
0 \to \lim \leftarrow \pi_0 \text{Hom}(ST, X_n) \to \pi_0 \text{Hom}(T, \lim \leftarrow X_n) \to \lim \leftarrow \pi_0 \text{Hom}(T, X_n) \to 0
\]

is exact.

In our situation we let \( T = S^p \), \( X_n = BGL(A_n)^+ \), and indeed change the \( X_n \)'s up to homotopy to turn the sequence into a fibration as done previously. Particularly having defined \( \lim \leftarrow (BGL(A_n)^+) \) in the homotopy category, as the limit of the sequence of fibrations associated to \( X_n \), we only have to check that the \( \lim \leftarrow \) term vanishes. This follows by the surjectivity, III.3, of the maps \( \pi_{p+1} BGL(A_{n+1})^+ \to \pi_{p+1} BGL(A_n)^+ \). \( \square \)

**References**


