WEAK UNIFORMIZATION OF ZARISKI-OPEN SUBSETS OF ALGEBRAIC SURFACES

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Abstract. We review some complex-analytic properties of deformation spaces of nodal curves, as introduced by Bers. This leads to a simultaneous partial uniformization theorem for the smooth locus of, possibly nodal, curves, which has applications to the complex geometry of algebraic surfaces.

This work draws its motivation from the works of Griffiths, [Gr71], and Bogomolov-Husemoller, [BH00], where the common theme is the search for higher dimensional analogues of the uniformization theorem. In this perspective, those papers are transversal. Griffiths proves that given any point on a smooth projective variety, one can find a Zariski-open neighborhood which is Eilenberg-MacLane, and more precisely it admits a tower of smooth fibrations by algebraic curves. This is proven, on surfaces, by taking a general pencil of curves and removing divisors till the pencil is smooth, and in general by induction. Moreover, the uniformization, for such towers of smooth fibrations by curves, is Bers’ celebrated simultaneous uniformization theorem, that one-dimensional uniformization is uniform in moduli. On the other hand, Bogomolov-Husemoller are more interested in the global aspect of the uniformization problem, which is however hopeless in higher dimensions. Therefore, their point of view becomes that of searching for a minimal class of projective varieties, such that any other variety is dominated by one in this class. Here dominated means by a generically finite map, in particular not necessarily flat or unramified. Conjecturally, a candidate for such a dominant class would be that of towers of smooth fibrations by compact curves. For results in the direction of this difficult conjecture, see [BB17] and references therein. Summing up, towers of smooth fibrations are the main characters both in the usual uniformization theory, local version, and in the dominant classes theory, global version. This is not a coincidence, in that these special varieties are the most natural examples of higher dimensional anabelian varieties.

Here, we push the strategy of Griffiths in the direction of what is dreamt by Bogomolov-Husemoller.
Theorem 1. Let $X$ be a smooth projective surface. Then there exists a diagram

$$
\begin{array}{ccc}
\mathbb{C} \times \Delta & \xleftarrow{i} & Y \\
\downarrow{p_2} & & \downarrow{q} \\
\Delta & \xrightarrow{p} & X
\end{array}
$$

where: $Y$ is an affine complex surface; $p$ is unramified; $i$ is open; for $s \in \Delta$ arbitrary, $p(q^{-1}(s))$ is an algebraic curve, which coincides with the smooth locus of its compactification inside $X$; the map $p_s : q^{-1}(s) \rightarrow p(q^{-1}(s))$ is a covering map.

In other words, the result could be stated as follows: given a general 1-dimensional family of curves in $X$, we can construct a simultaneous partial uniformization of their smooth loci. We say partial because $q^{-1}(s)$ is open in $\mathbb{C}$, but not simply connected in general. In particular, for a sufficiently general choice of such family, the map $p$ is surjective, and in any case it is surjective onto the complement of an at worst finite subset of $X$. We can view our statement as a middle ground between Griffiths and Bogomolov-Husemoller: indeed we gain control over the global aspect - $p$ is surjective - but we lose control of the uniformizing objects - the structures of $i$ and $p$ are very intricated.

We can now discuss the technique of proof, whose details appear in Section II. What is required is a version of Bers’ theorem for families of curves degenerating to a nodal one. Certainly this cannot be as neat as the usual simultaneous uniformization - for example, the uniformizing group must jump on subsets of the parameter space corresponding to nodal curves. However this can be done, and an outline of a very elegant, purely group-theoretic construction, has been given by Bers in [Be81]. For a different, more geometric proof, see [Kra90]. Our Section II is devoted to explaining Bers’ construction, while filling missing details and completing vague sketches of proof. We can summarize it as follows. The crucial point is to understand what is the correct analogue of Teichmüller space when nodal curves are allowed. This analogue is a deformation space: for a nodal Riemann surface $\Sigma$, the deformation space is the space of continuous deformations $S \rightarrow \Sigma$ of nodal Riemann surfaces onto $\Sigma$, modulo a suitable equivalence relation. Here deformation just means that the map is generically one-to-one, and might contract some loops onto nodes. Now we have to endow the deformation spaces with a complex structure, which is sufficiently functorial so that the profusion of natural maps - between each other, and towards the compact moduli of curves - are holomorphic. Let us assume $\Sigma$ is terminal of genus $g$. There exists a Kleinian group $\Gamma$, freely generated by free groups $\Gamma_i$, $i = 1, \cdots, 2g - 2$ on
two generators, each preserving a disk $\Delta_i$ in the Riemann sphere, such that $\Gamma$ uniformizes the smooth locus of $\Sigma$. In particular, the $\Gamma$-orbits of $\Delta_i$ are pairwise disjoint. Profiting from Maskit's combination theorems, one can add transformations, $\delta_i$, $i = 1, \cdots, 3g - 3$ - each depending naturally on a complex parameter - such that the orbits of $\Delta_i$ under the action of $\Theta := \langle \Gamma, \delta_i \rangle$ do intersect. Hence there is a Kleinian subgroup $\text{Stab} \subset \Theta$, such that $\text{Stab} \cdot \Delta_i$ are not all pairwise disjoint, which moreover uniformizes a Riemann surface $S$ along with a deformation onto $\Sigma$. Pairwise intersecting orbits $\text{Stab} \cdot \Delta_i$ naturally correspond to thickenings of the nodes of $\Sigma$. The simultaneous embeddings:

$$\Gamma \hookrightarrow \Theta \hookrightarrow \text{Stab}$$

can be thought of as a simultaneous uniformization of the smooth loci of $S$ and $\Sigma$, together with a deformation $S \rightarrow \Sigma$. That every deformation can be obtained in this way is straightforward. The dependence of $\delta_i$ on complex parameters induces a natural, functorial complex structure on every deformation space, which turn out to be a bounded domains of holomorphy in $\mathbb{C}^{3g-3}$.

\section*{I. Bers' theory of deformation spaces}

Let $T_{g,p}$ be the Teichmüller space of genus $g$ and $p$ marked points. We will write $T_g = T_{g,0}$ to avoid redundant notation. The purpose of this section is to extend Bers' theory of simultaneous uniformization, to the more general realm of strong deformation spaces, defined as follows:

\textbf{Definition I.1} (Strong deformation). Let $\Sigma_1, \Sigma_2$ be surfaces with nodes of genus $g$. A strong deformation $f : \Sigma_1 \rightarrow \Sigma_2$ is a continuous surjective map such that

- nodes of $\Sigma_1$ are mapped to nodes of $\Sigma_2$;
- the pre-image of nodes of $\Sigma_2$ are either nodes of $\Sigma_1$ or simple closed curves of $\Sigma_1$;
- $f$ restricted to the complement of the pre-image of nodes of $\Sigma_2$ in $\Sigma_1$ is an orientation-preserving homeomorphism.

In what follows, a nodal surface $\Sigma_0$ will be called terminal if every strong deformation defined on $\Sigma_0$ is injective. Equivalently, $\Sigma_0$ is terminal if it has the maximum number of nodes.
In order to define strong deformation spaces, we need to introduce an equivalence relation between strong deformation maps:

**Definition 1.2** (Strong deformation space). Let $\Sigma$ be a surface with nodes and consider two strong deformations with $\Sigma$ as target space, namely $f : S_1 \to \Sigma$ and $g : S_2 \to \Sigma$. We say $f$ and $g$ are equivalent if there exists a commutative diagram

$$
\begin{array}{c}
S_1 \xrightarrow{f} \Sigma \\
\downarrow \rho \quad \downarrow \psi \\
S_2 \xrightarrow{g} \Sigma
\end{array}
$$

such that $\psi$ is a homeomorphism of $\Sigma$ homotopic to the identity and the induced map $\rho$ is a conformal surjective map. We define the strong deformation space of $\Sigma$, $\mathcal{D}(\Sigma)$, to be the space of strong deformation maps onto $\Sigma$ modulo the equivalence relation defined above.

The topology on $\mathcal{D}(\Sigma)$ can be described in the following way (see [Be81]): a set $U$ is open if for every point $[f : S_1 \to \Sigma]$ there exist a set of curves $C_1, ..., C_r$ and $\epsilon > 0$ such that if the deformation map $h : S_0 \to S_1$ is such that

$$
|l_{S_0}(h^{-1}(C_j)) - l_{S_1}(C_j)| < \epsilon
$$

for $j = 1, ..., r$

$$
|l_{S_0}(h^{-1}(Q))| < \epsilon
$$

for all nodes $Q$ of $S_1$

then $[f \circ h : S_0 \to \Sigma]$ is in $U$. Here the length of the curve is taken as the infimum among all the lengths in the free homotopy class. Note that with this topology, $\mathcal{D}(\Sigma)$ is connected.

A central role in the following construction will be played by Kleinian groups, of which we quickly recall some properties. A Kleinian group $G$ is a discrete subgroup of $\text{PSL}(2, \mathbb{C})$ acting properly discontinuously on an open subset of $\hat{\mathbb{C}}$. The action on the sphere is induced by the action on the hyperbolic space (seen as a subset of the quaternions) in the following way:

$$
\begin{bmatrix} a & b \\ c & d \end{bmatrix} q = \frac{aq + b}{cq + d}, \quad q \in \mathbb{H}^3 = \{x + iy + jt \mid x, y \in \mathbb{R}, t > 0\}
$$

As this action leaves $\partial \mathbb{H}^3 = \hat{\mathbb{C}}$ invariant, it is useful to consider its restriction to it. In fact, by a result of Poincaré, Kleinian groups act properly discontinuously on $\mathbb{H}^3$ thus
the orbits have a limit set $\Lambda(\Gamma)$ on the boundary. By the definition of Kleinian group, its complement in $\hat{C}$ is nonempty and we call its interior $\Omega(\Gamma)$ the region of discontinuity of $\Gamma$.

In general, $\mathbb{H}^3/\Gamma$ is a 3-manifold, with $\Omega(\Gamma)/\Gamma$ being its boundary. The latter space is therefore a union of connected Riemann surfaces. The following fundamental result from Ahlfors tells us that for finitely generated groups the number of surfaces has to be finite:

**Theorem I.3 (Ahlfors finiteness Theorem).** Let $\Gamma$ be a finitely generated Kleinian group and $\Omega(\Gamma)$ its region of discontinuity. Then $\Omega(\Gamma)/\Gamma$ has a finite number of connected components, each of which is a Riemann surface of finite area.

In general, $\Omega(\Gamma)$ has different orbits for the action of $\Gamma$, $\{\Omega_1, ..., \Omega_n\}$, each with a stabilizer $\Gamma_i \subset \Gamma$, such that each connected surface is homeomorphic to the quotient $\Omega_i/\Gamma_i$. If $\Gamma_i = \Gamma$ for some $i$, we say that $\Omega_i$ is an invariant component. Notice that Fuchsian groups are a particular kind of Kleinian groups having two invariant components: the upper and the lower half plane. In general, groups whose domain of discontinuity consist of two invariant components are called quasi-Fuchsian. In the discussion that follows, we will study Riemann surfaces through the action of Kleinian groups on $\hat{C}$.

It is a classical result of Bers (see [Be81]) that any $T_{g,p}$ can be embedded, as a bounded domain of holomorphy, in $\mathbb{C}^{3g-3+p}$, so that a natural complex structure can be defined on the Teichmüller space. Its natural topological boundary $\partial T_{g,p}$ can be studied through boundary groups: more precisely, any point of $T_{g,p}$ can be identified with a Kleinian group derived from the conjugation of a Fuchsian group with a quasi-conformal map, thus a quasi-Fuchsian group. However, points at the boundary of the Teichmüller space are represented by Kleinian groups that are, in general, not quasi-Fuchsian. Despite the very interesting geometry of such boundary groups, a serious drawback is that $T_{g,p} \cup \partial T_{g,p}$ does not have the structure of a complex orbifold in any reasonable sense. On the other hand, the compactified moduli space $\bar{\mathcal{M}}_{g,p}$ is a compact orbifold. Since the moduli space $\mathcal{M}_{g,p}$ is obtained by quotienting $T_{g,p}$ by the action of its modular group, it is natural to try and seek for a similar structure on $\mathcal{M}_{g,p}$. In particular, we look for an open complex manifold, with an orbifold covering map onto an open $\mathcal{U} \subset \mathcal{M}_{g,p}$ with $\mathcal{U} \cap (\bar{\mathcal{M}}_{g,p}\setminus\mathcal{M}_{g,p})$ as big as possible. This would follow from the existence of a sufficiently functorial complex structure on the
previously-mentioned strong deformation spaces. By sufficiently functorial, we mean that all the possible tautological maps between various deformation / moduli spaces should be holomorphic. The main result of this section, Theorem I.6 stated below, it is that this is true when $p = 0$.

The literature on the subject contains a profusion of manuscripts, for example from Bers [Be81], [Be74], [Be74'], Maskit [Ma88] and Kra [Kra90]. While Kra proves the result through a so-called plumbing procedure, we will follow Bers’ approach, as it can be found in [Be81]. The reason for this is that, while Kra’s approach is rather explicit and constructive, Bers’ proof is purely group-theoretic. Moreover, his proof is rather a sketch of proof, and certainly it is interesting to review his argument while adding some missing details.

The remaining technical tools that are needed in order to understand the proofs are stated below without proof. In particular, in the context of finitely generated Kleinian groups, a key role is played by Maskit first and second combination theorems, which, in the form that follows, are tailored to our needs (we refer to [Ma88] for the general case).

**Theorem I.4** (Maskit first combination). Let $G_1$ and $G_2$ be two discrete groups in $PSL(2, \mathbb{C})$ and let $W$ be a simple closed curve dividing $\hat{C}$ into two simply connected sets, $B_1 \subset \Omega(G_1)$ and $B_2 \subset \Omega(G_2)$. Then if for each $i \neq j$, $G_i$ minus the identity maps $B_i$ to $B_j$ and for at least one $i$ there is a point in $B_i$ which is not the image of any point in $B_j$ through the action of $G_j$, then $G := \langle G_1, G_2 \rangle$ is the free product of the two groups and is a discrete group. Moreover, if $D_i$ is a fundamental set for $G_i$, maximal with respect to $B_i$, such that $D_i \cap B_j$ is either empty or has non-empty interior and $D_1 \cap W = D_2 \cap W$, then $D := (D_1 \cap B_2) \cup (D_2 \cap B_1)$ is a fundamental domain for $G$.

**Theorem I.5** (Maskit second combination). Let $J_1$ and $J_2$ be geometrically finite subgroups of a discrete group $G \subset PSL(2, \mathbb{C})$ and let an element $\delta$ have infinite order in $PSL(2, \mathbb{C})$. Let there be two simply connected closed sets $B_1$ and $B_2$, such that they are precisely invariant respectively under $J_1$ and $J_2$, $\delta$ maps the interior of $B_1$ to the exterior of $B_2$ and $\delta^{-1}J_2\delta = J_1$. Moreover, call $D_0$ the fundamental domain of $G$ and $A$ the common exterior of $B_1$ and $B_2$. Then $G' := \langle G, \delta \rangle = G \ast \delta$ (HNN extension), $G'$ is a discrete group and $D = D_0 \cap (A \cup \partial B_1)$ is a fundamental set for $G'$, where $D \cap \partial B_1$ and $D \cap \partial B_2$ are identified.

We are ready to state Bers’ theorem, whose proof occupies the rest of the present Section:
Theorem I.6. [Be81, Bers] If $\Sigma$ is a terminal surface of genus $g$, $\mathcal{D}(\Sigma)$ can be embedded as a bounded domain of holomorphy in $\mathbb{C}^{3g-3}$.

The idea of the proof is as follows: first, we pick a Kleinian group $\Gamma$ representing the surface $\Sigma$; then we describe a general procedure to modify $\Gamma$ into a Kleinian group representing a given point in $\mathcal{D}(\Sigma)$. Critically, such procedure will be parametrized by a bounded subset in a finite-dimensional complex vector space of expected dimension $3g-3$. Let us start by recalling that the fundamental domain of $\Gamma(2)$, the principal congruence subgroup modulo 2 of $\text{PSL}(2, \mathbb{Z})$, is given by:

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{}
\end{figure}

where the region labeled as 1 is the fundamental region of $\text{PSL}(2, \mathbb{Z})$ itself, and the other five hyperbolic triangles are obtained by the action on such region by representatives of $\text{PSL}(2, \mathbb{Z})/\Gamma(2)$. Such group has signature $(0, 3, \infty, \infty, \infty)$, meaning that $\mathbb{H}/\Gamma(2)$ is a thrice
punctured sphere, with the punctures being cusps. From now on, we want to keep track of changes in the fundamental domains and domains of discontinuity of those Kleinian group that will appear. Observe that if $D$ is a fundamental domain for a Fuchsian group on the upper half plane model and $\bar{D}$ is its reflection through the real axis, then $D \cup \bar{D}$ is the fundamental domain of the same group when considered as a Kleinian group acting on its domain of discontinuity inside the compactified complex plane. In the following discussion, we will denote by $D$ the fundamental domains, and by $\Omega$ the domains of discontinuity.

Recall that any terminal surface $\Sigma$ of genus $g$ can be seen as the union of $2g$ thrice punctured spheres, where the punctures are paired, each pair representing a node. Our current purpose is to construct a Kleinian group $\Gamma$ representing such disjoint union of $2g-2$, thrice punctured spheres.

We take $2g-2$ disjoint disks $\Delta_j$ and we consider a group $\Gamma_j$ generated by three parabolic elements $\gamma_{j1}, \gamma_{j2}, \gamma_{j3}$ fixing three different boundary points, respectively $Q_{j1}, Q_{j2}, Q_{j3}$, so that $\gamma_{j1} \circ \gamma_{j2} \circ \gamma_{j3} = I$. This group is conjugated to $\Gamma(2)$ through a transformation sending $\partial \Delta_j$ to the compactified real line and thus the fundamental domain passing through $Q_{j1}, Q_{j2}$ and $Q_{j3}$ to the one in figure 1. We denote by $\alpha_{jv}$ such a transformation, i.e. such that $\alpha_{jv} \circ \Gamma_j \circ \alpha_{jv}^{-1} = \Gamma(2)$, along with the property of sending $Q_{jv}$ to $\infty$.

Let $\Gamma := \langle \Gamma_1, ..., \Gamma_{2g-2} \rangle$. We have:

**Lemma I.7.** $\Gamma = \Gamma_1 \ast \ldots \ast \Gamma_{2g-2}$, is a Kleinian group and has one non-simply connected invariant component $\Omega_0$ such that $\Omega_0/\Gamma$ is a sphere with $6g-6$ punctures.

**Proof.** The proof of this lemma is simply an application of Maskit’s first combination theorem, [2.4]. We start with two disks to show the process and we get the general case by induction. Consider the disjoint disks $\Delta_1$ and $\Delta_2$ and the respective (Fuchsian) fundamental domains $D_1$ and $D_2$ of $\Gamma_1$ and $\Gamma_2$ inside such disks. Let $\bar{D}_1$ and $\bar{D}_2$ be the respective reflections of the fundamental domains through the circles. Note that we can always choose $\Delta_1$ and $\Delta_2$ so that the boundaries of such reflections don’t intersect. Let $W_1$ and $W_2$ be two circles separating the boundaries of $\bar{D}_1$ and $\bar{D}_2$ as in the figure 2.

We now consider $U$ as the disk inside $W_2$ and $V$ as the interior of its complement in $\hat{C}$. We have the following actions:

- $\Gamma_1 \setminus \{e\}$ sends $U$ to $V$ since $U \subset \bar{D}_1$;
- $\Gamma_2 \setminus \{e\}$ sends $V$ to $U$ since $V \subset \bar{D}_2$;
the only element fixing $U$ and $V$ is the identity;

- there are points in $U$ inside $\Delta_2$ that cannot be the image of a point in $V$ through an element of $\Gamma_2$.

Therefore by Maskit’s first combination theorem we have that $\langle \Gamma_1, \Gamma_2 \rangle = \Gamma_1 \ast \Gamma_2$, that this is a Kleinian group and that the fundamental domain is given by $D = D_1 \cup D_2 \cup A$, i.e. the white portion in figure 2.

The inductive step follows the same pattern: for an additional disk $\Delta_j$, consider the respective $W_j$ constructed as above which divides $\hat{C}$ into another so called interactive pair. The fundamental set $D$ can be visualized as in the case of two disks, while the domain of discontinuity is obtained by $\Omega(\Gamma) = \bigcup_{g \in \Gamma} gD$. We can also observe that $\Omega(\Gamma)$ has one connected, non-simply connected invariant component corresponding to the orbit of $A = \bar{D}_1 \cap \bar{D}_2$. Note that it is impossible for an element of $\Gamma$ to map $A$ inside one of the disks $\Delta_i$, since each $\Gamma_i$ tessellates $\bar{D}_i$ on $\hat{C} \setminus \Delta_i$, therefore $\Gamma_iA$ will be in the shadowed area outside the disk $\Delta_i$ and the orbit will be non-simply connected. Moreover, the other
components of $\Omega(\Gamma)$ will be the original disks $\Delta_j$ and each of them having stabilizer $\Gamma_j$ and representing a different thrice punctured sphere. □

Let us form $3g-3$ pairs of points $Q_{j^v}$ and $Q_{j^{v'}}$, with $j$ taking values in $\{1,\ldots,2g-2\}$ and $v$ in $\{1,2,3\}$. This way, points in a pair are identified to define one of the nodes in $\Sigma$. In what follows, we construct elements of $\mathcal{H}(\Sigma)$ through a thickening procedure parametrised by $3g-3$ complex numbers of absolute value less than one. Such construction will be purely algebraic, as we will only add group elements to $\Gamma$ to obtain Kleinian groups representing surfaces.

For $\zeta := (\zeta_1,\ldots,\zeta_{3g-3})$ in $\mathbb{C}^{3g-3}$, and any choice of logarithm, we define holomorphic functions $\beta_i : \mathbb{C} \to \mathbb{C}$ as follows:

\[
\beta_i(z) = \begin{cases} 
  z & \text{if } \zeta_i = 0 \\
  -z - i \log \zeta_i & \text{if } \zeta_i \neq 0
\end{cases}
\]

Let $\delta_i(\zeta) := \alpha_{j^v}^{-1} \circ \beta_i \circ \alpha_{j^v}$ be the element which associates a given pair $(j^v,j^{v'})$ to the $i$-th coordinate. For $\zeta_i \neq 0$, $\delta_i$ sends the interior of $\Delta_j$ to the upper half plane through $\alpha_{j^v}$, then the upper half plane to $\text{Im}(z) < \ln \frac{1}{|\zeta_i|}$ through $\beta_i$, and finally this domain to the set outside a horocycle in $\Delta_j$ through $\alpha_{j^v}^{-1}$.

We now define the space $\mathcal{H}(\Sigma)$ as a subset of the unit ball in $\mathbb{C}^{3g-3}$, as follows: we say that $\zeta \in \mathbb{C}^{3g-3}$, with $|\zeta| < 1$, belongs to $\mathcal{H}(\Sigma)$ if, for every $\zeta_i \neq 0$, there exist two simple closed curves $K_{j^v}$ and $K_{j^{v'}}$ with the following properties:

- each curve contains the respective point $Q_{j^v}$ or $Q_{j^{v'}}$ and is otherwise contained in $\Delta_j$ (or $\Delta_j'$);
- the interior $B$ of each curve is precisely invariant under the group generated by $\gamma_{j^v}$ ($\gamma_{j^{v'}}$), that is, such group is its stabiliser and for any other element $g$ of $\Gamma$, $gB \cap B = \emptyset$;
- $\delta_i$ maps the exterior of $K_{j^{v'}}$ onto the interior of $K_{j^v}$;
- curves on the same disk $\Delta_j$ are disjoint when projected onto $\Delta_j/\Gamma_j$.

We note that indeed there exist $\zeta$ satisfying the above requisites. In fact, it can be seen from the following picture that a small horocycle based at $Q_{j^v}$ mapped by $\alpha_{j^v}$ to an horizontal line, is mapped through $\beta_i$ to another horizontal line and finally by $\alpha_{j^{v'}}^{-1}$ to another
horocycle based at $Q_{j\nu'}$. The first condition is verified if the image through $\beta_i \circ \alpha_j^{\nu'}$ of the first horocycle ($K^\nu_{j_1}$ in our picture) is contained in the strip $\text{Im}(z) \in (0, \ln \frac{1}{|\zeta_i|})$, which is more plausible the smaller the modulo of $\zeta_i$. The second condition is verified for small horocycles based at $Q_{j\nu}$, while the third condition simply explains the reasoning behind the definition of $\delta_i$. The last one becomes "increasingly important" as $|\zeta_i|$ grows, since larger curves defined on the same $\Delta_j$ could intersect each other on the fundamental domain. Thus there will be some $\zeta$ in the unit ball which does not meet this requisite, and in fact the boundary of $\mathcal{D}(\Sigma)$ has Hausdorff dimension strictly bigger than its topological dimension $3g - 4$. The following picture explains the definition given above.

For each $\zeta$ in $\mathcal{D}(\Sigma)$, construct the group $\Theta_{\zeta_1, \ldots, \zeta_{3g-3}}:=<\Gamma, \delta_1, \ldots, \delta_{3g-3}>$. We claim that such group is a Kleinian group representing a surface with nodes of genus $g$.

**Lemma I.8.** For each $\zeta \in \mathcal{D}(\Sigma)$, $\Theta(\zeta_1, \ldots, \zeta_{3g-3})$ is a geometrically finite Kleinian group with $N+1$ nonconjugate components, where $N$ is the number of connected components of $\Sigma$ obtained by removing the nodes with corresponding $\zeta_i$ equal to 0. Those $\zeta_i \neq 0$ determine a thickening of the corresponding node in $\Sigma$. In case $\zeta_i \neq 0$, the corresponding curves $K_{j\nu}$ and $K_{j'\nu}$ project onto one simple closed curve in the thickening, and two such curves never intersect unless they coincide.

**Proof.** The proof of this theorem is an application of Maskit’s second combination theorem, I.5. We start with the simple case of two disks and generalize by induction. According to the construction of $\Gamma$, consider $\Gamma_1$ and $\Gamma_2$, the subgroups of $\Gamma$, stabilizing $\Delta_1$ and $\Delta_2$ respectively. For simplicity, we consider the case of two disks, say the one corresponding to the pair $(Q_{11}, Q_{21})$, so then $\zeta_i \neq 0$ for some $i$ corresponding to our pair, and $\zeta_j = 0$ for $j \neq i$. Define $J_1 = <\gamma_{11}>$ and $J_2 = <\gamma_{21}>$. In our case, since $\zeta \in \mathcal{D}(\Sigma)$, there exist curves $K_{11}$ and $K_{21}$ that satisfy the following conditions:

- the simply connected part in $C$, bounded by $K_{11}$ ($K_{21}$), along with the boundary, is precisely invariant under $J_1$ ($J_2$ respectively);
- the action of $\delta_1$ maps the interior of $K_{11}$ to the exterior of $K_{21}$;
- by definition of the element $\delta$, $\delta_1^{-1}J_2\delta_1$ fixes $Q_{11}$, so that it is either the group $J_1$ or a conjugate of it. Given its action on $K_{11}$, we have $\delta_1^{-1}J_2\delta_1 = J_1$.

This way, we can apply Maskit’s second combination theorem and have that $\Theta(\zeta_1, \ldots, \zeta_{3g-3})$ is a Kleinian group obtained by the HNN extension of $\Gamma$ by $\delta_1$. The projections of the two
curves on the respective fundamental domains are identified and the part in white in figure 4 represents a fundamental domain $D$ for this new group, where the boundary of the curves $K$ are identified in the respective fundamental domains:

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure4.png}
\caption{Figure 4.}
\end{figure}

The fact that two different curves never intersect comes from the definition of the space $\mathcal{D}(\Sigma)$. Finally, the induction step follows by taking, for each $\zeta_i \neq 0$, $G = \Theta(\zeta_1, ..., \zeta_{i-1}, 0, 0, 0)$ and $G' = \langle G, \delta_i \rangle = \Theta(\zeta_1, ..., \zeta_{i-1}, \zeta_i, 0, 0)$. \hfill \Box

Let us look closer at the two disks appearing in the first step of the previous induction. Through the identification of $D_1 \cap K_{11}$ with $D_2 \cap K_{21}$ we are now generating a new orbit, within the discontinuity set, that unifies those (previously disjoint) orbits corresponding to the thrice punctured spheres. Such new orbit defines some conformal structure on a sphere with four punctures. In fact, $\delta_1$ maps the white region contained in $\Delta_2$ - i.e. the part of the fundamental domain of $\Gamma_2$ lying inside $\Delta_2$ and outside $K_{21}$ - to one lying in the domain bounded by $K_{11}$. Therefore, the Ahlfors decomposition consists of two orbits $\Omega_0$ and $\Omega_1$, where the stabiliser of $\Omega_0$ is $\Gamma$, the amalgamated free product of $\Gamma_1$ and $\Gamma_2$, while the stabiliser of $\Omega_1$ represents a new surface. This way we can think of $\Theta(\zeta)$ as uniformizing simultaneously $\Omega_0/\Gamma$ and $\Omega_1/\text{Stab.}$
The next step in the process is to define a function $\phi : \mathcal{P}(\Sigma) \to \mathcal{P}(\Sigma)$ whose inverse is the desired embedding. In particular, observe that for each $\zeta$ we get a group $\Theta(\zeta)$, such that $\Omega(\Theta(\zeta))/\Theta(\zeta)$ is a disjoint union of surface with nodes $(\Omega_0/\Gamma) \cup \Sigma(\zeta)$. Furthermore, such construction enables us to identify pairwise disjoint simple closed curves on the surface, given by the corresponding identifications of the respective $K_j^\nu(\zeta)$. Another important consequence is that the surface $\Sigma(\zeta)$ has a conformal structure depending on $\zeta$. Note that a priori, in the proof of the previous Lemma the conformal structure of the resulting surface would depend on the choice of the curves. It is crucial to observe that the group $\Theta(\zeta)$ is defined independently of the choice of feasible curves $K(\zeta)$, so then the conformal structure on $\Sigma(\zeta)$ will no depend on the choice of the curves either. Therefore, $\Theta(\zeta)$ provides us with a surface with nodes, a conformal structure, and a collection of pairwise disjoint simple closed curves on it. It is a basic result that there are pairwise disjoint, simple closed geodesics homotopic to the configuration given by $K(\zeta)$.

We now define a function $\phi$ which associates to each $\zeta$ in $\mathcal{P}(\Sigma)$ a strong deformation $[f : \Sigma(\zeta) \to \Sigma]$ sending those geodesics to nodes, and the connected component bounded by three geodesics (which might be degenerate, if the corresponding $\zeta_i$ is 0) onto the corresponding thrice punctured sphere. More precisely:

**Definition 1.9.** We define $\phi : \mathcal{P}(\Sigma) \to \mathcal{P}(\Sigma)$ by the rule $\phi(\zeta) := [f : \Sigma(\zeta) \to \Sigma]$, where $f$ maps the interior of each pair of pants of $\Sigma(\zeta)$, determined by the geodesic curves $K(\zeta)$, homeomorphically to the corresponding thrice punctured sphere of $\Sigma$, and the curves $K(\zeta)$ to the corresponding nodes of $\Sigma$.

Note that since the modular group of a thrice punctured sphere is generated by permutation of the punctures, there is only one possible homeomorphism, up to homotopy equivalence, from the interior of a pair of pants to a thrice punctured sphere fixing the boundary as described above. The function $\phi$ is therefore well defined.

It is easy to check that $\phi$ is a continuous function away from the hyperplanes $\zeta_i = 0$. This is given by the fact that for each sequence of $\zeta_i$ not going to 0 the choice of curves is bounded within a strip. However, when some $\zeta_i$ tends to 0, even though there is no bound on the curve that could grant us a limit curve, the group degenerates to represent a node on the corresponding pair $(Q_j^\nu, Q_j^\nu)$. This happens because the structure of the surface is independent on the choice of curves and for $\zeta_i$ decreasing to 0, despite $\delta_i$ not being continuous, we can choose a sequence of $(K_j^\nu, K_j^\nu)$ whose hyperbolic lengths decrease to
0. The continuity of \( \phi \), in the above-mentioned topology of \( \mathscr{D}(\Sigma) \), holds true everywhere. We prove now that \( \phi \) is surjective.

**Lemma I.10.** \( \phi \) is open and closed, hence onto.

**Proof.** In order to prove openness, consider \( \phi(\zeta_0) = [f : \Sigma(\zeta_0) \to \Sigma] \). The transformations \( \delta_i(\zeta) \) define, for \( \zeta \) in a sufficiently small open neighborhood \( U \) of \( \zeta_0 \), a \( 3g - 3 \)-dimensional local subvariety of \( \mathcal{M}_g \) containing \( \Sigma(\zeta) \). Such subvariety must, therefore, be an open neighborhood of \( [\Sigma(\zeta)] \). It follows that \( U \subset \mathcal{D}_1 \).

As for closedness, consider a sequence \( \{\zeta^n\}_{n \in \mathbb{N}} \) such that \( \phi(\zeta^n) \to x \). Since \( \mathcal{D}_1 \) is compact in \( \mathbb{C}^{3g-3} \), we have that, up to a subsequence, \( \zeta^n \to \zeta \). We want to prove that \( \phi(\zeta) = x \), so that \( x \) is in the image of \( \phi \). By continuity, it is enough to show that \( \zeta \in \mathcal{D}_1(\Sigma) \). The existence of \( x \) implies that, at least for \( n \) large enough, curves homotopic to \( K_{j_v}(\zeta^n) \) accumulate to some, possibly degenerate, pairwise disjoint curves \( K_{j_v} \). Indeed, the contraction of their image inside \( S \) defines the point \( x \in \mathcal{D}_1(\Sigma) \). We have two cases:

- \( \zeta_i = 0 \);
- \( \zeta_i \neq 0 \).

In the first case, there is a choice of the corresponding curves \( K_{j_v}(\zeta^n) \) shrinking towards \( Q_{j_v} \), which therefore coincides with (degenerate, in this case) limit curve \( K_{j_v} \).

In the second case, we have \( \beta_i(\zeta^n) \to \beta_i(\zeta) + C \), where \( C \in \{0, 2\pi\} \) is appearing because of the lack of continuity of any choice of logarithm. Certainly \( K_{j_v} \) are \( \gamma_{j_v} \)-invariant, and contain \( Q_{j_v} \). We now check that \( \delta_i(\zeta) \) sends the interior of either one to the exterior of the other: the issue is that \( \delta_i \) fails to be continuous when \( \zeta_i \) approaches the real axis. However, the curves \( K_{j_v} \) are \( \gamma_{j_v} \)-invariant, and \( \gamma_{j_v} \) acts, on the upper half plane model, as a real translation, which we can normalize to be \( z \to z + 2\pi \). Our claim follows.

The only issue that might be left is, that \( K_{j_v} = \partial\Delta_j \) might happen. This is however impossible: for \( w \neq v \), \( Q_{j_w} \cap K_{j_v} \cap \partial\Delta_j \), while \( K_{j_v} \) and \( K_{j_w} \) project to disjoint curves in \( S \). This proves that \( \zeta \in \mathcal{D}(\Sigma) \), and concludes the proof of the Lemma. \( \square \)

On the other hand, we have:

**Lemma I.11.** \( \phi \) is injective.

**Proof.** Let \( H \) denote the restriction of \( 3g - 3 \) coordinate hyperplanes to \( \mathcal{D}(\Sigma) \). The usual Teichmuller space \( T_g \) is a universal cover of \( \mathcal{D}(\Sigma) \setminus H \). This way, it is easy to see that
Maskit’s coordinates, [Ma74], defined on $T_g$, provide an inverse to $\phi$ defined on $\mathcal{D}(\Sigma)\setminus H$. Since everything is Hausdorff, $\phi$ must be injective.

The embedding $\mathcal{D}(\Sigma) \to \mathbb{C}^{3g-3}$, provided by $\phi^{-1}$, gives the deformation space the structure of a bounded domain. We furthermore have:

**Proposition I.12.** $\mathcal{D}(\Sigma)$ is a domain of holomorphy.

**Proof.** We use Hejhal’s trick, [Hej75]. Denote by $H$ the union of the $3g - 3$ coordinate hyperplanes. Let $T_g$ the usual Teichmüller space of genus $g$, so then there is a covering map $u: T_g \to \mathcal{D}(\Sigma)\setminus H$ giving $T_g$ the structure of a Riemann domain. Quite generally, if $U \to \mathbb{C}^n$ is a Riemann domain with relatively compact image, denote by $d_U$ the induced distance to the boundary. Clearly we have $d_{T_g} = d_{\mathcal{D}(\Sigma)\setminus H} \circ u$. Since $T_g$ is a domain of holomorphy, $\log d_{T_g}^{-1}$ is psh, so the same holds for $\log d_{\mathcal{D}(\Sigma)\setminus H}^{-1}$, which implies that $\mathcal{D}(\Sigma)\setminus H$ is a domain of holomorphy, hence holomorphically convex. However, holomorphic convexity is characterized by the existence, locally around every point of the boundary, of unbounded holomorphic functions. It is clear then that $\mathcal{D}(\Sigma)$ is also holomorphically convex, hence a domain of holomorphy. □

We point out that the natural forgetful map $\mathcal{D}(\Sigma) \to \overline{\mathcal{M}}_g$ is holomorphic, the image contains $\mathcal{M}_g$, as well as the subset parametrizing irreducible curves - which will be proved in the next Section.

Let us conclude by introducing the notion of Bers fiber space. The homeomorphism $\phi$ previously considered, provides a purely group theoretic algorithm to reconstruct a deformation. Given $\zeta$, we build $\Theta(\zeta)$, along with an action on $\hat{\mathcal{C}}$ and corresponding region of discontinuity $\Omega(\Theta(\zeta))$. Denote by $\Omega^0(\Theta(\zeta)) \subset \Omega(\Theta(\zeta))$ the complement of the orbit of $\Omega(\Theta(\zeta))_0$, notation from Lemma [L8] with its elliptic fixed points further removed. If we glue the images of pairs of marked points in $\Omega^0(\Theta(\zeta))/\text{Stab}$, tautologically we obtain $\Sigma(\zeta)$.

**Definition I.13.** The Bers fiber space is the natural submersion $\Omega^0(\Sigma) \to \mathcal{D}(\Sigma)$, where

$$\Omega^0(\Sigma) := \{(\zeta, z) | \zeta \in \mathcal{D}(\Sigma), \quad z \in \Omega^0(\Theta(\zeta)) \} \subset \mathbb{C}^{3g-2}$$

and the map to $\mathcal{D}(\Sigma)$ is a composition of the projection onto the first factor, followed by $\phi$.

This open domain is extremely complicated, and properties of its boundary mysterious. Already for $\Sigma$ smooth, i.e. in the usual Teichmüller setting, the fibers of this fibration
are embedded disks with boundary Jordan curves of Hausdorff dimension strictly bigger than 1. Such curves vary holomorphically, so then their Hausdorff dimension defines a real-analytic function on $T_g$, see Ruelle [Ru82]. Correct analogues hold for the Bers fiber space over non-necessarily smooth $\Sigma$, where now the fibers are disconnected, non simply connected open Riemann surfaces. Their boundary curves vary holomorphically on Zariski-open subsets of the deformation space, and the type of discontinuity, which appears when crossing a nodal Riemann surface, is certainly of interest and will be subject to further investigation.

II. Proof of the main theorem

The idea behind the proof is quite simple: we use the existence of Lefschetz pencils, in conjunction with the structure of the deformation spaces discussed in the previous chapter, to obtain what we want. This is the natural generalization of the technique employed by Griffiths in [Gr71]. Let us start by setting up some notation. Let $j : X \hookrightarrow \mathbb{P}^d$ be a projective embedding of a smooth variety, and denote by $\mathbb{P} X$ the projective space of hyperplane sections of $X$, relative to $j$. It carries a natural tautological family $H \in X \times \mathbb{P} X$. Upon composing $j$ with a Veronese embedding, we can assume, [SGA7.2, Exposé XVII], that the dual variety $X^\vee \subset \mathbb{P} X$, parametrizing singular hyperplane sections of $X$, is an irreducible divisor. Moreover, the generic point of $X^\vee$ parametrizes an irreducible hyperplane section, whose singular locus is an isolated ordinary double point. In particular we have:

**Fact II.1.** Let $C \hookrightarrow \mathbb{P} X$ be a general curve, with corresponding family $H_C \to C$. Then the fibers are irreducible, the general one is smooth, and the singular ones have one isolated ordinary double point.

In order to connect this with Bers’ theory, we need a simple remark, whose proof was suggested by Dima Zakharov:

**Lemma II.2.** Let $\Sigma$ be any terminal Riemann surface, and $S$ any irreducible curve. Then there exists a deformation $S \to \Sigma$ if and only if they have the same genus.

**Proof.** One direction is obvious. For the other, denote by $g$ the common genus, and consider the natural morphism $\mathcal{M}_{g-1,2} \to \mathcal{M}_g$. Its image contains all those points parametrizing curves with at least one non-separating node. This is clear, if we take the partial normalization at such non-separating node. On the other hand, every terminal Riemann surface...
has at least one non-separating node: otherwise its dual graph would be a tree, and a tree of rational curves cannot be stable. □

We are now ready to start the proof of Theorem 1.

Proof. Fix a projective embedding of our surface $X$, and let $C$ be a general curve inside $\mathbf{P}_X$, with universal family $\mathcal{H}_C \to C$. We can assume, after a ramified cover of $C$, that such family is stable. The general fiber is a smooth curve, while the singular fiber is irreducible with one node. As such, for any terminal Riemann surface $\Sigma$, every fiber of $\mathcal{H}_C \to C$ deforms onto $\Sigma$. We can therefore pick a simultaneous deformation $t : \mathcal{H}_C \to \Sigma \times C$, i.e. a continuous $C$-map, restricting to a deformation over each point of $C$. This defines a holomorphic embedding $t : \Delta \to \mathcal{B}(\Sigma)$, and therefore a family of open surfaces $t^*\Omega^0(\Sigma) \to \Delta$ induced by the Bers fiber space, Definition I.13. Such family has a canonical, fibrewise unramified holomorphic map onto $\mathcal{U} \subset \mathcal{H}_C$, which is the complement of the nodes appearing in the fibers. □

Observe that, in the above proof, we can pick a curve $C$ such that the natural projection $\mathcal{U} \to X$ is surjective - this just means that every node appearing in curves parametrized by $C$, lies in the smooth locus of some other curve parametrized by $C$, and this can easily be achieved.

References


Weak uniformization of Zariski-open subsets of algebraic surfaces


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