

# POSITIVE ENTROPY ACTIONS OF COUNTABLE GROUPS FACTOR ONTO BERNOULLI SHIFTS

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ABSTRACT. We prove that if a free ergodic action of a countably infinite group has positive Rokhlin entropy (or, less generally, positive sofic entropy) then it factors onto all Bernoulli shifts of lesser or equal entropy. This extends to all countably infinite groups the well-known Sinai factor theorem from classical entropy theory.

## 1. INTRODUCTION

For a countable group  $G$  and a standard probability space  $(L, \lambda)$ , the *Bernoulli shift* over  $G$  with base space  $(L, \lambda)$  is the product measure space  $(L^G, \lambda^G)$  together with the left shift-action of  $G$ : for  $g \in G$  and  $x \in L^G$ ,  $g \cdot x$  is defined by  $(g \cdot x)(t) = x(g^{-1}t)$  for  $t \in G$ . For  $n \in \mathbb{N}$  we write  $n^G$  for  $\{0, \dots, n-1\}^G$  and  $u_n$  for the normalized counting measure on  $\{0, \dots, n-1\}$ . The *Shannon entropy* of  $(L, \lambda)$ , denoted  $H(L, \lambda)$ , is  $\sum_{\ell \in L} -\lambda(\ell) \log \lambda(\ell)$  if  $\lambda$  has countable support and is  $\infty$  otherwise. We note that  $H(n, u_n) = \log(n)$ .

Kolmogorov–Sinai entropy, which we denote  $h_{\mathbb{Z}}^{\text{KS}}$ , was introduced in 1958 in order to provide a negative answer to the roughly 20 year-old question of von Neumann that asked whether the Bernoulli shifts  $\mathbb{Z} \curvearrowright (2^{\mathbb{Z}}, u_2^{\mathbb{Z}})$  and  $\mathbb{Z} \curvearrowright (3^{\mathbb{Z}}, u_3^{\mathbb{Z}})$  are isomorphic [32, 33, 50]. Shortly after, Sinai proved the following famous theorem.

**Theorem** (Sinai’s factor theorem, 1962 [51]). *Let  $\mathbb{Z} \curvearrowright (X, \mu)$  be a free ergodic p.m.p. action. If  $(L, \lambda)$  is a probability space with  $h_{\mathbb{Z}}^{\text{KS}}(X, \mu) \geq H(L, \lambda)$ , then  $\mathbb{Z} \curvearrowright (X, \mu)$  factors onto the Bernoulli shift  $\mathbb{Z} \curvearrowright (L^{\mathbb{Z}}, \lambda^{\mathbb{Z}})$ .*

One reason for this theorem’s significance is its applicability. Specifically, the conclusion is quite strong as it is extremely difficult to construct Bernoulli factors. In fact, the conclusion was not even previously known in the case where  $\mathbb{Z} \curvearrowright (X, \mu)$  is itself a Bernoulli shift. On the other hand, the assumption of the theorem is by comparison much easier to verify, as many practical techniques are known for obtaining bounds on Kolmogorov–Sinai entropy.

This theorem is also important for aesthetic and philosophical reasons. Previously, entropy merely happened to be a tool capable of showing the non-isomorphism of some  $\mathbb{Z}$ -Bernoulli shifts. However, Sinai’s theorem indicates that there is a deeper connection at play, and that entropy and Bernoullicity are intricately linked to one another. His theorem reveals that in fact Bernoulli shifts are the source of all positive entropy (as the original action has 0 relative entropy over a full-entropy Bernoulli factor).

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Sinai's theorem also played a significant role in the historical development of entropy theory and our understanding of Bernoulli shifts. Specifically, Sinai's theorem inspired the profound work of Ornstein that followed. Building off of Sinai's theorem and the methods of its proof, Ornstein proved his famous isomorphism theorem stating that  $\mathbb{Z}$ -Bernoulli shifts are completely classified up to isomorphism by their Kolmogorov–Sinai entropy (equivalently, the Shannon entropy of their base) [37, 38]. Ornstein and other researchers continued to cultivate these techniques, ultimately creating ‘Ornstein theory,’ a collection of practical necessary-and-sufficient conditions for  $\mathbb{Z}$ -actions to be isomorphic to Bernoulli. This led to the surprising discovery that many natural examples of  $\mathbb{Z}$ -actions turn out to be isomorphic to Bernoulli, such as: factors of Bernoulli shifts [39], inverse limits of Bernoulli shifts [40], ergodic automorphisms of compact metrizable groups [34, 36], mixing Markov chains [16], geodesic flows on surfaces of negative curvature [41], Anosov flows with a smooth measure [43], and two-dimensional billiards with a convex scatterer [19]. These discoveries are generally considered to be the deepest results in entropy theory.

Over time, Sinai's theorem was generalized in a few different ways. First, it was generalized to the realm of relative entropy by Thouvenot.

**Theorem** (The relative factor theorem, Thouvenot 1975 [53] (see also [38])). *Let  $\mathbb{Z} \curvearrowright (X, \mu)$  be a free ergodic p.m.p. action, let  $\mathbb{Z} \curvearrowright (Y, \nu)$  be a factor action given by the map  $\phi : X \rightarrow Y$ , and set  $\mathcal{F} = \phi^{-1}(\mathcal{B}(Y))$ . If  $(L, \lambda)$  is a probability space with  $h_{\mathbb{Z}}^{\text{KS}}(X, \mu | \mathcal{F}) \geq H(L, \lambda)$ , then there is a map  $\psi : X \rightarrow L^{\mathbb{Z}}$  so that  $\psi \times \phi$  is a factor map from  $\mathbb{Z} \curvearrowright (X, \mu)$  onto the direct-product action  $\mathbb{Z} \curvearrowright (L^{\mathbb{Z}} \times Y, \lambda^{\mathbb{Z}} \times \nu)$ .*

There was also a perturbative factor theorem established in the course of Sinai's original proof which Thouvenot relativized in obtaining the above theorem. We refer the reader to Section 2 for the relevant definitions.

**Theorem** (The perturbative relative factor theorem, Thouvenot 1975 [53] (see also [38])). *There exists a function  $\delta : \mathbb{N}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$  with the following property. Let  $\mathbb{Z} \curvearrowright (X, \mu)$  be a free ergodic p.m.p. action, let  $\mathcal{F}$  be a  $\mathbb{Z}$ -invariant sub- $\sigma$ -algebra, let  $k \in \mathbb{N}$ , and let  $\epsilon > 0$ . If  $\xi$  is a  $k$ -piece ordered partition of  $X$  and  $\bar{p}$  is a length- $k$  probability vector with  $H(\bar{p}) \leq h_{\mathbb{Z}}^{\text{KS}}(X, \mu | \mathcal{F})$  and*

$$|\text{dist}(\xi) - \bar{p}| + |h_{\mathbb{Z}}^{\text{KS}}(\xi | \mathcal{F}) - H(\bar{p})| < \delta(k, \epsilon),$$

*then there is an ordered partition  $\alpha$  with  $\text{dist}(\alpha) = \bar{p}$  and  $d_{\mu}^{\text{alg}}(\alpha, \xi) < \epsilon$  such that the  $\mathbb{Z}$ -translates of  $\alpha$  are mutually independent and  $\sigma\text{-alg}_{\mathbb{Z}}(\alpha)$  is independent with  $\mathcal{F}$ .*

Later, the factor theorem was extended to non-ergodic actions by Kieffer and Rahe. Below, for a Borel action  $G \curvearrowright X$  we let  $\mathcal{E}_G(X)$  denote the set of invariant ergodic Borel probability measures.

**Theorem** (The non-ergodic factor theorem, Kieffer–Rahe 1981 [31]). *Let  $\mathbb{Z} \curvearrowright X$  be a free Borel action on a standard Borel space  $X$ . If  $\nu \mapsto \lambda_{\nu}$  is a Borel map associating to every  $\nu \in \mathcal{E}_{\mathbb{Z}}(X)$  a probability measure  $\lambda_{\nu}$  on  $[0, 1]$  satisfying  $h_{\mathbb{Z}}^{\text{KS}}(X, \nu) \geq H([0, 1], \lambda_{\nu})$ , then there is a  $\mathbb{Z}$ -equivariant factor map  $\phi : X \rightarrow [0, 1]^{\mathbb{Z}}$  such that  $\phi_*(\nu) = \lambda_{\nu}^{\mathbb{Z}}$  for every  $\nu \in \mathcal{E}_{\mathbb{Z}}(X)$ .*

Finally, the most stream-lined proof came from the residual theorem of Burton, Keane, and Serafin.

**Theorem** (The residual factor theorem, Burton–Keane–Serafin 2000 [9]). *Let  $A$  be a finite set and let  $\mathbb{Z} \curvearrowright (A^{\mathbb{Z}}, \mu)$  be a free ergodic p.m.p. action. If  $(L, \lambda)$  is a finite probability space with  $h_{\mathbb{Z}}^{\text{KS}}(A^{\mathbb{Z}}, \mu) = H(L, \lambda)$ , then within the space of ergodic joinings of  $\mu$  with  $\lambda^{\mathbb{Z}}$ , the set of joinings that come from graphs of factor maps,  $\{(\text{id} \times \phi)_*(\mu) : \phi : A^{\mathbb{Z}} \rightarrow L^{\mathbb{Z}}, \phi_*(\mu) = \lambda^{\mathbb{Z}}\}$ , is a dense  $G_{\delta}$  set in the weak\*-topology.*

Sinai’s theorem was also extended to actions of other groups. In 1972, Katznelson and Weiss extended the original factor theorem to free ergodic actions of  $\mathbb{Z}^n$  [25], and in 1987 Ornstein and Weiss generalized the original factor theorem to free ergodic actions of countable amenable groups [42]. Later, Danilenko and Park extended Thouvenot’s relative factor theorem to free ergodic actions of countable amenable groups in 2002 [10].

For countable non-amenable groups, the study of Bernoulli shifts and the development of an entropy theory were long considered unattainable. Today it is still open, for example, whether the Bernoulli shifts  $G \curvearrowright (2^G, u_2^G)$  and  $G \curvearrowright (3^G, u_3^G)$  are non-isomorphic for all countable groups  $G$ . However, dramatic advancements have recently occurred due to breakthrough work of Bowen in 2008.

Work of Bowen [4], combined with improvements by Kerr and Li [28, 29], created the notion of sofic entropy for p.m.p. actions of sofic groups. We remind the reader that the class of sofic groups contains the countable amenable groups, and it is an open problem whether every countable group is sofic. Sofic entropy is an extension of Kolmogorov–Sinai entropy, as when the acting group is amenable the two notions coincide [5, 29]. For sofic groups  $G$ , the Bernoulli shift  $G \curvearrowright (L^G, \lambda^G)$  has sofic entropy  $H(L, \lambda)$  as expected [4, 30], thus implying the non-isomorphism of many Bernoulli shifts.

In this paper, we will work with a different notion of entropy which was introduced by the author in 2014 [44] and is defined for actions of arbitrary (not necessarily sofic) countable groups. If  $G$  is a countable group,  $G \curvearrowright (X, \mu)$  is an ergodic p.m.p. action, and  $\xi \subseteq \mathcal{B}(X)$  is a collection of sets, then we define the (outer) Rokhlin entropy of  $\xi$  to be

$$h_G(\xi) = \inf \left\{ H(\alpha) : \alpha \text{ a countable partition with } \xi \subseteq \sigma\text{-alg}_G(\alpha) \right\},$$

where  $\sigma\text{-alg}_G(\alpha)$  is the smallest  $G$ -invariant  $\sigma$ -algebra containing  $\alpha$ . When  $\xi = \mathcal{B}(X)$  is the Borel  $\sigma$ -algebra of  $X$ , we simply write  $h_G(X, \mu)$  for the Rokhlin entropy of  $(X, \mu)$ . In this paper,  $h_G$  will always denote Rokhlin entropy. We caution the reader that Rokhlin entropy is defined in a distinct way for non-ergodic actions; see Section 2.

For free actions of amenable groups, Rokhlin entropy coincides with Kolmogorov–Sinai entropy [2]. Furthermore, for free actions of sofic groups, sofic entropy is a lower-bound to Rokhlin entropy [2], and it is an important open question whether the two coincide (excluding cases where sofic entropy is minus infinity). The equality  $h_G(L^G, \lambda^G) = H(L, \lambda)$  holds for sofic groups  $G$ , and it is conjectured to hold for all countably infinite groups [45].

At face value there’s not necessarily any good reason to expect the factor theorem to hold for non-amenable groups. Specifically, we know Bernoulli shifts behave in drastically different ways for non-amenable groups. For example, for amenable groups factors of Bernoulli shifts, inverse limits of Bernoulli shifts, and d-bar limits of Bernoulli shifts are all isomorphic to Bernoulli shifts, but all of these assertions are false for non-amenable groups. Furthermore, all prior proofs of Sinai’s factor

theorem and its variations rely critically upon three properties: the Rokhlin lemma, the Shannon-McMillan-Breiman theorem, and the monotonicity of entropy under factor maps. Yet all three of these properties fail miserably for non-amenable groups. Finally, a concerning fact is that there are simple examples showing that the residual factor theorem is false for non-amenable groups (see Proposition 9.2). Despite these warning signs, we generalize the factor theorem to all countably infinite groups. For simplicity, here we state our main theorem only in its most basic form. (For the strongest versions, see Theorems 5.4, 5.5, 9.1, and Corollary 10.2).

**Theorem 1.1.** *Let  $G$  be a countably infinite group and let  $G \curvearrowright (X, \mu)$  be a free ergodic p.m.p. action. If  $(L, \lambda)$  is a probability space with  $h_G(X, \mu) \geq H(L, \lambda)$  then  $G \curvearrowright (X, \mu)$  factors onto the Bernoulli shift  $G \curvearrowright (L^G, \lambda^G)$ .*

Let us recast the above theorem in a more concrete form that highlights the simplicity of the assumption. Recall that a Borel partition  $\alpha$  of  $X$  is *generating* if  $\sigma\text{-alg}_G(\alpha) = \mathcal{B}(X) \bmod \text{null sets}$ . The classical finite generator theorem of Krieger extends to Rokhlin entropy [44], allowing the following reformulation of Theorem 1.1.

**Corollary 1.2.** *Let  $G$  be a countably infinite group and let  $G \curvearrowright (X, \mu)$  be a free ergodic p.m.p. action. If  $G \curvearrowright (X, \mu)$  does not admit any  $n$ -piece generating partition then it factors onto the Bernoulli shift  $(n^G, u_n^G)$ . If  $G \curvearrowright (X, \mu)$  does not admit any finite generating partition, then it factors onto every  $G$ -Bernoulli shift.*

Since sofic entropy is a lower-bound to Rokhlin entropy [2], as a bonus we automatically obtain Sinai's factor theorem for sofic entropy as well.

**Corollary 1.3.** *Let  $G$  be a sofic group and let  $G \curvearrowright (X, \mu)$  be a free ergodic p.m.p. action. If  $(L, \lambda)$  is a probability space and the sofic entropy of  $G \curvearrowright (X, \mu)$  (with respect to some sofic approximation of  $G$ ) is greater than or equal to  $H(L, \lambda)$ , then  $G \curvearrowright (X, \mu)$  factors onto the Bernoulli shift  $G \curvearrowright (L^G, \lambda^G)$ .*

A recent striking result of Bowen states that when  $G$  is non-amenable, all  $G$ -Bernoulli shifts factor onto one another [7]. This leads to the following corollary.

**Corollary 1.4.** *Let  $G$  be a countable non-amenable group and let  $G \curvearrowright (X, \mu)$  be a free ergodic action. If  $h_G(X, \mu) > 0$  then  $G \curvearrowright (X, \mu)$  factors onto every  $G$ -Bernoulli shift.*

In particular, the converse of Theorem 1.1 fails for non-amenable groups (but holds for amenable groups by monotonicity of entropy).

As with the original factor theorem, we believe the conclusion of Theorem 1.1 is strong. In fact, the conclusion was not previously known in the case where  $G \curvearrowright (X, \mu)$  is itself a Bernoulli shift (though this was implied by independent results obtained later [7, 48]). To point to other concrete examples, we mention that Hayes has computed the sofic entropy of Gaussian actions and principal algebraic actions of sofic groups [20, 23], many of which have positive sofic entropy and thus possess Bernoulli factors by our theorem. Let us describe more precisely this latter class of actions. Let  $G$  be a sofic group and let  $f \in \mathbb{Z}G$  be an element of the group ring. Let  $X_f$  be the compact abelian group (using coordinate-wise addition) of  $x \in (\mathbb{R}/\mathbb{Z})^G$  such that the right-convolution  $x * f$  is trivial mod  $\mathbb{Z}^G$ , and let  $m$  be the Haar probability measure on  $X_f$ . Hayes proved that if  $f$  is injective

as a left-convolution operator on  $\ell^2(G)$ , then the sofic entropy of  $G \curvearrowright (X_f, m)$  equals  $\int_{[0, \infty)} \log(t) d\mu_{|f|}(t)$  (the logarithm of the Fuglede–Kadison determinant of  $f$ ), where  $\mu_{|f|}$  is the spectral measure of the operator  $|f|$  acting on  $\ell^2(G)$ . Many explicit examples will give positive entropy values, but directly finding Bernoulli factors would appear to be extremely difficult without our theorem.

To give a brief but intriguing glimpse into the applicability of Theorem 1.1, we mention the following. This corollary relies upon the recent powerful result of Bowen [7] that all Bernoulli shifts over non-amenable groups satisfy the measurable von Neumann–Day conjecture.

**Corollary 1.5.** *Let  $G$  be a countable non-amenable group and let  $G \curvearrowright (X, \mu)$  be a free ergodic action. If  $h_G(X, \mu) > 0$  then this action satisfies the measurable von Neumann–Day conjecture; specifically there is a free action of the rank 2 free group  $F_2 \curvearrowright (X, \mu)$  such that  $\mu$ -almost-every  $F_2$ -orbit is contained in a  $G$ -orbit.*

In analogy with the original factor theorem, we believe Theorem 1.1 holds philosophical and aesthetic value. While sofic entropy and Rokhlin entropy are useful for distinguishing Bernoulli shifts, it's not at all clear a priori what properties of the action are reflected in these entropies. Our theorem indicates the surprising fact that both sofic entropy and Rokhlin entropy continue to be intimately tied to Bernoullicity. Specifically, our theorem shows that all positive entropy phenomena comes from Bernoulli shifts. (However, the picture is not yet as clear as in the classical setting; see Section 10).

Theorem 1.1 is only the most basic form of our main theorem. Specifically, we prove relative, perturbative, and non-ergodic factor theorems. With the exception of the residual factor theorem which is known to be false for non-amenable groups, our work extends all prior versions of the factor theorem to general countably infinite groups. Furthermore, through the use of non-free variants of Bernoulli shifts, our theorems apply to non-free actions as well.

We believe Theorem 1.1 and Corollary 1.3 mark an important step forward in the development of both sofic entropy theory and Rokhlin entropy theory. An ambitious long-term goal in this field is the development of an Ornstein theory for non-amenable groups. To a large extent, our understanding of Bernoulli shifts over non-amenable groups is still quite poor. One peculiar exception is the (positive direction of) the Ornstein isomorphism theorem itself. Work of Bowen [6] and recent work of the author [48] (occurring after the present results were obtained) show that for every countably infinite group, Bernoulli shifts with base spaces of equal Shannon entropy must be isomorphic (in particular, Bernoulli shifts over sofic groups are completely classified by their sofic entropy, i.e. the Shannon entropy of their base). This result is peculiar in the sense that the proof relies upon very restrictive and specialized tricks that do not use any entropy theory aside from the original Ornstein isomorphism theorem for  $\mathbb{Z}$ . The proof is an isolated achievement, seemingly incapable of generalizing or of revealing more about the structure of Bernoulli shifts. On the other hand, we believe the techniques and theorems of this paper may lay a new path to studying Bernoulli shifts, just as the original factor theorem historically served as the foundation for Ornstein theory.

One immediate consequence of our main theorem is that Bernoulli shifts  $G \curvearrowright (L^G, \lambda^G)$  are finitely-determined whenever  $h_G(L^G, \lambda^G) = H(L, \lambda)$ . Recall that for a finite set  $L$  and two  $G$ -invariant probability measures  $\mu, \nu$  on  $L^G$ , their d-bar

distance  $\bar{d}(\mu, \nu)$  is defined to be the infimum of

$$\lambda(\{(x, y) \in L^G \times L^G : x(1_G) \neq y(1_G)\}),$$

where  $\lambda$  ranges over all joinings of  $\mu$  with  $\nu$ . The measure  $\mu$  is *finitely-determined* if for every  $\epsilon > 0$  there is a weak\*-open neighborhood  $U$  of  $\mu$  and  $\delta > 0$  such that whenever  $\nu \in U$  satisfies  $\text{Stab}_*(\nu) = \text{Stab}_*(\mu)$  and  $|h_G(L^G, \nu) - h_G(L^G, \mu)| < \delta$  we have  $\bar{d}(\mu, \nu) < \epsilon$ . The notion of finitely-determined played a prominent role in Ornstein theory, and it was proven that for an amenable group  $G$  a measure  $\mu$  on  $L^G$  is finitely-determined if and only if  $G \curvearrowright (L^G, \mu)$  is isomorphic to a Bernoulli shift [42].

**Corollary 1.6.** *Let  $G$  be a countably infinite group and let  $(L, \lambda)$  be a finite probability space. If  $h_G(L^G, \lambda^G) = H(L, \lambda)$  then  $\lambda^G$  is finitely-determined.*

The techniques we create to prove our main theorem also lead to new insights on the spectral structure of actions having positive Rokhlin entropy. This will be presented in upcoming work [47] and will mirror and expand upon similar results obtained by Hayes in the setting of sofic entropy [21, 22]. In fact, we would like to explicitly mention that our main theorem directly combines with the work of Hayes [21, 22] to produce the following corollary (see [22] for relevant notation). This corollary extends a similar result for actions of amenable groups obtained by Dooley–Golodets [11].

**Corollary 1.7.** *Let  $G$  be a sofic group. Let  $G \curvearrowright (X, \mu)$  be a free ergodic p.m.p. action and let  $G \curvearrowright (Y, \nu)$  and  $G \curvearrowright (Y_0, \nu_0)$  be the (sofic entropy) Pinsker factor and the (sofic entropy) outer Pinsker factor, respectively. Then, as a representation of  $\mathcal{L}^\infty(Y) \rtimes_{\text{alg}} G$ , we have that  $\mathcal{L}^2(X) \ominus \mathcal{L}^2(Y)$  is isomorphic to  $\mathcal{L}^2(Y, \ell^2(G))^\infty$ . Similarly,  $\mathcal{L}^2(X) \ominus \mathcal{L}^2(Y_0)$  is isomorphic to  $\mathcal{L}^2(Y_0, \ell^2(G))^\infty$ .*

In [47] this corollary will be adapted to Rokhlin entropy and extended to actions of general countable groups.

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## 2. PRELIMINARIES

Our probability spaces  $(X, \mu)$  will always be assumed to be standard, meaning  $X$  is a standard Borel space, with its Borel  $\sigma$ -algebra denoted  $\mathcal{B}(X)$ , and  $\mu$  is a Borel probability measure. The space  $\text{Prob}(X)$  of all Borel probability measures on  $X$  is itself a standard Borel space. Its Borel  $\sigma$ -algebra is the smallest  $\sigma$ -algebra making all of the maps  $\mu \mapsto \mu(A)$  measurable for every  $A \in \mathcal{B}(X)$ .

Recall that for every sub- $\sigma$ -algebra  $\Sigma$  there is a probability space  $(Y, \nu)$ , unique up to isomorphism, and a measure-preserving map  $\phi : X \rightarrow Y$  with  $\phi^{-1}(\mathcal{B}(Y)) = \Sigma$ . Furthermore, there is a  $\nu$ -almost-everywhere unique Borel map  $y \in Y \mapsto \mu_y \in$

$\text{Prob}(X)$  satisfying  $\mu_y(\phi^{-1}(y)) = 1$  and  $\mu = \int_Y \mu_y d\nu(y)$ . We refer to this as the *disintegration of  $\mu$  over  $\nu$*  or the *disintegration of  $\mu$  with respect to  $\phi$* . If  $\alpha$  is a countable Borel partition of  $X$  then the *Shannon entropy of  $\alpha$*  is

$$H(\alpha) = H_\mu(\alpha) = \sum_{A \in \alpha} -\mu(A) \log \mu(A),$$

and the *conditional Shannon entropy of  $\alpha$  relative to  $\Sigma$*  is

$$H(\alpha | \Sigma) = \int_Y H_{\mu_y}(\alpha) d\nu(y).$$

If  $\beta$  is another countable Borel partition of  $X$  then we write  $H(\alpha | \beta)$  for  $H(\alpha | \sigma\text{-alg}(\beta))$ . We will assume familiarity with the basic properties of conditional Shannon entropy; see [13] for reference.

A collection of  $\sigma$ -algebras  $(\mathcal{F}_i)_{i \in I}$  on  $(X, \mu)$  are said to be *mutually independent* if for any disjoint subsets  $J_1, J_2 \subseteq I$  and sets  $A_i \in \bigvee_{k \in J_i} \mathcal{F}_k$  we have  $\mu(A_1 \cap A_2) = \mu(A_1)\mu(A_2)$ . More generally, if  $\Sigma$  is another  $\sigma$ -algebra,  $\phi : (X, \mu) \rightarrow (Y, \nu)$  is the associated factor, and  $\mu = \int_Y \mu_y d\nu(y)$  is the disintegration of  $\mu$  over  $\nu$ , then we say that  $(\mathcal{F}_i)_{i \in I}$  are *mutually independent relative to  $\Sigma$*  if for any disjoint subsets  $J_1, J_2 \subseteq I$  and sets  $A_i \in \bigvee_{k \in J_i} \mathcal{F}_k$  we have  $\mu_y(A_1 \cap A_2) = \mu_y(A_1)\mu_y(A_2)$  for  $\nu$ -almost-every  $y \in Y$ . In the case  $|I| = 2$ , we simply say that the two  $\sigma$ -algebras are independent (or independent relative to  $\Sigma$ ). By associating partitions with the  $\sigma$ -algebras they generate and functions with the smallest  $\sigma$ -algebra making them measurable, we can also speak of independence of partitions and of functions. We record a basic lemma.

**Lemma 2.1.** *Let  $(X, \mu)$  be a standard probability space, let  $\alpha$  be a partition, and let  $\Sigma$  and  $\mathcal{F}$  be sub- $\sigma$ -algebras. If  $\alpha$  is independent with  $\Sigma$  relative to  $\mathcal{F}$  and  $\alpha$  is independent with  $\mathcal{F}$ , then  $\alpha$  is independent with  $\Sigma$ .*

*Proof.* Let  $\phi : (X, \mu) \rightarrow (Y, \nu)$  be the factor associated to  $\mathcal{F}$ , and let  $\mu = \int_Y \mu_y d\nu(y)$  be the corresponding disintegration of  $\mu$ . Fix  $A \in \sigma\text{-alg}(\alpha)$  and  $C \in \Sigma$ . Since  $\alpha$  is independent with  $\Sigma$  relative to  $\mathcal{F}$ , we have that  $\mu_y(A \cap C) = \mu_y(A)\mu_y(C)$  for  $\nu$ -almost-every  $y \in Y$ . Since  $\alpha$  is also independent with  $\mathcal{F}$ , we have that  $\mu_y(A) = \mu(A)$  for  $\nu$ -almost-every  $y \in Y$ . Thus  $\mu_y(A \cap C) = \mu(A)\mu_y(C)$  for  $\nu$ -almost-every  $y \in Y$ . Therefore

$$\mu(A \cap C) = \int_Y \mu_y(A \cap C) d\nu(y) = \int_Y \mu(A)\mu_y(C) d\nu(y) = \mu(A)\mu(C).$$

We conclude that  $\alpha$  and  $\Sigma$  are independent.  $\square$

A *probability vector  $\bar{p}$*  is a finite or countable tuple  $(p_i)_{0 \leq i < |\bar{p}|}$  of non-negative real numbers that sum to 1. We write  $|\bar{p}|$  for the length of  $\bar{p}$ . If  $\bar{q}$  is another probability vector then their  $\ell^1$ -distance is

$$|\bar{p} - \bar{q}| = \sum_i |p_i - q_i|,$$

with the convention that  $p_i = 0$  when  $i \geq |\bar{p}|$  and  $q_j = 0$  when  $j \geq |\bar{q}|$ .

In this paper we will implicitly assume that all countable partitions  $\alpha$  of  $(X, \mu)$  are both Borel and ordered:  $\alpha = \{A_i : 0 \leq i < |\alpha|\}$ . We write  $\text{dist}(\alpha)$ , or  $\text{dist}_\mu(\alpha)$ , for the *distribution of  $\alpha$* , which is the probability vector having  $i^{\text{th}}$  term  $\mu(A_i)$ . If

$\beta = \{B_i : 0 \leq i < |\beta|\}$  is another ordered partition, then their distance in the measure algebra is

$$d_\mu^{\text{alg}}(\alpha, \beta) = \sum_i \mu(A_i \Delta B_i),$$

with the convention  $A_i = \emptyset$  when  $i \geq |\alpha|$  and  $B_j = \emptyset$  when  $j \geq |\beta|$ . Notice that

$$|\text{dist}(\alpha) - \text{dist}(\beta)| \leq d_\mu^{\text{alg}}(\alpha, \beta).$$

We say that  $\beta$  refines  $\alpha$ , written  $\beta \geq \alpha$ , if each  $A \in \alpha$  is a union of members of  $\beta$ . Similarly, we say  $\bar{q}$  is a refinement of  $\bar{p}$  if there is a function  $f : \{0, \dots, |\bar{q}|\} \rightarrow \{0, \dots, |\bar{p}|\}$  satisfying

$$\forall 0 \leq i < |\bar{p}| \quad \sum_{j \in f^{-1}(i)} q_j = p_i.$$

For a countable group  $G$  and a Borel action  $G \curvearrowright X$  we let  $\mathfrak{M}_G(X)$  and  $\mathcal{E}_G(X)$  denote the sets of invariant Borel probability measures and invariant ergodic Borel probability measures, respectively. We also write  $\mathcal{I} = \mathcal{I}_G$  for the  $\sigma$ -algebra of  $G$ -invariant Borel subsets of  $X$ . The action  $G \curvearrowright X$  is *aperiodic* if every orbit is infinite. We similarly call a p.m.p. action  $G \curvearrowright (X, \mu)$  *aperiodic* if  $\mu$ -almost-every orbit is infinite. We write  $\text{Stab}(x) = \{g \in G : g \cdot x = x\}$  for the stabilizer of  $x$ . If  $\Sigma$  is a  $G$ -invariant sub- $\sigma$ -algebra and  $(Y, \nu)$  is the corresponding factor via  $\phi : X \rightarrow Y$ , then there is a  $\nu$ -almost-everywhere unique p.m.p. action of  $G$  on  $(Y, \nu)$  so that  $\phi$  is  $G$ -equivariant. We say that  $\Sigma$  (or  $\phi$ ) is *class-bijective* if  $\text{Stab}(\phi(x)) = \text{Stab}(x)$  for  $\mu$ -almost-every  $x \in X$ . For a partition  $\alpha$  of  $X$  and  $g \in G$  we set  $g \cdot \alpha = \{g \cdot A : A \in \alpha\}$ , and for a finite set  $W \subseteq G$  we set  $\alpha^W = \bigvee_{w \in W} w^{-1} \cdot \alpha$ . For  $\mathcal{C} \subseteq \mathcal{B}(X)$  we write  $\sigma\text{-alg}(\mathcal{C})$  for the  $\sigma$ -algebra on  $X$  generated by  $\mathcal{C}$  and  $\sigma\text{-alg}_G(\mathcal{C})$  for the  $G$ -invariant  $\sigma$ -algebra generated by  $\mathcal{C}$ . A sub- $\sigma$ -algebra  $\Sigma$  is *countably generated* if there is a countable collection  $\mathcal{C}$  so that  $\sigma\text{-alg}(\mathcal{C}) = \Sigma$  (equality in the purely Borel sense). Of course, for any fixed probability measure  $\mu$  on  $X$ , every sub- $\sigma$ -algebra  $\Sigma$  coincides, modulo  $\mu$ -null sets, with a countably generated  $\sigma$ -algebra.

If  $G$  is a countable group,  $G \curvearrowright (X, \mu)$  is a p.m.p. action,  $\mathcal{F}$  is a  $G$ -invariant sub- $\sigma$ -algebra, and  $\xi \subseteq \mathcal{B}(X)$  is a collection of sets, then we define the (*outer*) *Rokhlin entropy of  $\xi$  relative to  $\mathcal{F}$*  to be

$$h_G(\xi | \mathcal{F}) = \inf \left\{ H(\alpha | \mathcal{F} \vee \mathcal{I}) : \alpha \text{ a countable partition with } \xi \subseteq \sigma\text{-alg}_G(\alpha) \vee \mathcal{F} \vee \mathcal{I} \right\}.$$

When we wish to emphasize the measure  $\mu$  we write  $h_{G, \mu}(\xi | \mathcal{F})$ . Notice that for ergodic actions  $\mathcal{I} = \{\emptyset, X\}$  modulo null sets. When  $\xi = \mathcal{B}(X)$  is the Borel  $\sigma$ -algebra of  $X$ , we simply write  $h_G(X, \mu | \mathcal{F})$  for the *Rokhlin entropy of  $(X, \mu)$  relative to  $\mathcal{F}$* .

One of the key features of Rokhlin entropy that we will use is that it is countably sub-additive [2]. Specifically, fix a p.m.p. action  $G \curvearrowright (X, \mu)$ , let  $\xi \subseteq \mathcal{B}(X)$ , let  $\mathcal{F}$  be a  $G$ -invariant  $\sigma$ -algebra, and let  $(\Sigma_n)_{n \geq 1}$  be an increasing sequence of  $G$ -invariant  $\sigma$ -algebras with  $\xi \subseteq \mathcal{F} \vee \bigvee_n \Sigma_n$ . Then

$$h_G(\xi | \mathcal{F}) \leq h_G(\Sigma_1 | \mathcal{F}) + \sum_{n=2}^{\infty} h_G(\Sigma_n | \Sigma_{n-1} \vee \mathcal{F}).$$

For a  $G$ -invariant sub- $\sigma$ -algebra  $\mathcal{F}$  we define the outer Pinsker algebra of  $(X, \mu)$  relative to  $\mathcal{F}$  to be

$$\Pi(\mathcal{F}) = \{A \in \mathcal{B}(X) : h_G(\{A, X \setminus A\} | \mathcal{F}) = 0\}.$$



Of course, by sub-additivity  $h_G(\Pi(\mathcal{F}) | \mathcal{F}) = 0$ . We say that  $G \curvearrowright (X, \mu)$  has *completely positive outer Rokhlin entropy relative to  $\mathcal{F}$*  if  $\Pi(\mathcal{F}) = \mathcal{F}$ . We note a simple lemma.

**Lemma 2.2.** *Let  $G \curvearrowright (X, \mu)$  be a p.m.p. action, let  $\xi \subseteq \mathcal{B}(X)$ , and let  $\mathcal{F}$  be a  $G$ -invariant sub- $\sigma$ -algebra. If  $\Pi$  is the outer Pinsker algebra of  $X$  relative to  $\mathcal{F}$  then*

$$h_G(\xi | \mathcal{F}) = h_G(\xi | \Pi).$$

*Proof.* Monotonicity is clear from the definitions, so  $h_G(\xi | \Pi) \leq h_G(\xi | \mathcal{F})$ . On the other hand, Rokhlin entropy is sub-additive, giving  $h_G(\xi | \mathcal{F}) \leq h_G(\Pi | \mathcal{F}) + h_G(\xi | \Pi) = h_G(\xi | \Pi)$ .  $\square$

We will also need the following fact.

**Lemma 2.3.** *Let  $G \curvearrowright (X, \mu)$  be an aperiodic p.m.p. action, and let  $\Sigma$  be a  $G$ -invariant sub- $\sigma$ -algebra. For every  $\epsilon > 0$  there is a class-bijective  $G$ -invariant sub- $\sigma$ -algebra  $\mathcal{F} \supseteq \Sigma$  such that  $G \curvearrowright (X, \mu)$  has completely positive outer Rokhlin entropy relative to  $\mathcal{F}$  and  $h_G(\mathcal{F} | \Sigma) < \epsilon$ . In particular, for any  $\xi \subseteq \mathcal{B}(X)$*

$$h_G(\xi | \Sigma) - \epsilon \leq h_G(\xi | \mathcal{F}) \leq h_G(\xi | \Sigma).$$

*Proof.* By work of the author and Tucker-Drob (see [49] for free actions; the case of non-free actions is in preparation), there is a class-bijective factor  $G \curvearrowright (Y, \nu)$  of  $(X, \mu)$  with  $h_G(Y, \nu) < \epsilon$ . Say the factor map is  $\phi : (X, \mu) \rightarrow (Y, \nu)$ . Set  $\mathcal{F} = \Pi(\Sigma \vee \phi^{-1}(\mathcal{B}(Y)))$ . Then  $\mathcal{F}$  is class-bijective since it contains  $\phi^{-1}(\mathcal{B}(Y))$ , and it is immediate that  $G \curvearrowright (X, \mu)$  has completely positive outer Rokhlin entropy relative to  $\mathcal{F}$ . Also, Lemma 2.2 gives

$$h_G(\mathcal{F} | \Sigma) = h_G(\phi^{-1}(\mathcal{B}(Y)) | \Sigma) \leq h_G(Y, \nu) < \epsilon.$$

If  $\xi \subseteq \mathcal{B}(X)$  then  $h_G(\xi | \mathcal{F}) \leq h_G(\xi | \Sigma)$  since  $\mathcal{F} \supseteq \Sigma$ , and the other inequality follows from sub-additivity:

$$h_G(\xi | \Sigma) \leq h_G(\mathcal{F} | \Sigma) + h_G(\xi | \mathcal{F}) \leq \epsilon + h_G(\xi | \mathcal{F}) \quad \square$$

### 3. NON-FREE BERNOULLI SHIFTS AND BERNOULLI PARTITIONS

For a countable group  $G$  we let  $\text{Sub}(G)$  denote the space of all subgroups of  $G$ . A base for the topology on  $\text{Sub}(G)$  is given by the basic open sets  $\{H \in \text{Sub}(G) : H \cap T = F\}$  as  $F \subseteq T$  range over the finite subsets of  $G$ . An *invariant random subgroup*, or *IRS*, of  $G$  is a Borel probability measure  $\theta$  on  $\text{Sub}(G)$  which is invariant under the conjugation action of  $G$ . This concept was first introduced in [1]. Every p.m.p. action  $G \curvearrowright (X, \mu)$  produces an IRS  $\text{Stab}_*(\mu)$  via the push-forward of  $\mu$  under the stabilizer map  $\text{Stab} : X \rightarrow \text{Sub}(G)$ . We call  $\text{Stab}_*(\mu)$  the *stabilizer type* of  $G \curvearrowright (X, \mu)$ .

A  $G$ -equivariant factor map  $\phi : X \rightarrow Y$  between two  $G$ -actions  $G \curvearrowright (X, \mu)$  and  $G \curvearrowright (Y, \nu)$  is called *stabilizer-preserving* if  $\text{Stab}(\phi(x)) = \text{Stab}(x)$  for  $\mu$ -almost-every  $x \in X$ .

**Lemma 3.1.** *Let  $G \curvearrowright (X, \mu)$  and  $G \curvearrowright (Y, \nu)$  be p.m.p. actions and let  $\phi : X \rightarrow Y$  be a factor map. The following are equivalent:*

- (i)  $\phi$  is class-bijective;
- (ii)  $\text{Stab}(\phi(x)) = \text{Stab}(x)$  for  $\mu$ -almost-every  $x \in X$ ;
- (iii)  $\text{Stab}_*(\mu) = \text{Stab}_*(\nu)$ .

*Proof.* (i) and (ii) are equivalent by definition, and (ii) clearly implies (iii). We will argue (iii) implies (ii). So assume  $\text{Stab}_*(\mu) = \text{Stab}_*(\nu)$ . Then for every  $g \in G$  we have

$$\begin{aligned} 0 &= \text{Stab}_*(\nu)(\{\Gamma \leq G : g \in \Gamma\}) - \text{Stab}_*(\mu)(\{\Gamma \leq G : g \in \Gamma\}) \\ &= \mu(\{x \in X : g \in \text{Stab}(\phi(x)) \setminus \text{Stab}(x)\}). \end{aligned}$$

As  $G$  is countable it follows that  $\mu(\{x \in X : \text{Stab}(\phi(x)) \neq \text{Stab}(x)\}) = 0$ .  $\square$

Every IRS of  $G$  is the stabilizer type of some p.m.p. action of  $G$  [1]. This fact follows from the construction of non-free Bernoulli shifts which we now discuss. Let  $(L, \lambda)$  be a standard probability space. We let  $G$  act on  $L^G$  by the standard left-shift action:  $(g \cdot x)(t) = x(g^{-1}t)$  for  $g, t \in G$  and  $x \in L^G$ . For  $H \in \text{Sub}(G)$ , we identify  $L^{H \setminus G}$  with the set of points  $x \in L^G$  with  $H \subseteq \text{Stab}(x)$ , and we consider the corresponding Borel probability measure  $\lambda^{H \setminus G}$  on  $L^G$  which is supported on  $L^{H \setminus G}$ . If  $\theta$  is an IRS of  $G$  which is supported on the infinite-index subgroups of  $G$ , then we define the *type- $\theta$  Bernoulli shift with base space  $(L, \lambda)$*  to be the standard shift-action of  $G$  on  $L^G$  equipped with the  $G$ -invariant probability measure

$$\lambda^{\theta \setminus G} := \int_{H \in \text{Sub}(G)} \lambda^{H \setminus G} d\theta(H).$$

We say simply ‘Bernoulli shift’ when we are referring to the type- $\delta_{\{1_G\}}$  Bernoulli shift, where  $\delta_{\{1_G\}}$  is the single-point mass. If  $H \in \text{Sub}(G)$  has infinite index in  $G$  and  $\lambda$  is non-trivial, then  $\text{Stab}(x) = H$  for  $\lambda^{H \setminus G}$ -almost-every  $x \in L^G$ . Thus  $\theta$  is indeed the stabilizer type of  $G \curvearrowright (L^G, \lambda^{\theta \setminus G})$ . Note that if  $\theta = \text{Stab}_*(\mu)$  for a p.m.p. action  $G \curvearrowright (X, \mu)$ , then  $\theta$  is supported on the infinite-index subgroups of  $G$  if and only if the action  $G \curvearrowright (X, \mu)$  is aperiodic.

It is well known that Bernoulli factors are in one-to-one correspondence with partitions possessing certain independence properties, commonly referred to as Bernoulli partitions. As we work in the less familiar setting of non-free Bernoulli shifts, we review this correspondence in detail.

**Definition 3.2.** Let  $G \curvearrowright (X, \mu)$  be an aperiodic p.m.p. action. Set  $\theta = \text{Stab}_*(\mu)$ , let  $\mu = \int \mu_\Gamma d\theta(\Gamma)$  be the disintegration of  $\mu$  over  $\theta$ . We say that a partition  $\alpha$  is  *$G$ -Bernoulli* if: (i)  $\alpha$  is independent with the stabilizer map; and (ii) for  $\theta$ -almost-every  $\Gamma \leq G$  and every finite set  $W \subseteq G$  which maps injectively into  $G/\Gamma$ , the partitions  $w^{-1} \cdot \alpha$ ,  $w \in W$ , are mutually  $\mu_\Gamma$ -independent. Furthermore, if  $\mathcal{F}$  is a  $G$ -invariant sub- $\sigma$ -algebra then we say  $\alpha$  is  *$G$ -Bernoulli over  $\mathcal{F}$*  if in addition we have: (iii)  $\sigma\text{-alg}_G(\alpha)$  is independent with  $\mathcal{F}$  relative to the stabilizer map.

We remark that it is standard in the literature to say that a partition  $\alpha$  as in the above definition is  *$G$ -Bernoulli relative to  $\mathcal{F}$* . We have no intention of overturning this convention. However, as we frequently use the term ‘relatively independent’ we feel that in the present paper its best to say ‘ $G$ -Bernoulli over’ in order to avoid any potential misunderstanding.

We verify the correspondence between Bernoulli factors and Bernoulli partitions.

**Lemma 3.3.** *Let  $G \curvearrowright (X, \mu)$  be an aperiodic p.m.p. action with stabilizer type  $\theta$ , and let  $\mu = \int_{\Gamma \leq G} \mu_\Gamma d\theta(\Gamma)$  be the disintegration of  $\mu$  with respect to  $\text{Stab}$ . Also let  $G \curvearrowright (Y, \nu)$  be a factor via the map  $f : X \rightarrow Y$ , set  $\nu_\Gamma = f_*(\mu_\Gamma)$ , and set  $\mathcal{F} = f^{-1}(\mathcal{B}(Y))$ . Fix a non-trivial (possibly uncountable) probability space  $(L, \lambda)$*

and let  $\beta$  be the partition of  $L$  into points. Then the following two classes of objects are naturally in one-to-one correspondence:

- (1)  $G$ -equivariant maps  $q : X \rightarrow L^G$  such that  $q \times f$  is a factor map from  $G \curvearrowright (X, \mu)$  onto  $G \curvearrowright (L^G \times Y, \int \lambda^{\Gamma \setminus G} \times \nu_\Gamma d\theta(\Gamma))$ ;
- (2) Borel measure-preserving maps  $Q : (X, \mu) \rightarrow (L, \lambda)$  such that the partition  $\alpha = Q^{-1}(\beta)$  is  $G$ -Bernoulli over  $\mathcal{F}$ .

Specifically, the correspondence is given by  $q(x)(g) = Q(g^{-1} \cdot x)$  for  $g \in G, x \in X$ .

*Proof.* (1)  $\rightarrow$  (2). Fix  $q : X \rightarrow L^G$  with the stated property. Setting  $Q(x) = q(x)(1_G)$  for all  $x \in X$ , it is immediate that  $Q_*(\mu) = \lambda$ . Let us abuse notation slightly by letting  $\beta = \{B_\ell : \ell \in L\}$  also denote the partition of  $L^G \times Y$  defined by  $B_\ell = \{(z, y) : z(1_G) = \ell\}$ , and by letting  $\mathcal{B}(Y)$  also denote the  $\sigma$ -algebra of Borel subsets of  $L^G \times Y$  measurable with respect to the second factor. Set  $\kappa = \int \lambda^{\Gamma \setminus G} \times \nu_\Gamma d\theta(\Gamma)$ , set  $\phi = \text{Stab}_*(\kappa)$ , and let  $\kappa = \int \kappa_\Gamma d\phi(\Gamma)$  be the disintegration of  $\kappa$  relative to  $\text{Stab}$ . Notice that  $\lambda^{\Gamma \setminus G} \times \nu_\Gamma$ -almost-every point in  $L^G \times Y$  has stabilizer  $\Gamma$ , so in fact  $\kappa_\Gamma = \lambda^{\Gamma \setminus G} \times \nu_\Gamma$ . From this observation it now quickly follows that  $\beta$  is  $G$ -Bernoulli over  $\mathcal{B}(Y)$ . Since  $(q \times f)$  preserves stabilizers, meaning  $\text{Stab}(x) = \text{Stab}((q \times f)(x))$  for  $\mu$ -almost-every  $x \in X$ , we conclude that  $\alpha = (q \times f)^{-1}(\beta)$  is  $G$ -Bernoulli over  $\mathcal{F} = (q \times f)^{-1}(\mathcal{B}(Y))$ .

(2)  $\rightarrow$  (1). Suppose that  $Q : (X, \mu) \rightarrow (L, \lambda)$  and  $\alpha = Q^{-1}(\beta)$  have the stated properties, and define  $q(x)(g) = Q(g^{-1} \cdot x)$  for  $g \in G, x \in X$ . Let's say  $\alpha = \{A_\ell : \ell \in L\}$  where  $A_\ell = Q^{-1}(\ell)$ . It suffices to show that  $(q \times f)_*(\mu_\Gamma) = \lambda^{\Gamma \setminus G} \times \nu_\Gamma$  for  $\theta$ -almost-every  $\Gamma$ . Property (ii) of Definition 3.2 says that, for  $\theta$ -almost-every  $\Gamma$ ,  $\sigma\text{-alg}_G(\alpha)$  and  $\mathcal{F}$  are  $\mu_\Gamma$ -independent and thus  $(q \times f)_*(\mu_\Gamma) = q_*(\mu_\Gamma) \times \nu_\Gamma$ . Finally, properties (i) and (iii) immediately imply that  $q_*(\mu_\Gamma) = \lambda^{\Gamma \setminus G}$  for  $\theta$ -almost-every  $\Gamma$ .  $\square$

We will also work with the following closely related notion.

**Definition 3.4.** Let  $G \curvearrowright (X, \mu)$  be an aperiodic p.m.p. action, and let  $\mathcal{F}$  be a  $G$ -invariant sub- $\sigma$ -algebra. Set  $\theta = \text{Stab}_*(\mu)$  and let  $\mu = \int \mu_\Gamma d\theta(\Gamma)$  be the disintegration of  $\mu$  over  $\theta$ . We say that a partition  $\alpha$  is *weakly  $G$ -Bernoulli over  $\mathcal{F}$*  if (a)  $\alpha$  is independent with the stabilizer map relative to  $\mathcal{F}$ , and (b) for  $\theta$ -almost-every  $\Gamma \leq G$  and every finite set  $W \subseteq G$  which maps injectively into  $G/\Gamma$ , the partitions  $w^{-1} \cdot \alpha, w \in W$ , are mutually  $\mu_\Gamma$ -independent relative to  $\mathcal{F}$ .

We would like to explicitly observe how these two definitions simplify in the case of free actions. We leave the proof of the following lemma as an easy exercise.

**Lemma 3.5.** *Let  $G \curvearrowright (X, \mu)$  be a free p.m.p. action (such as an aperiodic action with  $G = \mathbb{Z}$ ), let  $\mathcal{F}$  be a  $G$ -invariant sub- $\sigma$ -algebra, and let  $\alpha$  be a Borel partition. Then:*

- (1)  $\alpha$  is  $G$ -Bernoulli over  $\mathcal{F}$  if and only if the  $G$ -translates of  $\alpha$  are mutually independent and  $\sigma\text{-alg}_G(\alpha)$  is independent with  $\mathcal{F}$ ;
- (2)  $\alpha$  is weakly  $G$ -Bernoulli over  $\mathcal{F}$  if and only if the  $G$ -translates of  $\alpha$  are mutually independent relative to  $\mathcal{F}$ .

The notion of weak Bernoullicity will be particularly convenient when discussing free non-ergodic actions, as the following lemma illustrates.

**Lemma 3.6.** *Let  $G \curvearrowright (X, \mu)$  be a free p.m.p. action, let  $\alpha$  be a partition, and let  $\mathcal{F}$  be a  $G$ -invariant sub- $\sigma$ -algebra. The following are equivalent:*

- (1)  $\alpha$  is weakly  $(G, \mu)$ -Bernoulli over  $\mathcal{I}_G$  and  $\sigma\text{-alg}_G(\alpha)$  is independent with  $\mathcal{F}$  relative to  $\mathcal{I}_G$ ;
- (2)  $\alpha$  is  $(G, \nu)$ -Bernoulli over  $\mathcal{F}$  for almost-every ergodic component  $\nu$  of  $\mu$ .

*Proof.* The equivalence is based off of the simple fact that, for a  $\sigma$ -algebra  $\Sigma$ , the property of independence relative to  $\Sigma$  is equivalent to the property of  $\nu$ -independence for almost-every fiber measure  $\nu$  in the disintegration of  $\mu$  relative to  $\Sigma$ . One needs to only recall Definitions 3.2 and 3.4 and observe that the disintegration of  $\mu$  relative to  $\mathcal{I}_G$  coincides with the ergodic decomposition of  $\mu$ .  $\square$

Finally, we clarify the difference between Bernoullicity and weak Bernoullicity in a special case.

**Lemma 3.7.** *Let  $G \curvearrowright (X, \mu)$  be an aperiodic p.m.p. action, let  $\mathcal{F}$  be a  $G$ -invariant sub- $\sigma$ -algebra, and let  $\alpha$  be a Borel partition. If  $\text{Stab}$  is  $\mathcal{F}$ -measurable then the following are equivalent:*

- (1)  $\alpha$  is independent with  $\mathcal{F}$  and weakly  $G$ -Bernoulli over  $\mathcal{F}$ ;
- (2)  $\alpha$  is  $G$ -Bernoulli over  $\mathcal{F}$ .

*Proof.* We will refer to properties (i), (ii), and (iii) of Definition 3.2 and properties (a) and (b) of Definition 3.4.

(1)  $\Rightarrow$  (2). Since  $\text{Stab}$  is  $\mathcal{F}$ -measurable and  $\alpha$  is independent with  $\mathcal{F}$ , we get that  $\alpha$  is independent with  $\text{Stab}$ , establishing (i). An equivalent formulation of property (b) is that for  $\theta$ -almost-every  $\Gamma \leq G$ , every finite  $W \subseteq G$ , and every  $g \in G$  with  $g\Gamma \not\subseteq W\Gamma$ , we have that  $g^{-1} \cdot \alpha$  and  $\alpha^W$  are  $\mu_\Gamma$ -independent relative to  $\mathcal{F}$ . By  $G$ -invariance,  $g^{-1} \cdot \alpha$  is independent with  $\mathcal{F}$ . Furthermore, since  $\text{Stab}$  is  $\mathcal{F}$ -measurable, it follows that  $g^{-1} \cdot \alpha$  is  $\mu_\Gamma$ -independent with  $\mathcal{F}$  for  $\theta$ -almost-every  $\Gamma \leq G$ . So it follows from Lemma 2.1 that  $g^{-1} \cdot \alpha$  and  $\alpha^W$  are  $\mu_\Gamma$ -independent. This establishes property (ii). So for  $\theta$ -almost-every  $\Gamma \leq G$  and every finite  $W \subseteq G$  mapping injectively into  $G/\Gamma$ , the partitions  $w^{-1} \cdot \alpha$ ,  $w \in W$ , are  $\mu_\Gamma$ -independent with  $\mathcal{F}$  and mutually  $\mu_\Gamma$ -independent relative to  $\mathcal{F}$ . Hence  $\sigma\text{-alg}_G(\alpha)$  is  $\mu_\Gamma$ -independent with  $\mathcal{F}$  for  $\theta$ -almost-every  $\Gamma \leq G$ . Equivalently,  $\sigma\text{-alg}_G(\alpha)$  and  $\mathcal{F}$  are independent relative to  $\text{Stab}$ , proving (iii).

(2)  $\Rightarrow$  (1). Property (iii) implies  $\alpha$  is independent with  $\mathcal{F}$  relative to  $\text{Stab}$ . Combined with property (i) and Lemma 2.1, it follows that  $\alpha$  is independent with  $\mathcal{F}$ . Since  $\text{Stab}$  is  $\mathcal{F}$ -measurable, property (a) is trivially true. Properties (ii) and (iii) imply that for  $\theta$ -almost-every  $\Gamma \leq G$  and every finite  $W \subseteq G$  mapping injectively into  $G/\Gamma$ , the partitions  $w^{-1} \cdot \alpha$ ,  $w \in W$ , are mutually  $\mu_\Gamma$ -independent and  $\alpha^W$  is  $\mu_\Gamma$ -independent with  $\mathcal{F}$ . It follows immediately that the partitions  $w^{-1} \cdot \alpha$ ,  $w \in W$ , are mutually  $\mu_\Gamma$ -independent relative to  $\mathcal{F}$ , establishing (b).  $\square$

#### 4. TRANSFORMATIONS IN THE FULL-GROUP

For a p.m.p. action  $G \curvearrowright (X, \mu)$ , we let  $E_G^X$  denote the *induced orbit equivalence relation*:

$$E_G^X = \{(x, y) : \exists g \in G \ g \cdot x = y\}.$$

The *full-group* of  $E_G^X$ , denoted  $[E_G^X]$ , is the set of all Borel bijections  $T : X \rightarrow X$  satisfying  $T(x) E_G^X x$  for all  $x \in X$ . In order to prove our main theorem we will apply classical results, such as Sinai's original factor theorem, to the  $\mathbb{Z}$ -actions induced by aperiodic elements  $T \in [E_G^X]$ . It's a simple exercise to check that every  $T \in [E_G^X]$  preserves  $\mu$ .

The group  $[E_G^X]$  is quite large, but the following definition introduces an important constraint that is useful in entropy theory.

**Definition 4.1.** Let  $G \curvearrowright (X, \mu)$  be a p.m.p. action, let  $T \in [E_G^X]$ , and let  $\mathcal{F}$  be a  $G$ -invariant sub- $\sigma$ -algebra. We say that  $T$  is  $\mathcal{F}$ -expressible if there is a  $\mathcal{F}$ -measurable partition  $\{Q_g : g \in G\}$  of  $X$  such that  $T(x) = g \cdot x$  for every  $x \in Q_g$  and all  $g \in G$ .

We recall two elementary lemmas from [45].

**Lemma 4.2.** Let  $G \curvearrowright (X, \mu)$  be a p.m.p. action, let  $\mathcal{F}$  be a  $G$ -invariant sub- $\sigma$ -algebra, and let  $T \in [E_G^X]$  be  $\mathcal{F}$ -expressible. Then  $T(A) \in \sigma\text{-alg}_G(A) \vee \mathcal{F}$  for every set  $A \in \mathcal{B}(X)$ . In particular, every  $G$ -invariant  $\sigma$ -algebra containing  $\mathcal{F}$  must be  $T$ -invariant.

**Lemma 4.3.** Let  $G \curvearrowright (X, \mu)$  be a p.m.p. action and let  $\mathcal{F}$  be a  $G$ -invariant sub- $\sigma$ -algebra. Then the set of  $\mathcal{F}$ -expressible elements of  $[E_G^X]$  is a group under the operation of composition.

In the more familiar language of cocycles, the condition that  $T$  be  $\mathcal{F}$ -expressible is equivalent to the existence of a cocycle  $c : \mathbb{Z} \times X \rightarrow G$  that connects the action of  $T$  to the action of  $G$  and is  $\mathcal{F}$ -measurable in the second coordinate. Since we work with non-free actions, the partition described in Definition 4.1 (and the cocycle) might not be unique. However the following fact will be useful to us.

**Lemma 4.4.** Let  $G \curvearrowright (X, \mu)$  be a p.m.p. action, let  $\mathcal{F}$  be a  $G$ -invariant class-bijective sub- $\sigma$ -algebra, and let  $T \in [E_G^X]$  be  $\mathcal{F}$ -expressible. Then for each  $h \in G$  the set  $\{x \in X : T(x) = h \cdot x\}$  is  $\mathcal{F}$ -measurable.

*Proof.* Fix a  $\mathcal{F}$ -measurable partition  $\{Q_g : g \in G\}$  as in Definition 4.1. Our assumption that  $\mathcal{F}$  is class-bijective implies that the map  $\text{Stab}$  is  $\mathcal{F}$ -measurable. Thus

$$\{x \in X : T(x) = h \cdot x\} = \bigcup_{g \in G} \left( Q_g \cap \{y \in X : g^{-1}h \in \text{Stab}(y)\} \right) \in \mathcal{F}. \quad \square$$

The next lemma illustrates how Bernoullicity can be transferred between the action of  $G$  and the action of a  $\mathcal{F}$ -expressible transformation. Ultimately, when we prove our main theorem we will need to use two elements  $S, T \in [E_G^X]$ . For ease of later reference, we will denote the transformation by  $S$  in the following lemma.

**Lemma 4.5.** Let  $G \curvearrowright (X, \mu)$  be an aperiodic p.m.p. action, let  $\mathcal{F}$  be a  $G$ -invariant class-bijective sub- $\sigma$ -algebra, and let  $S \in [E_G^X]$  be aperiodic and  $\mathcal{F}$ -expressible. Assume  $\beta$  is a partition which is  $G$ -weakly Bernoulli over  $\mathcal{F}$ . Then

- (i)  $\beta$  is  $S$ -weakly Bernoulli over  $\mathcal{F}$ ;
- (ii) if  $\beta$  is furthermore  $G$ -Bernoulli over  $\mathcal{F}$  then it is  $S$ -Bernoulli over  $\mathcal{F}$ ;
- (iii) if  $\alpha \subseteq \sigma\text{-alg}_S(\beta) \vee \mathcal{F}$  is any partition which is  $S$ -Bernoulli over  $\mathcal{F}$ , then  $\alpha$  is  $G$ -Bernoulli over  $\mathcal{F}$ .

*Proof.* (i). Since  $S$  acts freely on  $X$ , it's enough to show that for every finite  $K \subseteq \mathbb{Z}$  the  $S^k$ -translates of  $\beta$ ,  $k \in K$ , are mutually independent relative to  $\mathcal{F}$ . Let  $G \curvearrowright (Y, \nu)$  be the factor of  $(X, \mu)$  associated to  $\mathcal{F}$ , and let  $\mu = \int \mu_y d\nu(y)$  be the disintegration of  $\mu$  over  $\nu$ . Since  $\mathcal{F}$  is class-bijective, the stabilizer map  $\text{Stab} : X \rightarrow \text{Sub}(G)$  is constant  $\mu_y$ -almost-everywhere for  $\nu$ -almost-every  $y$ . Also observe that  $S$  descends to a transformation of  $Y$  since  $S$  is  $\mathcal{F}$ -expressible. Let  $Y_0$

be the set of  $y \in Y$  which have an infinite  $S$ -orbit and which have the property that for every finite  $W \subseteq G$  that maps injectively to  $W \cdot y$ , the  $W^{-1}$ -translates of  $\beta$  are mutually  $\mu_y$ -independent. Our assumptions imply that  $\nu(Y_0) = 1$ . Now fix  $y \in Y_0$  and a finite set  $K \subseteq \mathbb{Z}$ . By Lemma 4.3, each  $S^k$  is  $\mathcal{F}$ -expressible. So there are  $w(k) \in G$ ,  $k \in K$ , with  $w(k) \cdot x = S^k(x)$  for  $\mu_y$ -almost-every  $x \in X$ . This implies that  $w(k)^{-1} \cdot \beta = S^{-k}(\beta)$   $\mu_y$ -almost-everywhere. Additionally, since the  $S$ -orbit of  $y$  is infinite, the map  $\{w(k) : k \in K\} \rightarrow \{w(k) \cdot y : k \in K\}$  is injective. So the partitions  $w(k)^{-1} \cdot \beta = S^{-k}(\beta)$ ,  $k \in K$ , are mutually  $\mu_y$ -independent. We conclude  $\beta$  is  $S$ -weakly Bernoulli over  $\mathcal{F}$ .

(ii). Note that the  $G$ -stabilizer map is  $\mathcal{F}$ -measurable and that the  $S$ -stabilizer map is essentially trivial. Thus the equivalence in Lemma 3.7 holds for both the actions of  $G$  and  $S$ . In particular, our assumption implies  $\beta$  is independent with  $\mathcal{F}$ , and (i) tells us that  $\beta$  is  $S$ -weakly Bernoulli over  $\mathcal{F}$ . Therefore  $\beta$  is  $S$ -Bernoulli over  $\mathcal{F}$ .

(iii). Now suppose that  $\alpha \subseteq \sigma\text{-alg}_S(\beta) \vee \mathcal{F}$  is  $S$ -Bernoulli over  $\mathcal{F}$ . Since  $S$  acts freely, Lemma 3.7 implies that  $\alpha$  is independent with  $\mathcal{F}$ . Additionally, since  $\mathcal{F}$  and  $\mu$  are  $G$ -invariant, we have that for fixed  $g \in G$  the partitions  $g \cdot S^k(\alpha)$ ,  $k \in \mathbb{Z}$ , are mutually independent and  $g \cdot \sigma\text{-alg}_S(\alpha)$  is independent with  $\mathcal{F}$ . Let  $Y_0$  be the set of  $y \in Y$  such that: (a) for every  $g \in G$  the partitions  $g \cdot S^k(\alpha)$ ,  $k \in \mathbb{Z}$ , are mutually  $\mu_y$ -independent, (b) the partitions  $w^{-1} \cdot \beta$ ,  $w \in W$ , are mutually  $\mu_y$ -independent whenever  $W$  is a finite subset of  $G$  that maps injectively to  $W \cdot y$ , and (c) such that  $g \cdot \alpha \subseteq g \cdot \sigma\text{-alg}_S(\beta)$  modulo  $\mu_y$ -null sets for every  $g \in G$ . Note  $\nu(Y_0) = 1$ . Fix  $y \in Y_0$  and fix a finite  $W \subseteq G$  that maps injectively to  $W \cdot y$ . Let  $\{H_i : i \in \mathbb{N}\}$  be the partition of  $G$  where  $g, g' \in G$  lie in the same piece of this partition if and only if there is  $k \in \mathbb{Z}$  with  $S^k(g \cdot y) = g' \cdot y$ . Set  $V_i = H_i \cap W$ , fix  $v_i \in V_i$  for each  $i$ , and set  $K_i = \{k \in \mathbb{Z} : S^k(v_i \cdot y) \in V_i \cdot y\}$ . Note that  $|K_i| = |V_i|$ . For each  $i$ , modulo  $\mu_y$ -null sets we have

$$\{w^{-1} \cdot \alpha : w \in V_i\} = \{v_i^{-1} \cdot S^{-k}(\alpha) : k \in K_i\}.$$

Since  $y \in Y_0$ , for each fixed  $i$  the partitions  $w^{-1} \cdot \alpha$ ,  $w \in V_i$ , are mutually  $\mu_y$ -independent. Furthermore, we have (modulo  $\mu_y$ -null sets)

$$\bigvee_{w \in V_i} w^{-1} \cdot \alpha \subseteq \bigvee_{w \in V_i} w^{-1} \cdot \sigma\text{-alg}_S(\beta) = \bigvee_{g \in H_i} g^{-1} \cdot \beta.$$

As  $y \in Y_0$  it follows that the  $\sigma$ -algebras  $\bigvee_{g \in H_i} g^{-1} \cdot \beta$ ,  $i \in \mathbb{N}$ , are mutually  $\mu_y$ -independent. Therefore the  $\sigma$ -algebras  $\bigvee_{w \in V_i} w^{-1} \cdot \alpha$ ,  $i \in \mathbb{N}$ , are mutually  $\mu_y$ -independent. We conclude that the  $W^{-1}$ -translates of  $\alpha$  are mutually  $\mu_y$ -independent. Since the  $G$ -stabilizer map is  $\mathcal{F}$ -measurable, its also trivially true that  $\alpha$  is independent with this map relative to  $\mathcal{F}$ . So  $\alpha$  is  $G$ -weakly Bernoulli over  $\mathcal{F}$ . We previously noted that  $\alpha$  is independent with  $\mathcal{F}$ , so Lemma 3.7 implies that  $\alpha$  is  $G$ -Bernoulli over  $\mathcal{F}$ .  $\square$

Finally, we draw a connection between  $\mathcal{F}$ -expressibility and entropy theory by recalling the following simple lemma.

**Lemma 4.6.** [2, Lem. 9.5] *Let  $G \curvearrowright (X, \mu)$  be a p.m.p. ergodic action, let  $\mathcal{F}$  be a  $G$ -invariant sub- $\sigma$ -algebra, and let  $T \in [E_G^X]$  be aperiodic and  $\mathcal{F}$ -expressible. Then  $h_G(\alpha | \mathcal{F}) \leq h_T^{\text{KS}}(\alpha | \mathcal{F})$  for every partition  $\alpha$ .*

In the corollary below, we write  $\mathcal{I}_T$  for the  $\sigma$ -algebra of  $T$ -invariant Borel sets.

**Corollary 4.7.** *Let  $G \curvearrowright (X, \mu)$  be an p.m.p. action, let  $\mathcal{F}$  be a  $G$ -invariant sub- $\sigma$ -algebra, and let  $T \in [E_G^X]$  be aperiodic and  $\mathcal{F}$ -expressible. If  $G \curvearrowright (X, \mu)$  has completely positive outer Rokhlin entropy relative to  $\mathcal{F}$ , then  $\mathcal{I}_T \subseteq \mathcal{F}$  modulo  $\mu$ -null sets.*

*Proof.* By Lemma 4.6 we have  $h_G(\mathcal{I}_T | \mathcal{F}) \leq h_T^{\text{KS}}(\mathcal{I}_T | \mathcal{F}) = 0$ . Since  $G \curvearrowright (X, \mu)$  has completely positive outer Rokhlin entropy relative to  $\mathcal{F}$ , we must have  $\mathcal{I}_T \subseteq \mathcal{F}$ .  $\square$

**Corollary 4.8.** *Let  $G \curvearrowright (X, \mu)$  be an aperiodic p.m.p. action. Let  $\mathcal{F}$  be a  $G$ -invariant class-bijective sub- $\sigma$ -algebra such that  $G \curvearrowright (X, \mu)$  has completely positive outer Rokhlin entropy relative to  $\mathcal{F}$ . Then there exists an aperiodic ergodic  $\mathcal{F}$ -expressible  $S \in [E_G^X]$ .*

*Proof.* Let  $G \curvearrowright (Y, \nu)$  be the factor associated to  $\mathcal{F}$ , with factor map  $\phi : X \rightarrow Y$ . By [26, Theorem 3.5] there exists an aperiodic ergodic  $\bar{S} \in [E_G^Y]$ . Fix a Borel partition  $\{\bar{Q}_g : g \in G\}$  satisfying  $\bar{S}(y) = g \cdot y$  for all  $y \in \bar{Q}_g$  and all  $g \in G$ . Set  $Q_g = \phi^{-1}(\bar{Q}_g) \in \mathcal{F}$  and define  $S : X \rightarrow X$  by the rule  $S(x) = g \cdot x$  for  $x \in Q_g$ ,  $g \in G$ . Its an easy exercise to check that  $S$  is a Borel bijection, and clearly  $S$  is  $\mathcal{F}$ -expressible. Finally, to demonstrate that  $S$  is ergodic we must show that  $\mathcal{I}_S = \{\emptyset, X\}$ . By Corollary 4.7 we have  $\mathcal{I}_S \subseteq \mathcal{F}$ . But the restriction of  $S$  to  $\mathcal{F}$  is isomorphic (as a transformation on a measure-algebra) with the ergodic transformation  $\bar{S}$ . Thus  $\mathcal{I}_S$  is trivial and  $S$  is ergodic.  $\square$

## 5. NON-ERGODIC ACTIONS

In this section we retrace and elaborate upon the work of Kieffer and Rahe which revealed the non-ergodic factor theorem to be a direct consequence of the perturbative factor theorem [31]. The purpose of this section is two-fold. Firstly, we will later require a factor theorem for  $\mathbb{Z}$ -actions which is simultaneously relative, non-ergodic, and perturbative, and no such result is formally stated in the literature (though it is known to experts). Secondly, we will directly prove our main theorems only for ergodic actions, but the methods in this section will immediately give their non-ergodic counter-parts for free.

For a standard Borel space  $X$  we write  $\mathcal{P}$  for the set of countable ordered Borel partitions  $\xi = \{C_i : 0 \leq i < |\xi|\}$  of  $X$ . If  $\mu$  is a Borel probability measure on  $X$ , then we write  $\mathcal{P}_H(\mu)$  for the set of  $\xi \in \mathcal{P}$  with  $H_\mu(\xi) < \infty$ . For  $\mathcal{Q} \subseteq \mathcal{P}$  and  $\gamma \in \mathcal{P}$  we set

$$d_\mu^{\text{alg}}(\gamma, \mathcal{Q}) = \inf_{\xi \in \mathcal{Q}} d_\mu^{\text{alg}}(\gamma, \xi).$$

Recall that for a Borel probability measure  $\mu$ , the Rokhlin distance between  $\xi, \gamma \in \mathcal{P}$  is

$$d_\mu^{\text{Rok}}(\xi, \gamma) = H_\mu(\xi | \gamma) + H_\mu(\gamma | \xi)$$

(this may be infinite). The Rokhlin distance doesn't take the order of the partitions into account, so we adjust it by setting

$$d_\mu^*(\xi, \gamma) = d_\mu^{\text{Rok}}(\xi, \gamma) + d_\mu^{\text{alg}}(\xi, \gamma).$$

Observe that the map  $\xi \in \mathcal{P} \mapsto H_\mu(\xi)$  is  $d_\mu^*$ -continuous.

For an ergodic aperiodic p.m.p. action  $G \curvearrowright (X, \mu)$ , a  $G$ -invariant sub- $\sigma$ -algebra  $\mathcal{F}$ , and a probability vector  $\bar{p}$ , set

$$\mathcal{B}_\mu(\bar{p}; \mathcal{F}) = \left\{ \alpha \in \mathcal{P} : \alpha \text{ is } (G, \mu)\text{-Bernoulli over } \mathcal{F} \text{ and } \text{dist}_\mu(\alpha) = \bar{p} \right\}.$$

Additionally, if  $\xi \in \mathcal{P}$  and  $H(\bar{p}) < \infty$  then we define the deficiency

$$\mathcal{D}_\mu^{\bar{p}}(\xi | \mathcal{F}) = |\text{dist}_\mu(\xi) - \bar{p}| + |H_\mu(\xi) - H(\bar{p})| + |h_{G,\mu}(\xi | \mathcal{F}) - H(\bar{p})|.$$

By [2], the map  $\xi \in \mathcal{P} \mapsto h_{G,\mu}(\xi | \mathcal{F})$  is  $d_\mu^*$ -continuous. Hence  $\xi \in \mathcal{P} \mapsto \mathcal{D}_\mu^{\bar{p}}(\xi | \mathcal{F})$  is  $d_\mu^*$ -continuous as well. Similarly, the map  $\mu \mapsto H_\mu(\xi)$  is Borel and by [2] the map  $\mu \in \{\nu \in \mathfrak{M}_G(X) : H_\nu(\xi) < \infty\} \mapsto h_{G,\mu}(\xi | \mathcal{F})$  is Borel provided  $\mathcal{F}$  is countably generated. So the map  $\mu \in \mathfrak{M}_G(X) \mapsto \mathcal{D}_\mu^{\bar{p}}(\xi | \mathcal{F})$  is Borel whenever  $\mathcal{F}$  is countably generated.

**Definition 5.1.** We say that  $G$  satisfies the relative perturbative factor theorem if there exists a function  $\delta : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$  so that the following holds. For all aperiodic ergodic p.m.p. actions  $G \curvearrowright (X, \mu)$ , all  $G$ -invariant sub- $\sigma$ -algebras  $\mathcal{F}$ , and all probability vectors  $\bar{p}$  with  $H(\bar{p}) < \infty$  and  $H(\bar{p}) \leq h_G(X, \mu | \mathcal{F})$  we have  $\mathcal{B}_\mu(\bar{p}; \mathcal{F}) \neq \emptyset$  and furthermore for all countable Borel partitions  $\xi$  of  $X$ , and all  $M, \epsilon > 0$  we have

$$H(\bar{p}) \leq M \text{ and } \mathcal{D}_\mu^{\bar{p}}(\xi | \mathcal{F}) < \delta(M, \epsilon) \implies d_\mu^{\text{alg}}(\xi, \mathcal{B}_\mu(\bar{p}; \mathcal{F})) < \epsilon.$$

While it has never been stated in the literature this way, it is known that all countable amenable groups satisfy the relative perturbative factor theorem (we will explicitly recall this for the group of integers in the next section). Our work in this paper will show that all countably infinite groups satisfy the relative perturbative factor theorem (in fact  $\delta$  can be chosen independent of  $G$ ).

While the property described in Definition 5.1 is stated in terms of ergodic actions, by using the methods of Kieffer and Rahe we will see that this property automatically extends to non-ergodic actions as well. We first recall two of their results.

**Lemma 5.2.** [31, Lem. 3] *Let  $X$  be a standard Borel space and let  $G \curvearrowright X$  be a Borel action. Pick a subset  $\mathcal{Q}_\nu \subseteq \mathcal{P}_H(\nu)$  for each  $\nu \in \mathcal{E}_G(X)$ . Assume that  $\{\nu \in \mathcal{E}_G(X) : \gamma \in \mathcal{Q}_\nu\}$  is Borel for every  $\gamma \in \mathcal{P}$  and that  $\mathcal{Q}_\nu$  is  $d_\nu^*$ -open for every  $\nu \in \mathcal{E}_G(X)$ . Then for fixed  $\gamma \in \mathcal{P}$  the map  $\nu \in \mathcal{E}_G(X) \mapsto d_\nu^{\text{alg}}(\gamma, \mathcal{Q}_\nu)$  is Borel.*

*Proof.* For fixed  $k \in \mathbb{N}$ , this is proven in [31] in the setting of  $k$ -piece partitions with the assumption that each  $\mathcal{Q}_\nu$  is  $d_\nu^{\text{alg}}$ -open. Their proof works in our setting with obvious minor modifications.  $\square$

**Theorem 5.3.** [31, Theorem 3] *Let  $X$  be a standard Borel space and let  $G \curvearrowright X$  be a Borel action. Pick a non-empty subset  $\mathcal{Q}_\nu \subseteq \mathcal{P}_H(\nu)$  for each  $\nu \in \mathcal{E}_G(X)$ . Also pick functions  $\psi_n : \mathcal{E}_G(X) \times \mathcal{P} \rightarrow [0, \infty)$ ,  $n \in \mathbb{N}$ , with the property that for all  $\nu \in \mathcal{E}_G(X)$  the map  $\gamma \mapsto \psi_n(\nu, \gamma)$  is  $d_\nu^{\text{alg}}$ -continuous and for all  $\gamma \in \mathcal{P}$  the map  $\nu \mapsto \psi_n(\nu, \gamma)$  is Borel. Assume that for every  $\nu \in \mathcal{E}_G(X)$  and  $\gamma \in \mathcal{P}$ ,  $\inf_n \psi_n(\nu, \gamma) = 0$  if and only if  $\gamma \in \mathcal{Q}_\nu$ , and assume that for all  $\gamma \in \mathcal{P}$  the map  $\nu \in \mathcal{E}_G(X) \mapsto d_\nu^{\text{alg}}(\gamma, \mathcal{Q}_\nu)$  is Borel. Then there is a Borel partition  $\alpha$  with  $\alpha \in \bigcap_{\nu \in \mathcal{E}_G(X)} \mathcal{Q}_\nu$ .*

*Proof.* This is proved in [31] for  $k$ -piece partitions, but their proof works in our setting without modification.  $\square$

We can now show that the relative perturbative factor theorem passes from ergodic actions to non-ergodic actions. We point out that our main theorem will imply that all countably infinite groups satisfy the initial assumption stated below.

**Theorem 5.4** (Essentially the non-ergodic, relative, perturbative factor theorem). *Suppose that  $G$  satisfies the relative perturbative factor theorem via the function*



$\delta : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$ . Let  $X$  be a standard Borel space, let  $G \curvearrowright X$  be an aperiodic Borel action, and let  $\mathcal{F}$  be a countably generated  $G$ -invariant sub- $\sigma$ -algebra. Fix  $M, \epsilon > 0$  and a countable partition  $\xi$  of  $X$ . If  $\nu \mapsto \bar{p}^\nu$  is a Borel map assigning to each  $\nu \in \mathcal{E}_G(X)$  a probability vector  $\bar{p}^\nu$  satisfying  $H(\bar{p}^\nu) < \infty$  and  $H(\bar{p}^\nu) \leq h_G(X, \nu | \mathcal{F})$ , then there is a Borel partition  $\alpha$  with  $\alpha \in \mathcal{B}_\nu(\bar{p}^\nu; \mathcal{F})$  for every  $\nu \in \mathcal{E}_G(X)$  and satisfying

$$\forall \nu \in \mathcal{E}_G(X) \quad \left( H(\bar{p}^\nu) \leq M \text{ and } \mathcal{D}_\nu^{\bar{p}^\nu}(\xi | \mathcal{F}) < \delta(M, \epsilon) \implies d_\nu^{\text{alg}}(\alpha, \xi) < \epsilon \right).$$

*Proof.* Set  $N = \{\nu : H(\bar{p}^\nu) \leq M \text{ and } \mathcal{D}_\nu^{\bar{p}^\nu}(\xi | \mathcal{F}) < \delta(M, \epsilon)\}$ . For  $\nu \in N$  set  $\mathcal{Q}_\nu = \{\beta \in \mathcal{B}_\nu(\bar{p}^\nu; \mathcal{F}) : d_\nu^{\text{alg}}(\xi, \beta) < \epsilon\}$  and for  $\nu \notin N$  set  $\mathcal{Q}_\nu = \mathcal{B}_\nu(\bar{p}^\nu; \mathcal{F})$ . Our assumption on  $G$  implies each  $\mathcal{Q}_\nu \neq \emptyset$ . It suffices to build an ordered partition  $\alpha$  with  $\alpha \in \mathcal{Q}_\nu$  for every  $\nu \in \mathcal{E}_G(X)$ .

We claim that for every  $\gamma \in \mathcal{P}$  the map  $\nu \mapsto d_\nu^{\text{alg}}(\gamma, \mathcal{Q}_\nu)$  is Borel. For  $n \in \mathbb{N}$  set

$$\mathcal{Q}_\nu^n = \begin{cases} \{\gamma : \mathcal{D}_\nu^{\bar{p}^\nu}(\gamma | \mathcal{F}) < \delta(M, 1/n) \text{ and } d_\nu^{\text{alg}}(\xi, \gamma) < \epsilon - 1/n\} & \text{if } \nu \in N \\ \{\gamma : \mathcal{D}_\nu^{\bar{p}^\nu}(\gamma | \mathcal{F}) < \delta(H(\bar{p}^\nu), 1/n)\} & \text{if } \nu \notin N. \end{cases}$$

For each  $n$  the family  $(\mathcal{Q}_\nu^n)_{\nu \in \mathcal{E}_G(X)}$  satisfies the assumptions of Lemma 5.2, so the map  $\nu \mapsto d_\nu^{\text{alg}}(\gamma, \mathcal{Q}_\nu^n)$  is Borel for every  $\gamma \in \mathcal{P}$ . Now fix  $\gamma \in \mathcal{P}$ . Each  $\beta \in \mathcal{Q}_\nu$  lies in  $\mathcal{Q}_\nu^n$  for all but finitely many  $n$ , so  $d_\nu^{\text{alg}}(\gamma, \mathcal{Q}_\nu) \geq \limsup_{n \rightarrow \infty} d_\nu^{\text{alg}}(\gamma, \mathcal{Q}_\nu^n)$ . On the other hand, by definition of  $\delta$  we have  $d_\nu^{\text{alg}}(\gamma, \mathcal{Q}_\nu) \leq d_\nu^{\text{alg}}(\gamma, \mathcal{Q}_\nu^n) + 1/n$ . Therefore  $d_\nu^{\text{alg}}(\gamma, \mathcal{Q}_\nu) = \lim_{n \rightarrow \infty} d_\nu^{\text{alg}}(\gamma, \mathcal{Q}_\nu^n)$ . We conclude that the map  $\nu \mapsto d_\nu^{\text{alg}}(\gamma, \mathcal{Q}_\nu)$  is Borel for every  $\gamma \in \mathcal{P}$ .

For  $\gamma = \{C_i : i \in \mathbb{N}\} \in \mathcal{P}$  and  $n \in \mathbb{N}$ , let  $\gamma_{(n)}$  be the partition of  $X$  into the sets  $C_i$ ,  $0 \leq i < n$ , and  $\bigcup_{i=n}^\infty C_i$ . For  $n \in \mathbb{N}$  define  $\psi_n(\nu, \gamma) = \mathcal{D}_\nu^{\bar{p}^\nu}(\gamma_{(n)} | \mathcal{F})$  when  $\nu \notin N$ , and define

$$\psi_n(\nu, \gamma) = \mathcal{D}_\nu^{\bar{p}^\nu}(\gamma_{(n)} | \mathcal{F}) + \max(0, n \cdot d_\nu^{\text{alg}}(\xi, \gamma) - n \cdot \epsilon + 1)$$

when  $\nu \in N$ . Then  $\psi_n(\cdot, \gamma)$  is Borel and  $\psi_n(\nu, \cdot)$  is  $d_\nu^{\text{alg}}$ -continuous. Also,  $\beta \in \mathcal{Q}_\nu$  if and only if  $\inf_n \psi_n(\nu, \beta) = 0$ . Therefore by Theorem 5.3 there is a partition  $\alpha \in \bigcap_{\nu \in \mathcal{E}_G(X)} \mathcal{Q}_\nu$ .  $\square$

Finally, we observe that the non-ergodic relative factor theorem follows as well. Again, it will follow from our main theorem that the initial assumption on  $G$  can be dropped and that this theorem holds for all countably infinite groups.

**Theorem 5.5** (Essentially the non-ergodic relative factor theorem). *Assume that  $G$  satisfies the relative perturbative factor theorem. Let  $X$  be a standard Borel space, let  $G \curvearrowright X$  be an aperiodic Borel action, and let  $\mathcal{F}$  be a countably generated  $G$ -invariant sub- $\sigma$ -algebra. If  $\nu \in \mathcal{E}_G(X) \mapsto \lambda_\nu$  is a Borel map assigning to each  $\nu \in \mathcal{E}_G(X)$  a Borel probability measure  $\lambda_\nu$  on  $[0, 1]$  satisfying  $h_G(X, \nu | \mathcal{F}) \geq H([0, 1], \lambda_\nu)$ , then there is a  $G$ -equivariant Borel map  $\phi : X \rightarrow [0, 1]^G$  such that, for every  $\nu \in \mathcal{E}_G(X)$ ,  $\phi_*(\nu) = \lambda_\nu^{\text{Stab}_*(\nu) \setminus G}$  and  $\phi^{-1}(\mathcal{B}([0, 1]^G))$  is  $\nu$ -independent with  $\mathcal{F}$  relative to  $\text{Stab}^{-1}(\mathcal{B}(\text{Sub}(G)))$ .*

*Proof.* By the ergodic decomposition theorem [14, 54], there is a  $G$ -invariant Borel map  $e : X \rightarrow \mathcal{E}_G(X)$  such that  $\nu(e^{-1}(\nu)) = 1$  for every  $\nu \in \mathcal{E}_G(X)$ . Setting  $X_{\text{fin}} = \{x \in X : H([0, 1], \lambda_{e(x)}) < \infty\}$  and  $X_\infty = \{x \in X : H([0, 1], \lambda_{e(x)}) = \infty\}$ , we obtain a  $G$ -invariant Borel partition  $\{X_{\text{fin}}, X_\infty\}$  of  $X$ . It suffices to define

$\phi$  separately on  $X_{\text{fin}}$  and on  $X_\infty$ . So without loss of generality we may assume  $X = X_{\text{fin}}$  or  $X = X_\infty$ .

First suppose that  $H([0, 1], \lambda_\nu) < \infty$  for all  $\nu \in \mathcal{E}_G(X)$ . Then every  $\lambda_\nu$  has countable support. Enumerate the atoms of  $\lambda_\nu$  as  $a_\nu^1, a_\nu^2, \dots$  so that  $\lambda_\nu(a_\nu^k) \geq \lambda_\nu(a_\nu^{k+1})$  and  $a_\nu^k \leq a_\nu^{k+1}$  when  $\lambda_\nu(a_\nu^k) = \lambda_\nu(a_\nu^{k+1})$ . Note that the numbers  $a_\nu^k \in [0, 1]$  can be determined in a Borel way from the  $\lambda_\nu$ -measures of all the intervals with rational endpoints. Therefore the map  $\nu \in \mathcal{E}_G(X) \mapsto (a_\nu^k)_{k \in \mathbb{N}} \in [0, 1]^\mathbb{N}$  is Borel. Set  $\bar{p}^\nu = (\lambda_\nu(a_\nu^k))_{k \in \mathbb{N}}$ . Since  $\lambda_\nu(a_\nu^k)$  is the infimum of the  $\lambda_\nu$ -measure of the intervals with rational endpoints containing  $a_\nu^k$ , the map  $\nu \mapsto \bar{p}^\nu$  is Borel as well. By Theorem 5.4 there is a Borel partition  $\beta$  with  $\beta \in \mathcal{B}_\nu(\bar{p}^\nu; \mathcal{F})$  for all  $\nu \in \mathcal{E}_G(X)$ . Say  $\beta = \{B_n : n \in \mathbb{N}\}$ . Define  $\phi : X \rightarrow [0, 1]^G$  by  $\theta(x)(g) = a_{e(x)}^k$  if  $g^{-1} \cdot x \in B_k$ . Then Lemmas 3.1 and 3.3 imply that  $\phi$  has the desired properties for every  $\nu \in \mathcal{E}_G(X)$ .

Now suppose that  $H([0, 1], \lambda_\nu) = \infty$  for all  $\nu \in \mathcal{E}_G(X)$ . Then  $h_G(X, \nu | \mathcal{F}) = \infty$  for every  $\nu \in \mathcal{E}_G(X)$ . Set  $\bar{p} = (\frac{1}{2}, \frac{1}{2})$ . Inductively for  $n \in \mathbb{N}$  we build a sequence of Borel partitions  $\beta^n \in \bigcap_{\nu \in \mathcal{E}_G(X)} \mathcal{B}_\nu(\bar{p}; \mathcal{F} \vee \bigvee_{k < n} \sigma\text{-alg}_G(\beta^k))$ . Theorem 5.4 provides  $\beta^0 \in \bigcap_{\nu \in \mathcal{E}_G(X)} \mathcal{B}_\nu(\bar{p}; \mathcal{F})$ . Now assume  $\beta^0$  through  $\beta^{n-1}$  have been constructed. Since  $h_{G, \nu}(\bigvee_{k < n} \beta^k | \mathcal{F}) \leq n \cdot \log(2) < \infty$ , by sub-additivity of entropy [2] we have  $h_G(X, \nu | \mathcal{F} \vee \bigvee_{k < n} \sigma\text{-alg}_G(\beta^k)) = \infty$ . Thus again Theorem 5.4 produces  $\beta^n$ . This constructs  $(\beta^n)_{n \in \mathbb{N}}$ . Say  $\beta^n = \{B_0^n, B_1^n\}$ , write  $2 = \{0, 1\}$ , and define the  $G$ -equivariant Borel map  $\phi_0 : X \rightarrow (2^\mathbb{N})^G$  by  $\phi_0(x)(g)(n) = i$  if and only if  $g^{-1} \cdot x \in B_i^n$ . Then by Lemmas 3.1 and 3.3  $\phi_0$  pushes every  $\nu$  forward to  $(u_2^\mathbb{N})^{\text{Stab}^*(\nu) \setminus G}$ , where  $u_2$  is the normalized counting measure on  $2$ ,  $\phi_0$  is  $\nu$ -class-bijective, and  $\phi_0^{-1}(\mathcal{B}((2^\mathbb{N})^G)) = \bigvee_{n \in \mathbb{N}} \sigma\text{-alg}_G(\beta^n)$  is  $\nu$ -independent with  $\mathcal{F}$  relative to  $\text{Stab}^{-1}(\mathcal{B}(\text{Sub}(G)))$  for every  $\nu$ . Now fix a Borel isomorphism  $\pi$  from  $2^\mathbb{N}$  to  $[0, 1]$  which takes  $u_2^\mathbb{N}$  to Lebesgue measure  $\text{Leb}$ . Define the Borel map  $(\nu, y) \in \mathcal{E}_G(X) \times [0, 1] \mapsto \psi_\nu(y) \in [0, 1]$  by  $\psi_\nu(y) = \inf\{q \in \mathbb{Q} : \lambda_\nu([0, q]) \geq y\}$ . Each  $\psi_\nu$  pushes  $\text{Leb}$  forward to  $\lambda_\nu$  since for  $r \in [0, 1]$ ,  $B = [0, r]$ , and  $y \in [0, 1]$  we have

$$\lambda_\nu(B) \geq y \iff r \geq \psi_\nu(y) \iff y \in \psi_\nu^{-1}(B) \iff \text{Leb}(\psi_\nu^{-1}(B)) \geq y.$$

Finally, we define  $\phi : X \rightarrow [0, 1]^G$  by  $\phi(x)(g) = \psi_{e(x)} \circ \pi \circ \phi_0(x)(g)$ . This completes the proof.  $\square$

## 6. A SPECIAL PERTURBATION

In this section we develop a specialized version of Theorem 5.4 that, in the case of  $\mathbb{Z}$ -actions, will be important for our main theorem. Unlike Theorem 5.4, the main result in this section will be stated entirely in terms of a fixed non-ergodic measure  $\mu$ , rather than in terms of its ergodic components. A key feature will be the perturbation of a partition  $\xi$  to a Bernoulli partition  $\alpha$  in a manner which only slightly changes its Shannon entropy.

In our proof it will be important that  $\mathbb{Z}$  satisfies the relative perturbative factor theorem, so we explicitly record this fact now.

**Lemma 6.1** (essentially Thouvenot [53] (see also [38])). *Let  $\mathbb{Z} \curvearrowright (X, \mu)$  be an aperiodic p.m.p. ergodic action, let  $\mathcal{F}$  be a  $\mathbb{Z}$ -invariant sub- $\sigma$ -algebra, and let  $\bar{p}$  be a probability vector. If  $H(\bar{p}) \leq h_\mathbb{Z}^{\text{KS}}(X, \mu | \mathcal{F})$  then there is a partition  $\alpha$  with  $\text{dist}(\alpha) = \bar{p}$  which is  $G$ -Bernoulli over  $\mathcal{F}$ .*

Moreover, there is a function  $\delta_{\mathbb{Z}} : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$  such that, for any  $\mathbb{Z} \curvearrowright (X, \mu)$ ,  $\mathcal{F}$ , and  $\bar{p}$  satisfying the assumptions above, if  $H(\bar{p}) \leq M$  and  $\xi$  is a partition satisfying

$$|\text{dist}(\xi) - \bar{p}| + |H(\xi) - H(\bar{p})| + |h_{\mathbb{Z}}^{\text{KS}}(\xi | \mathcal{F}) - H(\bar{p})| < \delta_{\mathbb{Z}}(M, \epsilon)$$

then the partition  $\alpha$  above may be chosen so that  $d_{\mu}^{\text{alg}}(\alpha, \xi) < \epsilon$ .

In order to explain the proof of the above lemma, we will need the following fact.

**Lemma 6.2.** [13, Fact 1.1.11] *For every  $M, \epsilon > 0$  there exists  $\eta = \eta(M, \epsilon) > 0$  with the following property. Let  $(X, \mu)$  be a standard probability space, let  $\mathcal{F}$  be a sub- $\sigma$ -algebra, let  $(Y, \nu)$  be the factor of  $(X, \mu)$  associated to  $\mathcal{F}$ , and let  $\mu = \int \mu_y \, d\nu(y)$  be the disintegration of  $\mu$  over  $\nu$ . If  $\gamma$  is a partition of  $X$  with  $H(\gamma) \leq M$  and  $H(\gamma | \mathcal{F}) > H(\gamma) - \eta(M, \epsilon)$ , then*

$$\nu(\{y \in Y : |\text{dist}_{\mu_y}(\gamma) - \text{dist}(\gamma)| < \epsilon\}) > 1 - \epsilon.$$

*Proof sketch of Lemma 6.1.* The first statement is well-known and is implied by the work of Thouvenot [53]. The second statement is stated a tad differently than it has previously appeared in the literature but is certainly known to experts. We briefly describe how to obtain the precise statement above, but we leave the details to the reader.

Working with  $k$ -piece partitions and length- $k$  probability vectors, Thouvenot proved the above statement with two modifications: (i) in place of  $\delta_{\mathbb{Z}}$  he used a quantity  $\delta_0(k, \epsilon)$  depending only on  $k$  and  $\epsilon$ ; and (ii) he omitted the  $|H(\xi) - H(\bar{p})|$  term from the inequality. Inspection of his proof reveals that, in the special case where  $h_{\mathbb{Z}}^{\text{KS}}(\xi | \mathcal{F}) < H(\bar{p})$  [53, Prop. 2], the quantity  $\delta_0(k, \epsilon)$  ultimately depends upon the uniform continuity of Shannon entropy on the space of length- $k$  probability vectors, and on a weakened version of Lemma 6.2. By insisting that  $|H(\xi) - H(\bar{p})|$  be small, and by using the stronger Lemma 6.2, we can instead choose a  $\delta_1(M, \epsilon)$  depending only upon  $M$  and  $\epsilon$ . Next, in the case  $h_{\mathbb{Z}}^{\text{KS}}(\xi | \mathcal{F}) \geq H(\bar{p})$ , one takes a continuous path of partitions  $(\xi_t)_{t \in [0,1]}$  with  $\xi_0 = \xi$  and  $H(\xi_1) < H(\bar{p})$ . Then there will be  $s \in [0, 1]$  with  $h_{\mathbb{Z}}^{\text{KS}}(\xi_s | \mathcal{F}) < H(\bar{p})$ , allowing the prior argument to be applied. Along this path, the Shannon entropy does not need to drop by much, so Lemma 6.3 from the next section implies that there is a path where the distribution of  $\xi_s$  does not change too much either, in fact the change in distribution is bounded by a function of the change in Shannon entropy. So our statement above holds for  $k$ -piece partitions and length- $k$  probability vectors simultaneously for every  $k$ . Finally, given a countable partition  $\xi$  we can find finite partitions  $\xi'$  coarser than  $\xi$  which approximate  $\text{dist}(\xi)$ ,  $H(\xi)$ , and  $h_{\mathbb{Z}}^{\text{KS}}(\xi | \mathcal{F})$  arbitrarily well. Similarly, if  $|\bar{p}| = \infty$  we can choose a sequence of finite probability vectors  $\bar{p}^n$  satisfying  $|\bar{p}^n - \bar{p}| + 2|H(\bar{p}^n) - H(\bar{p})| < (1/2)\delta_1(M, 2^{-n-1}\epsilon)$ . One can then perform a standard construction of a Cauchy sequence of partitions  $\alpha^n$  which are  $\mathbb{Z}$ -Bernoulli over  $\mathcal{F}$  and have distribution  $\bar{p}^n$ , and the limiting partition  $\alpha$  will have the desired properties. It suffices to set  $\delta_{\mathbb{Z}}(M, \epsilon) = \delta_1(M, \epsilon/2)$  for this construction.  $\square$

Next is a technical lemma. Below we write  $\mathbf{1}$  for the trivial probability vector which has 0<sup>th</sup>-coordinate 1 and all other coordinates 0.

**Lemma 6.3.** *There is a monotone increasing continuous function  $\theta : [0, \infty) \rightarrow [0, 2)$  with  $\theta(0) = 0$  satisfying the following property. If  $\bar{p} = (p_i)$  is a probability vector with  $p_0 = \max(\bar{p})$  and  $H(\bar{p}) < \infty$  then, setting  $\bar{p}^t = t\mathbf{1} + (1-t)\bar{p}$ , we have  $H(\bar{p}^t) \leq H(\bar{p})$  and  $|\bar{p} - \bar{p}^t| \leq \theta(H(\bar{p}) - H(\bar{p}^t))$  for all  $t \in [0, 1]$ .*

*Proof.* For  $x \in [0, 1]$  set  $f(x) = H(x, 1 - x)$ . For  $r \in [0, 1]$  define

$$\psi(r) = \inf\{f(s) + r \cdot f'(s) - f(s + r) : s \in (0, 1 - r)\}.$$

It may be helpful to note that  $f(s) + r \cdot f'(s)$  is a tangent-line approximation to  $f(s + r)$ . Notice that when  $r$  is 0 every value in the above set is 0 and hence  $\psi(0) = 0$ . Furthermore, for fixed  $s$  the function  $r \mapsto f(s) + r \cdot f'(s) - f(s + r)$  is continuous, and it is strictly increasing since  $f$  is concave down. Therefore  $\psi$  is continuous and strictly increasing. Notice that  $\text{rng}(\psi) = [0, \infty)$  since  $f'(s) \rightarrow \infty$  as  $s \rightarrow 0$ . Set  $\theta = 2 \cdot \psi^{-1}$ .

Fix  $\bar{p}$  with  $p_0 = \max(\bar{p})$ . Note that for any  $i \neq 0$  we have

$$H(\bar{p}) = f(p_0 + p_i) + (p_0 + p_i) \cdot f\left(\frac{p_0}{p_0 + p_i}\right) + (1 - p_0 - p_i) \cdot H\left(\frac{1}{1 - p_0 - p_i} \cdot (p_u)_{u \neq 0, i}\right).$$

Thus if we increase the value of  $p_0$ , decrease the value of  $p_i$  by the same amount, and leave all other coordinates the same, then  $H(\bar{p})$  will decrease. By repeating this with  $i$  varying, we find that if we increase  $p_0$  and decrease all other coordinates then  $H(\bar{p})$  will decrease.

Set  $\bar{q} = \frac{1}{1 - p_0} \cdot (p_i)_{i > 0}$  and  $\bar{p}^t = t\mathbf{1} + (1 - t)\bar{p}$ . Note that  $|\bar{p} - \bar{p}^t| = 2t(1 - p_0)$  and

$$(6.1) \quad H(\bar{p}^t) = f(t + (1 - t)p_0) + (1 - t - (1 - t)p_0) \cdot H(\bar{q}).$$

From the previous paragraph we know that  $H(\bar{p}^t)$  is decreasing. In particular,  $f'(p_0) \leq H(\bar{q})$  since the derivative of  $H(\bar{p}^t)$  at  $t = 0$  is

$$(1 - p_0)f'(p_0) - (1 - p_0) \cdot H(\bar{q}) \leq 0.$$

Now fix  $t \in [0, 1]$ , set  $r = \frac{1}{2}|\bar{p} - \bar{p}^t| = t(1 - p_0)$  and set  $s = p_0 \leq 1 - r$ . Using  $f'(p_0) \leq H(\bar{q})$  and (6.1), we obtain

$$\begin{aligned} H(\bar{p}) - H(\bar{p}^t) &= f(s) + (1 - s) \cdot H(\bar{q}) - f(s + r) - (1 - s - r) \cdot H(\bar{q}) \\ &\geq f(s) - f(s + r) + r \cdot f'(s) \geq \psi\left(\frac{1}{2}|\bar{p} - \bar{p}^t|\right). \end{aligned}$$

Therefore  $|\bar{p} - \bar{p}^t| \leq \theta(H(\bar{p}) - H(\bar{p}^t))$  as claimed.  $\square$

The previous lemma allows us to choose probability vectors in an intelligent way.

**Corollary 6.4.** *For every  $\epsilon > 0$  there is  $\kappa = \kappa(\epsilon) > 0$  having the following property. If  $(X, \mu)$  is a probability space and  $x \mapsto \bar{q}^x$  and  $x \mapsto h_x \leq H(\bar{q}^x) < \infty$  are Borel maps, then there is a Borel map  $x \mapsto \bar{p}^x$  such that  $H(\bar{p}^x) = h_x$ ,  $H(\int_X \bar{p}^x d\mu(x)) \leq H(\int_X \bar{q}^x d\mu(x))$ , and*

$$\forall x \in X \quad \left( \left| \int_X \bar{q}^y d\mu(y) - \bar{q}^x \right| < \kappa \text{ and } |H(\bar{q}^x) - h_x| < \kappa \right) \implies |\bar{p}^x - \bar{q}^x| < \epsilon.$$

*Proof.* Set  $f(t) = H(t, 1 - t)$  for  $t \in [0, 1]$  and let  $\theta$  be as in Lemma 6.3. Choose  $0 < \kappa < \epsilon/2$  so that  $2\kappa + \theta(f(\kappa) + \kappa) < \epsilon$ . Set  $\bar{q} = \int_X \bar{q}^x d\mu(x)$ . By uniformly reordering coordinates for all  $x \in X$  simultaneously, we may assume that  $q_0 = \max(\bar{q})$ . For  $x \in X$  and  $t \in [0, 1]$  set  $\bar{q}^{x,t} = t\mathbf{1} + (1 - t)\bar{q}^x$ . The value  $H(\bar{q}^{x,t})$  moves continuously from  $H(\bar{q}^x)$  to 0, so we may define  $\bar{p}^x = \bar{q}^{x,v}$  where  $v$  is least satisfying  $H(\bar{q}^{x,v}) = h_x$ . Then  $x \mapsto \bar{p}^x$  is Borel and  $H(\bar{p}^x) = h_x$ . Set  $\bar{p} = \int_X \bar{p}^x d\mu(x)$ . Then  $p_0 \geq q_0$  while  $p_i \leq q_i$  for all  $i > 0$ . As argued in the proof of Lemma 6.3, this implies that  $H(\bar{p}) \leq H(\bar{q})$ .

Finally, fix  $x$  with  $|\bar{q} - \bar{q}^x| < \kappa$  and  $H(\bar{q}^x) - \kappa < h_x \leq H(\bar{q}^x)$ . Define  $v$  as above. We must have  $\max(\bar{q}^x) - q_0^x < \kappa$ , so there is  $u$  with  $u(1 - q_0^x) < \kappa$  and

$q_0^{x,u} = \max(\bar{q}^{x,u})$ . If  $u \geq v$  then  $|\bar{p}^x - \bar{q}^x| \leq |\bar{q}^{x,u} - \bar{q}^x| = 2u(1 - q_0^x) < 2\kappa < \epsilon$  and we are done. So assume  $u < v$  and set  $\bar{r}^x = \frac{1}{1 - q_0^x} \cdot (q_i^x)_{i>0}$ . Notice

$$\mathbb{H}(\bar{q}^{x,t}) = f(t + (1 - t)q_0^x) + (1 - t - (1 - t)q_0^x) \cdot \mathbb{H}(\bar{r}^x).$$

Using the concavity of  $f$  and the inequality  $u + (1 - u)q_0^x = q_0^x + u(1 - q_0^x) < q_0^x + \kappa$ , we observe that

$$\begin{aligned} \mathbb{H}(\bar{q}^{x,u}) - h_x &= \mathbb{H}(\bar{q}^{x,u}) - \mathbb{H}(\bar{q}^x) + \mathbb{H}(\bar{q}^x) - h_x \\ &\leq f(u + (1 - u)q_0^x) - f(q_0^x) + \kappa \\ &\leq f(\kappa) - f(0) + \kappa. \end{aligned}$$

Noting that  $f(0) = 0$  and applying Lemma 6.3 to  $\bar{q}^{x,u}$  we conclude

$$\begin{aligned} |\bar{q}^x - \bar{p}^x| &\leq |\bar{q}^x - \bar{q}^{x,u}| + |\bar{q}^{x,u} - \bar{q}^{x,v}| \leq 2u(1 - q_0^x) + \theta(\mathbb{H}(\bar{q}^{x,u}) - h_x) \\ &< 2\kappa + \theta(f(\kappa) + \kappa) < \epsilon. \quad \square \end{aligned}$$

Now we present the main result of this section, a specialized version of Theorem 5.4. It may be helpful to recall Lemma 3.6.

**Corollary 6.5.** *Assume  $G$  satisfies the relative perturbative factor theorem. Let  $G \curvearrowright (X, \mu)$  be a free p.m.p. action and let  $\mathcal{F}$  be a  $G$ -invariant sub- $\sigma$ -algebra. If  $\xi$  is a countable partition of  $X$  with  $\mathbb{H}(\xi | \mathcal{F}) < \infty$  then there is a partition  $\alpha$  such that  $\alpha$  is weakly  $G$ -Bernoulli over  $\mathcal{I}_G$ ,  $\sigma\text{-alg}_G(\alpha)$  is independent with  $\mathcal{F}$  relative to  $\mathcal{I}_G$ , and  $\mathbb{H}(\alpha | \mathcal{I}_G) = h_G(\xi | \mathcal{F})$ .*

*In fact, for  $M, \epsilon > 0$  there exists  $v > 0$  (depending only on  $M, \epsilon$ , and  $G^\dagger$ ) with the following property. If  $\xi$  is a partition satisfying  $\mathbb{H}(\xi) \leq M$  and  $\mathbb{H}(\xi) - h_G(\xi | \mathcal{F}) < v$ , then the partition  $\alpha$  above may be chosen with the additional properties that  $d_\mu^{\text{alg}}(\alpha, \xi) < \epsilon$  and  $|\mathbb{H}(\alpha) - \mathbb{H}(\xi)| < \epsilon$ .*

*Proof.* The first statement is nearly an immediate consequence of Theorem 5.4, as our argument will reveal below. We focus on proving the second claim. Let  $\delta : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$  witness that  $G$  satisfies the relative perturbative factor theorem. Choose  $M' > 12M/\epsilon$  and set  $\epsilon' = (1/2)\delta(M', \epsilon/2)$ . Next, let  $\kappa = \kappa(\epsilon')$  be as in Corollary 6.4. If necessary, we may shrink  $\kappa$  so that  $\kappa < \epsilon/12$  and  $\kappa < (1/2)\delta(M', \epsilon/2)$ . Finally, let  $v = \eta(M, \kappa)$  be as in Lemma 6.2. If necessary, shrink  $v$  so that  $v < (1/12)\kappa\epsilon$ .

Let  $G \curvearrowright (X, \mu)$ ,  $\mathcal{F}$ , and  $\xi$  be as in the statement of the corollary. If necessary, replace  $\mathcal{F}$  with a countably generated  $\sigma$ -algebra that is equivalent modulo  $\mu$ -null sets. Let  $\mu = \int_{\mathcal{E}_G(X)} \nu \, d\tau(\nu)$  be the ergodic decomposition of  $\mu$ . Rokhlin entropy is an affine function on the space of invariant measures [2], so

$$\int_{\mathcal{E}_G(X)} h_{G,\nu}(\xi | \mathcal{F}) \, d\tau(\nu) = h_{G,\mu}(\xi | \mathcal{F}).$$

Combined with the inequalities  $h_{G,\mu}(\xi | \mathcal{F}) \leq \mathbb{H}_\mu(\xi) \leq M$ , this implies

$$(6.2) \quad \tau(\{\nu \in \mathcal{E}_G(X) : h_{G,\nu}(\xi | \mathcal{F}) \leq M'\}) \geq 1 - \frac{M}{M'} > 1 - \frac{\epsilon}{12}.$$

Next, by applying Lemma 6.2 to the inequality

$$\mathbb{H}_\mu(\xi | \mathcal{I}_G) \geq h_{G,\mu}(\xi | \mathcal{F} \vee \mathcal{I}_G) = h_{G,\mu}(\xi | \mathcal{F}) > \mathbb{H}_\mu(\xi) - v$$

<sup>†</sup>Actually, the proof of our main theorem will show that  $v$  can be chosen independent of  $G$ .

we find that

$$(6.3) \quad \tau(\{\nu \in \mathcal{E}_G(X) : |\text{dist}_\nu(\xi) - \text{dist}_\mu(\xi)| < \kappa\}) > 1 - \kappa > 1 - \epsilon/12.$$

Finally, since  $\mathbf{H}_\nu(\xi) - h_{G,\nu}(\xi | \mathcal{F}) \geq 0$  for all  $\nu$  and

$$\begin{aligned} \int_{\mathcal{E}_G(X)} \left( \mathbf{H}_\nu(\xi) - h_{G,\nu}(\xi | \mathcal{F}) \right) d\tau(\nu) &= \mathbf{H}_\mu(\xi | \mathcal{I}_G) - h_{G,\mu}(\xi | \mathcal{F}) \\ &\leq \mathbf{H}_\mu(\xi) - h_{G,\mu}(\xi | \mathcal{F}) < \nu, \end{aligned}$$

we must have

$$(6.4) \quad \tau(\{\nu \in \mathcal{E}_G(X) : |\mathbf{H}_\nu(\xi) - h_{G,\nu}(\xi | \mathcal{F})| < \kappa\}) > 1 - \frac{\nu}{\kappa} > 1 - \frac{\epsilon}{12}.$$

Let  $N$  be the set of  $\nu$  with  $h_{G,\nu}(\xi | \mathcal{F}) \leq M'$ ,  $|\text{dist}_\nu(\xi) - \text{dist}_\mu(\xi)| < \kappa$ , and  $|\mathbf{H}_\nu(\xi) - h_{G,\nu}(\xi | \mathcal{F})| < \kappa$ . Inequalities (6.2) (6.3) (6.4) show that  $\tau(N) > 1 - \epsilon/4$ .

Set  $\bar{q}^\nu = \text{dist}_\nu(\xi)$  and  $h_\nu = h_{G,\nu}(\xi | \mathcal{F})$ . Then  $\nu \mapsto \bar{q}^\nu$  is Borel and, since  $\mathbf{H}_\nu(\xi) < \infty$  for  $\tau$ -almost-every  $\nu \in \mathcal{E}_G(X)$ , the map  $\nu \mapsto h_\nu$  is Borel on a  $\tau$ -conull Borel subset of  $\mathcal{E}_G(X)$  [2]. Apply Corollary 6.4 to obtain a Borel map  $\nu \mapsto \bar{p}^\nu$  such that  $\mathbf{H}(\int_{\mathcal{E}_G(X)} \bar{p}^\nu d\tau(\nu)) \leq \mathbf{H}_\mu(\xi)$ ,  $\mathbf{H}(\bar{p}^\nu) = h_{G,\nu}(\xi | \mathcal{F})$  for all  $\nu$ , and such that for  $\tau$ -almost-every  $\nu \in \mathcal{E}_G(X)$  we have

$$\left( |\text{dist}_\nu(\xi) - \text{dist}_\mu(\xi)| < \kappa \text{ and } |\mathbf{H}_\nu(\xi) - h_{G,\nu}(\xi | \mathcal{F})| < \kappa \right) \implies |\bar{p}^\nu - \text{dist}_\nu(\xi)| < \epsilon'.$$

Note that if  $\nu \in N$  then  $\mathbf{H}(\bar{p}^\nu) = h_{G,\nu}(\xi | \mathcal{F}) \leq M'$  and

$$\begin{aligned} \mathcal{D}_\nu^{\bar{p}^\nu}(\xi | \mathcal{F}) &= |\text{dist}_\nu(\xi) - \bar{p}^\nu| + |\mathbf{H}_\nu(\xi) - \mathbf{H}(\bar{p}^\nu)| + |h_{G,\nu}(\xi | \mathcal{F}) - \mathbf{H}(\bar{p}^\nu)| \\ &< \epsilon' + |\mathbf{H}_\nu(\xi) - h_{G,\nu}(\xi | \mathcal{F})| + 0 \\ &< \epsilon' + \kappa < \delta(M', \epsilon/2). \end{aligned}$$

Now apply Theorem 5.4 to obtain a Borel partition  $\alpha \in \bigcap_\nu \mathcal{B}_\nu(\bar{p}^\nu; \mathcal{F})$  satisfying

$$\forall \nu \in \mathcal{E}_G(X) \quad \left( \mathbf{H}(\bar{p}^\nu) \leq M' \text{ and } \mathcal{D}_\nu^{\bar{p}^\nu}(\xi | \mathcal{F}) < \delta(M', \epsilon/2) \implies d_\nu^{\text{alg}}(\alpha, \xi) < \epsilon/2 \right).$$

In particular,  $\alpha$  is weakly  $G$ -Bernoulli over  $\mathcal{I}_G$  and  $\sigma\text{-alg}_G(\alpha)$  is independent with  $\mathcal{F}$  relative to  $\mathcal{I}_G$  by Lemma 3.6. Also, since  $\tau(N) > 1 - \epsilon/4$  and  $d_\nu^{\text{alg}}(\alpha, \xi) < \epsilon/2$  for all  $\nu \in N$ , we obtain

$$d_\mu^{\text{alg}}(\xi, \alpha) = \int_{\mathcal{E}_G(X)} d_\nu^{\text{alg}}(\xi, \alpha) d\tau(\nu) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

The ergodic decomposition of  $\mu$  coincides with the disintegration of  $\mu$  relative to  $\mathcal{I}_G$ , so

$$\mathbf{H}_\mu(\alpha | \mathcal{I}_G) = \int_{\mathcal{E}_G(X)} \mathbf{H}_\nu(\alpha) d\tau(\nu) = \int_{\mathcal{E}_G(X)} h_{G,\nu}(\xi | \mathcal{F}) = h_{G,\mu}(\xi | \mathcal{F}).$$

Finally,

$$\mathbf{H}_\mu(\xi) - \epsilon < h_{G,\mu}(\xi | \mathcal{F}) = \mathbf{H}_\mu(\alpha | \mathcal{I}_G) \leq \mathbf{H}_\mu(\alpha) = \mathbf{H} \left( \int_{\mathcal{E}_G(X)} \bar{p}^\nu d\tau(\nu) \right) \leq \mathbf{H}_\mu(\xi),$$

and thus  $|\mathbf{H}_\mu(\alpha) - \mathbf{H}_\mu(\xi)| < \epsilon$ .  $\square$

## 7. A SMOOTH DIVISION INTO HYPERFINITE RELATIONS

One of the key ingredients in the proof of the main theorem is a new construction related to countable Borel equivalence relations. Specifically, for an aperiodic countable Borel equivalence relation  $E$  on a standard Borel space  $X$ , we show that there is a smooth division  $X = \sqcup_{0 \leq r \leq 1} X_r$  of  $X$  into Borel sets  $X_r$  with the property that the restriction of  $E$  to each  $X_r$  is aperiodic and hyperfinite. This section is devoted to proving this fact.

Recall that for a standard Borel space  $X$ , a *Borel graph* on  $X$  is a Borel, symmetric, anti-reflexive subset  $\Gamma \subseteq X \times X$ . For  $A \subseteq X$  we write  $\mathcal{N}_\Gamma(A)$  for the  $\Gamma$ -neighborhood of  $A$ , i.e. the set of  $x \in X$  such that there is  $a \in A$  with  $(x, a) \in \Gamma$ . The *degree* of  $x \in X$  is  $\deg_\Gamma(x) = |\mathcal{N}_\Gamma(x)|$ , and  $\Gamma$  is *locally finite* if  $\deg_\Gamma(x) < \infty$  for all  $x \in X$ . A set  $A \subseteq X$  is  $\Gamma$ -independent if for all  $a, b \in A$  we have  $(a, b) \notin \Gamma$ , and a function  $f : X \rightarrow \mathbb{N}$  is a  $\Gamma$ -coloring if  $f^{-1}(i)$  is  $\Gamma$ -independent for every  $i \in \mathbb{N}$ . It was proven by Kechris–Solecki–Todorćević in [27] that every locally finite Borel graph  $\Gamma$  admits a Borel  $\Gamma$ -coloring  $f : X \rightarrow \mathbb{N}$  and also admits a maximal  $\Gamma$ -independent set that is Borel. We write  $E_\Gamma^X$  for the equivalence relation on  $X$  given by the connected components of  $\Gamma$ , and we say  $\Gamma$  is *aperiodic* if every class of  $E_\Gamma^X$  is infinite.

**Lemma 7.1.** *Let  $E$  be an aperiodic countable Borel equivalence relation on a standard Borel space  $X$ . Then there is a Borel function  $f : X \rightarrow \mathbb{N}$  such that  $|[x]_E \cap f^{-1}(i)| = \infty$  for all  $x \in X$  and all  $i \in \mathbb{N}$ .*

*Proof.* We claim that under these assumptions there is a Borel set  $A \subseteq X$  with  $|[x]_E \cap A| = |[x]_E \cap (X \setminus A)| = \infty$  for all  $x \in X$ . By [24, Lem. 3.25] there exists an aperiodic  $S \in [E]$ . Let  $\Gamma$  be the graph  $\{(x, S(x)) : x \in X\} \cup \{(S(x), x) : x \in X\}$ . By [27], there is a maximal  $\Gamma$ -independent Borel set  $A \subseteq X$ . Independence gives  $A \cap S(A) = \emptyset$  and maximality implies  $X = S^{-1}(A) \cup A \cup S(A)$ . Clearly  $A$  has the desired properties, establishing our preliminary claim.

Set  $X_0 = X$ . Inductively on  $n$ , given  $X_n \subseteq X$  having infinite intersection with every  $E$ -class, apply the previous claim to the relation  $E_n = E \upharpoonright (X_n \times X_n)$  to get a Borel set  $A_n \subseteq X_n$  with  $|[x]_{E_n} \cap A_n| = |[x]_{E_n} \cap (X_n \setminus A_n)| = \infty$  for all  $x \in X_n$ . Then  $A_n$  and  $X_n \setminus A_n$  have infinite intersection with every  $E$ -class. Set  $X_{n+1} = X_n \setminus A_n$  and continue the induction. This creates a sequence  $(A_n)_{n \in \mathbb{N}}$  of pairwise-disjoint Borel sets having infinite intersection with every  $E$ -class. To complete the proof, define  $f(x) = n$  if  $x \in A_n$  and  $f(x) = 0$  if  $x \notin \bigcup_n A_n$ .  $\square$

**Lemma 7.2.** *Let  $X$  be a standard Borel space, let  $E$  be an aperiodic countable Borel equivalence relation on  $X$ , and let  $\Gamma \subseteq E$  be a locally-finite Borel graph on  $X$ . Then there is a Borel  $\Gamma$ -coloring  $f : X \rightarrow \mathbb{N}$  such that  $|[x]_E \cap f^{-1}(i)| = \infty$  for all  $x \in X$  and all  $i \in \mathbb{N}$ .*

*Proof.* By [27] there is a  $\Gamma$ -coloring  $h : X \rightarrow \{n \in \mathbb{N} : n \geq 1\}$  and a maximal  $\Gamma$ -independent set  $P_0$  that is Borel. For  $i \geq 1$  set  $P_i = h^{-1}(i) \setminus P_0$ , so that  $\{P_n : n \in \mathbb{N}\}$  is a partition of  $X$  into  $\Gamma$ -independent sets. Since  $E$  is aperiodic,  $\Gamma$  is locally finite, and  $P_0$  is a maximal  $\Gamma$ -independent set, we must have that  $[x]_E \cap P_0$  is infinite for every  $x \in X$ . Consider the restriction of  $E$  to  $P_0 \times P_0$  and apply the previous lemma to obtain a Borel function  $f : P_0 \rightarrow \mathbb{N}$  with  $|[x]_E \cap P_0 \cap f^{-1}(n)| = \infty$  for every  $x \in X$  and  $n \in \mathbb{N}$ . Now suppose  $f$  is defined on  $\bigcup_{0 \leq j < i} P_j$  and gives distinct values to  $\Gamma$ -adjacent points. We will extend  $f$  to  $\bigcup_{0 \leq j \leq i} P_j$ . For each

$y \in P_i$  define  $f(y)$  to be the least element of  $\mathbb{N} \setminus f(\mathcal{N}_\Gamma(y) \cap \bigcup_{0 \leq j < i} P_j)$ . Then  $f$  continues to be a Borel function, and since  $(P_i \times P_i) \cap \Gamma = \emptyset$ ,  $f$  continues to assign distinct values to  $\Gamma$ -adjacent points in  $\bigcup_{0 \leq j \leq i} P_j$ . Continuing by induction, we obtain a Borel  $\Gamma$ -coloring  $f : X \rightarrow \mathbb{N}$ . For each  $x \in X$  and  $n \in \mathbb{N}$  we have  $|[x]_E \cap f^{-1}(n)| \geq |[x]_E \cap P_0 \cap f^{-1}(n)| = \infty$ .  $\square$

Recall that a Borel equivalence relation  $F$  on  $X$  is *smooth* if there is a standard Borel space  $Y$  and a Borel function  $f : X \rightarrow Y$  such that  $x F x' \Leftrightarrow f(x) = f(x')$ . Also recall that a Borel equivalence relation  $R$  on  $X$  is *hyperfinite* if there is an increasing sequence of Borel equivalence relations  $(R_n)_{n \in \mathbb{N}}$  such that  $R = \bigcup_n R_n$  and each  $R_n$  is finite (i.e. each class of  $R_n$  is finite).

**Theorem 7.3.** *Let  $E$  be an aperiodic countable Borel equivalence relation on a standard Borel space  $X$ . Then there is a smooth Borel equivalence relation  $F$  such that  $E \cap F$  is aperiodic and hyperfinite.*

*Proof.* By a result of Feldman and Moore [15], there is a countable group  $G$  and a Borel action  $G \curvearrowright X$  such that  $E = \{(x, y) : \exists g \in G \ g \cdot x = y\}$ . Fix an increasing sequence of finite symmetric sets  $K_n \subseteq G$  with  $\bigcup_n K_n = G$ . We will build a sequence of Borel functions  $f_n : X \rightarrow \mathbb{N}$  and an increasing sequence of finite Borel equivalence relations  $R_n \subseteq E$  satisfying:

- (i) for all  $n \in \mathbb{N}$ ,  $k \leq n$ , and  $x, y \in X$ , if  $x R_n y$  then  $f_k(x) = f_k(y)$ ;
- (ii) for all  $x \in X$ ,  $n \in \mathbb{N}$ , and  $v \in \mathbb{N}^n$ ,  $|[x]_E \cap (f_0 \times \cdots \times f_{n-1})^{-1}(v)| = \infty$ ;
- (iii) for all  $n \geq 1$  and  $x, y \in X$ , if  $y \in K_n \cdot x$  then either  $x R_{n-1} y$  or else  $f_n(x) \neq f_n(y)$ ;
- (iv) for all  $n \in \mathbb{N}$ , every  $R_{n+1}$ -class contains at least two  $R_n$ -classes.

To begin let  $f_0 : X \rightarrow \mathbb{N}$  be given by Lemma 7.1 and let  $R_0 = \{(x, x) : x \in X\}$  be the equivalence relation of equality.

Now assume that  $f_k$  and  $R_k$  have been constructed for all  $k < n$  and satisfy (i) through (iv). Since  $R_{n-1}$  is finite, there is a Borel set  $Y \subseteq X$  that meets every  $R_{n-1}$ -class in precisely one point and there is a Borel function  $t : X \rightarrow Y$  satisfying  $t(x) R_{n-1} x$  for every  $x$ . Define an equivalence relation  $H$  on  $Y$  by the rule  $y H y' \Leftrightarrow (y E y') \wedge (\forall k < n \ f_k(y) = f_k(y'))$ . Then  $H$  is Borel,  $H \subseteq E \cap (Y \times Y)$ , and clauses (i) and (ii) imply that  $H$  is aperiodic. Define a Borel graph  $\Gamma$  on  $Y$  by the rule

$$(y, y') \in \Gamma \iff (y \neq y') \wedge (y H y') \wedge ([y]_{R_{n-1}} \cap K_n \cdot [y']_{R_{n-1}} \neq \emptyset).$$

Note that  $\Gamma$  is locally finite since  $R_{n-1}$  and  $K_n$  are finite. By Lemma 7.2 there is a Borel  $\Gamma$ -coloring  $h : Y \rightarrow \mathbb{N}$  such that  $|[y]_H \cap h^{-1}(i)| = \infty$  for all  $y \in Y$  and  $i \in \mathbb{N}$ . Then the equivalence relation  $H'$  defined by  $y H' y' \Leftrightarrow (y H y') \wedge (h(y) = h(y'))$  is an aperiodic countable Borel equivalence relation. Pick any finite Borel equivalence relation  $S \subseteq H'$  with the property that every  $S$ -class contains at least two points. Now define  $f_n(x) = h(t(x))$ , and set  $x R_n x'$  if and only if  $(t(x), t(x')) \in S$ . Then  $f_n$  and  $R_n$  are Borel, and it is immediately seen that (i), (ii), and (iv) continue to hold. Also (iii) holds due to how the graph  $\Gamma$  was defined and the fact that the function  $h$  is a  $\Gamma$ -coloring.

Now define the smooth Borel equivalence relation  $F$  by the rule

$$x F y \iff f_\infty(x) = f_\infty(y),$$



where  $f_\infty : X \rightarrow \mathbb{N}^{\mathbb{N}}$  is defined by  $f_\infty(x)(n) = f_n(x)$ . Also set  $R_\infty = \bigcup_n R_n$ . Clearly  $R_\infty$  is hyperfinite, and since every  $R_{n+1}$ -class contains at least two  $R_n$ -classes,  $R_\infty$  is aperiodic. To finish the proof it suffices to show that  $R_\infty = E \cap F$ . Clause (i) immediately gives the containment  $R_\infty \subseteq E \cap F$ . For the reverse direction, suppose  $(x, y) \in E \cap F$ . Let  $n$  be sufficiently large that  $y \in K_n \cdot x$ . Then clause (iii) implies that either  $(x, y) \in R_{n-1} \subseteq R_\infty$  or else  $(x, y) \notin F$ . The former contradicts our assumption, so the latter must hold and  $E \cap F \subseteq R_\infty$ .  $\square$

We will rely on the following corollary to Theorem 7.3.

**Corollary 7.4.** *Let  $G \curvearrowright (X, \mu)$  be an aperiodic p.m.p. action, and let  $\mathcal{F}$  be a  $G$ -invariant class-bijective sub- $\sigma$ -algebra. Then there is an aperiodic  $\mathcal{F}$ -expressible  $T \in [E_G^X]$  and a  $T$ -invariant  $\mathcal{F}$ -measurable function  $f : X \rightarrow [0, 1]$  with the property that  $x, y \in X$  lie in the same  $G$ -orbit and have equal  $f$  values if and only if they lie in the same  $T$ -orbit.*

*Proof.* Let  $G \curvearrowright (Y, \nu)$  be the factor of  $(X, \mu)$  associated to  $\mathcal{F}$ , and let  $\phi : (X, \mu) \rightarrow (Y, \nu)$  be the factor map. Let  $E_G^Y$  be the orbit equivalence relation on  $Y$ , and let  $F'$  be the smooth Borel equivalence relation given by Theorem 7.3. Since  $F'$  is smooth and every standard Borel space embeds into  $[0, 1]$ , there is a Borel function  $h : Y \rightarrow [0, 1]$  such that  $y F' y' \Leftrightarrow h(y) = h(y')$ . Since  $F' \cap E_G^Y$  is hyperfinite and aperiodic, there is an aperiodic automorphism  $S$  of  $(Y, \nu)$  such that its induced orbit equivalence relation,  $E_S^Y$ , coincides with  $F' \cap E_G^Y$ . In particular,  $S \in [E_G^Y]$  so there is a Borel partition  $\{B_g^S : g \in G\}$  of  $Y$  with  $S(y) = g \cdot y$  for all  $g \in G$  and  $y \in B_g^S$ . Define a  $\mathcal{F}$ -measurable function  $f : X \rightarrow [0, 1]$  by  $f(x) = h \circ \phi(x)$ , and define an aperiodic  $\mathcal{F}$ -expressible  $T \in [E_G^X]$  by setting  $T(x) = g \cdot x$  if  $\phi(x) \in B_g^S$ . Notice that  $\phi \circ T = S \circ \phi$  and hence  $f$  is  $T$ -invariant since  $h$  is  $S$ -invariant. Clearly any two points lying in the same  $T$ -orbit will also lie in the same  $G$ -orbit and have equal  $f$  values. So consider the reverse scenario where  $x \in X$ ,  $x' \in G \cdot x$ , and  $f(x) = f(x')$ . Setting  $y = \phi(x)$  and  $y' = \phi(x')$ , we have  $y' \in G \cdot y$  and  $h(y) = h(y')$ . It follows there is  $n \in \mathbb{Z}$  with  $S^n(y) = y'$ . Then  $\phi(T^n(x)) = y' = \phi(x')$ . Since  $\phi$  is class-bijective this implies  $T^n(x) = x'$ .  $\square$

The transformation  $T$  in the previous corollary cannot be  $\mu$ -ergodic. In fact, from the perspective of the  $G$ -action  $T$  must be highly non-ergodic, as the  $T$ -invariant function  $f$  must separate all  $T$ -orbits within any common  $G$ -orbit.

## 8. THE EXTERNAL PAST

The notion of ‘past’ plays a significant role in the classical entropy theory of actions of  $\mathbb{Z}$ . An important consequence of Corollary 7.4 is that it allows the creation of a useful notion of past. Specifically, the action of  $T$  imbues a natural order to every  $T$ -orbit, and the function  $f$  creates a linear order on the collection of  $T$ -orbits contained in any  $G$ -orbit. Combined, these create a total ordering of every  $G$ -orbit and hence a notion of ‘past’ for each point. Given  $x$ , we refer to the points  $g \cdot x$  with  $f(g \cdot x) < f(x)$  as the ‘external past’ of  $x$  (as this portion of  $x$ ’s past is external to the  $T$ -orbit of  $x$ ).

For a partition  $\alpha$  of  $X$  and  $Y \subseteq X$  we define another partition of  $X$  by

$$\alpha \upharpoonright Y = \{X \setminus Y\} \cup \{A \cap Y : A \in \alpha\}.$$

Similarly, for a  $\sigma$ -algebra  $\Sigma$  we write  $\Sigma \upharpoonright Y$  for the  $\sigma$ -algebra on  $X$  generated by the sets  $\{X\} \cup \{B \cap Y : B \in \Sigma\}$ .

**Definition 8.1.** Let  $G \curvearrowright (X, \mu)$  be an aperiodic p.m.p. action, let  $\mathcal{F}$  be a  $G$ -invariant class-bijective sub- $\sigma$ -algebra, and let  $T \in [E_G^X]$  and  $f : X \rightarrow [0, 1]$  be as in Corollary 7.4. For a countable partition  $\xi$  of  $X$  and  $S \subseteq [0, 1]$  we will write  $\xi_S$  for the partition  $\xi_S = \xi \upharpoonright f^{-1}(S)$ . We define the *external past* of  $\xi$  as

$$\mathcal{P}_\xi = \mathcal{F} \vee \bigvee_{t \in [0, 1]} \left( \sigma\text{-alg}_G(\xi_{[0, t]}) \upharpoonright f^{-1}([t, 1]) \right).$$

In other words, the function  $f$  gives a quasi-ordering to the  $T$ -orbits (which, on every  $G$ -orbit restricts to a total ordering), and the external past of  $\xi$  consists of the sets which you can measure by using  $G$  to travel to strictly “smaller”  $T$ -orbits and looking at the partition  $\xi$  (we also include  $\mathcal{F}$  in this  $\sigma$ -algebra for technical reasons).

**Lemma 8.2.** *Let  $G \curvearrowright (X, \mu)$  be an aperiodic p.m.p. action, let  $\mathcal{F}$  be a  $G$ -invariant class-bijective sub- $\sigma$ -algebra, and let  $T \in [E_G^X]$  and  $f : X \rightarrow [0, 1]$  be as in Corollary 7.4. Fix a countable partition  $\xi$ .*

- (i) *If  $\mathcal{F}$  is countably generated then so is  $\mathcal{P}_\xi$ .*
- (ii)  *$\mathcal{P}_\xi$  is  $T$ -invariant.*
- (iii)  *$\sigma\text{-alg}_T(\xi) \vee \mathcal{P}_\xi \subseteq \mathcal{F} \vee \sigma\text{-alg}_G(\xi)$ .*
- (iv)  *$(\sigma\text{-alg}_T(\xi) \vee \mathcal{P}_\xi) \upharpoonright f^{-1}([0, t]) \subseteq \mathcal{F} \vee \sigma\text{-alg}_G(\xi_{[0, t]})$  for every  $t \in [0, 1]$ .*
- (v) *If  $\beta \subseteq \sigma\text{-alg}_T(\xi) \vee \mathcal{P}_\xi$ , then  $\mathcal{P}_\beta \subseteq \mathcal{P}_\xi$ .*

*Proof.* (i). For any set  $C$  we have  $C \cap f^{-1}([0, t]) = \bigcup_{t > q \in \mathbb{Q}} C \cap f^{-1}([0, q])$ . So

$$\begin{aligned} \sigma\text{-alg}_G(\xi_{[0, t]}) \upharpoonright f^{-1}([t, 1]) &\subseteq \bigvee_{t > q \in \mathbb{Q}} \sigma\text{-alg}_G(\xi_{[0, q]}) \upharpoonright f^{-1}([t, 1]) \\ &\subseteq \mathcal{F} \vee \bigvee_{t > q \in \mathbb{Q}} \sigma\text{-alg}_G(\xi_{[0, q]}) \upharpoonright f^{-1}([q, 1]). \end{aligned}$$

Thus in the definition of  $\mathcal{P}_\xi$  one may take the join over  $[0, 1] \cap \mathbb{Q}$  rather than  $[0, 1]$ .

(ii). Since  $f$  is  $\mathcal{F}$ -measurable, we can write  $\mathcal{P}_\xi = \bigvee_{t \in [0, 1]} (\mathcal{F} \vee \sigma\text{-alg}_G(\xi_{[0, t]})) \upharpoonright f^{-1}([t, 1])$ . By Lemma 4.2  $\mathcal{F} \vee \sigma\text{-alg}_G(\xi_{[0, t]})$  is  $T$ -invariant. Also  $f$  is  $T$ -invariant, so the claim follows.

(iii). The  $\sigma$ -algebra  $\mathcal{F} \vee \sigma\text{-alg}_G(\xi)$  is  $G$  and  $T$  invariant by Lemma 4.2, and it contains the sets  $f^{-1}([t, 1])$  and the partitions  $\xi_{[0, t]}$ . So the claim is immediate.

(iv). Fix  $t$ . Using Lemma 4.2 and the  $T$ -invariance of  $f$  we get  $\sigma\text{-alg}_T(\xi) \upharpoonright f^{-1}([0, t]) = \sigma\text{-alg}_T(\xi_{[0, t]}) \subseteq \mathcal{F} \vee \sigma\text{-alg}_G(\xi_{[0, t]})$ . Next, consider  $\mathcal{P}_\xi$ . If  $s \geq t$  then  $(\sigma\text{-alg}_G(\xi_{[0, s]}) \upharpoonright f^{-1}([s, 1])) \upharpoonright f^{-1}([0, t]) \subseteq \mathcal{F}$ . On the other hand, when  $s < t$  we have  $\sigma\text{-alg}_G(\xi_{[0, s]}) \subseteq \mathcal{F} \vee \sigma\text{-alg}_G(\xi_{[0, t]})$ . This establishes the claim.

(v).  $\beta_{[0, t]} = \beta \upharpoonright f^{-1}([0, t])$  is contained in  $\mathcal{F} \vee \sigma\text{-alg}_G(\xi_{[0, t]})$  by (iv). So  $\sigma\text{-alg}_G(\beta_{[0, t]}) \subseteq \mathcal{F} \vee \sigma\text{-alg}_G(\xi_{[0, t]})$ . Now take restrictions to  $f^{-1}([t, 1])$  and join over  $t \in [0, 1]$ .  $\square$

For  $x \in X$  we define a quasi-order on  $G$  by setting  $u \leq_x v$  if either  $f(u^{-1} \cdot x) < f(v^{-1} \cdot x)$  or  $f(u^{-1} \cdot x) = f(v^{-1} \cdot x)$  and there is  $m \geq 0$  with  $T^{-m}(u^{-1} \cdot x) = v^{-1} \cdot x$ . When  $x$  has trivial stabilizer  $\leq_x$  is a total order on  $G$ . We write  $u <_x v$  when  $u \leq_x v$  but  $\neg(v \leq_x u)$ . Its a simple consequence of Lemmas 4.3 and 4.4 that for every  $u, v \in G$  the set  $\{x \in X : u <_x v\}$  is  $\mathcal{F}$ -measurable.

**Lemma 8.3.** *For all  $u \neq v \in G$  and every  $D \in \mathcal{F}$  with  $D \subseteq \{x \in X : u <_x v\}$  we have*

$$\left[ u \cdot \left( \bigvee_{k \leq 0} T^k(\xi) \vee \mathcal{P}_\xi \right) \right] \upharpoonright D \subseteq v \cdot \left( \bigvee_{k < 0} T^k(\xi) \vee \mathcal{P}_\xi \right).$$

*Proof.*  $D$  is the union of the sets

$$\begin{aligned} D_m &= \{x \in D : u^{-1} \cdot x = T^m(v^{-1} \cdot x)\} & m \geq 1 \\ D_q &= \{x \in D : f(u^{-1} \cdot x) < q \leq f(v^{-1} \cdot x)\} & q \in \mathbb{Q} \cap [0, 1]. \end{aligned}$$

So it suffices to prove the claim for each  $D_m$  and  $D_q$ .

First consider  $D_m$ . Note  $D_m \in \mathcal{F}$  by Lemma 4.4. For any set  $A$  we have

$$\begin{aligned} (u \cdot A) \cap D_m &= \{x \in D_m : u^{-1} \cdot x \in A\} \\ &= \{x \in D_m : T^m(v^{-1} \cdot x) \in A\} = (v \cdot T^{-m}(A)) \cap D_m. \end{aligned}$$

Since  $\mathcal{P}_\xi$  is  $T$ -invariant, it immediately follows that

$$\begin{aligned} \left[ u \cdot \left( \bigvee_{k \leq 0} T^k(\xi) \vee \mathcal{P}_\xi \right) \right] \upharpoonright D_m &= \left[ v \cdot \left( \bigvee_{k \leq 0} T^{k-m}(\xi) \vee \mathcal{P}_\xi \right) \right] \upharpoonright D_m \\ &\subseteq v \cdot \left( \bigvee_{k < 0} T^k(\xi) \vee \mathcal{P}_\xi \right). \end{aligned}$$

Now consider  $D_q$ . Again note  $D_q \in \mathcal{F}$ . Since  $u^{-1} \cdot D_q \subseteq f^{-1}([0, q])$ , from Lemma 8.2.(iv) we get

$$\left( \bigvee_{k \leq 0} T^k(\xi) \vee \mathcal{P}_\xi \right) \upharpoonright u^{-1} \cdot D_q \subseteq \left( \mathcal{F} \vee \sigma\text{-alg}_G(\xi_{[0, q]}) \right) \upharpoonright u^{-1} \cdot D_q.$$

By multiplying throughout by  $v^{-1}u$  and using the fact that  $v^{-1} \cdot D_q \subseteq f^{-1}([q, 1])$ , we obtain

$$\begin{aligned} \left[ v^{-1}u \cdot \left( \bigvee_{k \leq 0} T^k(\xi) \vee \mathcal{P}_\xi \right) \right] \upharpoonright v^{-1} \cdot D_q &\subseteq \left[ v^{-1}u \cdot \left( \mathcal{F} \vee \sigma\text{-alg}_G(\xi_{[0, q]}) \right) \right] \upharpoonright v^{-1} \cdot D_q \\ &\subseteq \left( \mathcal{F} \vee \sigma\text{-alg}_G(\xi_{[q, 1]}) \right) \upharpoonright f^{-1}([q, 1]) \\ &\subseteq \mathcal{P}_\xi. \end{aligned} \quad \square$$

In analogy with the earlier Lemma 4.5, we relate  $T$ -Bernoullicity over  $\mathcal{P}_\xi$  with  $G$ -Bernoullicity over  $\mathcal{F}$ .

**Lemma 8.4.** *Let  $G \curvearrowright (X, \mu)$  be an aperiodic p.m.p. action. Let  $\mathcal{F}$  be a  $G$ -invariant class-bijective sub- $\sigma$ -algebra such that  $G \curvearrowright (X, \mu)$  has completely positive outer Rokhlin entropy relative to  $\mathcal{F}$ . Let  $T \in [E_G^X]$  and  $f : X \rightarrow [0, 1]$  be as given by Corollary 7.4.*

- (1) *Let  $\alpha$  be a partition of  $X$ . If  $\alpha$  is  $G$ -Bernoulli over  $\mathcal{F}$  then  $\alpha$  is  $T$ -Bernoulli over  $\mathcal{P}_\alpha$ .*
- (2) *Let  $\xi$  be a partition of  $X$  and let  $\beta \subseteq \sigma\text{-alg}_T(\xi) \vee \mathcal{P}_\xi$  be a partition. If  $\beta$  is  $T$ -weakly Bernoulli over  $\mathcal{I}_T$  and  $\sigma\text{-alg}_T(\beta)$  is independent with  $\mathcal{P}_\xi$  relative to  $\mathcal{I}_T$ , then  $\beta$  is  $G$ -weakly Bernoulli over  $\mathcal{F}$ .*

*Proof.* Let  $G \curvearrowright (Y, \nu)$  be the factor associated with  $\mathcal{F}$ , and let  $\mu = \int \mu_y d\nu(y)$  be the disintegration of  $\mu$  over  $\nu$ . Since  $\mathcal{F}$  is class-bijective we have that  $\text{Stab} : X \rightarrow \text{Sub}(G)$  is  $\mathcal{F}$ -measurable and that  $\text{Stab}(y) = \text{Stab}(x)$  for  $\mu_y$ -almost-every  $x \in X$ . Also note that  $T$  and  $f$  both descend to  $Y$  since they are  $\mathcal{F}$ -expressible and  $\mathcal{F}$ -measurable, respectively.

(1). Lemma 4.5.(ii) tells us that  $\alpha$  is  $T$ -Bernoulli over  $\mathcal{F}$ . By Lemma 3.5, this means the  $T$ -translates of  $\alpha$  are mutually independent and  $\sigma\text{-alg}_T(\alpha)$  is independent

with  $\mathcal{F}$ . Lemmas 2.1 and 3.5 imply that it will be sufficient to check that  $\sigma\text{-alg}_T(\alpha)$  is independent with  $\mathcal{P}_\alpha$  relative to  $\mathcal{F}$ .

Let us express  $\mathcal{P}_\alpha$  as

$$\mathcal{P}_\alpha = \mathcal{F} \vee \bigvee_{t \in [0,1]} \left( \bigvee_{g \in G} g \cdot \alpha_{[0,t)} \right) \upharpoonright f^{-1}([t, 1]).$$

Notice that  $f^{-1}([t, 1])$  is  $\mu_y$ -conull when  $t \leq f(y)$  and otherwise is  $\mu_y$ -null. Similarly  $g \cdot \alpha_{[0,t)}$  equals  $\alpha$  mod  $\mu_y$ -null sets when  $f(g^{-1} \cdot y) < t$  and is  $\mu_y$ -trivial otherwise. Thus  $\mathcal{P}_\alpha = \alpha^{\{g \in G: f(g \cdot y) < f(y)\}}$  mod  $\mu_y$ -null sets.

Let  $Y_0$  be the set of  $y \in Y$  that have the property that for every finite  $W \subseteq G$  that maps injectively to  $W \cdot y$ , the  $W^{-1}$ -translates of  $\alpha$  are mutually  $\mu_y$ -independent. Note  $\nu(Y_0) = 1$ . Fix  $y \in Y_0$ . Let  $P \subseteq \{g \in G : f(g \cdot y) < f(y)\}$  and  $K \subseteq \mathbb{Z}$  be finite sets. By Lemma 4.3 each  $T^k$  is  $\mathcal{F}$ -expressible, hence there is  $W \subseteq G$  with  $W \cdot y = \{T^k(y) : k \in K\}$ . In particular  $\alpha^W = \bigvee_{k \in K} T^{-k}(\alpha)$  mod  $\mu_y$ -null sets. Since for every  $w \in W$  there is  $k \in K$  with  $f(w \cdot y) = f(T^k(y)) = f(y)$ , we see that  $W \cdot y \cap P \cdot y = \emptyset$ . Therefore the partitions  $\bigvee_{k \in K} T^{-k}(\alpha) = \alpha^W$  and  $\alpha^P$  are  $\mu_y$ -independent. We conclude that  $\sigma\text{-alg}_T(\alpha)$  is independent with  $\mathcal{P}_\alpha$  relative to  $\mathcal{F}$ , completing the proof of (1).

(2). Since the stabilizer map is  $\mathcal{F}$ -measurable, it is trivially true that  $\beta$  is independent with  $\text{Stab}^{-1}(\mathcal{B}(\text{Sub}(G)))$  relative to  $\mathcal{F}$ . For the same reason, the second condition in Definition 3.4 becomes equivalent to the statement: for  $\nu$ -almost-every  $y \in Y$  and every finite  $W \subseteq G$  that maps injectively to  $W^{-1} \cdot y$ , the  $W$ -translates of  $\beta$  are mutually  $\mu_y$ -independent.

From Lemma 8.2.(v) we see that  $\sigma\text{-alg}_T(\beta)$  is independent with  $\mathcal{P}_\beta$  relative to  $\mathcal{I}_T$ . Combined with our other assumption on  $\beta$  and Lemma 3.6, we get that for almost-every  $T$ -ergodic component  $\nu$  of  $\mu$ ,  $\beta$  is  $(T, \nu)$ -Bernoulli over  $\mathcal{P}_\beta$ , meaning that the  $T$ -translates of  $\beta$  are mutually  $\nu$ -independent and  $\sigma\text{-alg}_T(\beta)$  is  $\nu$ -independent with  $\mathcal{P}_\beta$ . In particular,  $\beta$  is  $\nu$ -independent with  $\bigvee_{k < 0} T^k(\beta) \vee \mathcal{P}_\beta$  for almost-every  $T$ -ergodic component  $\nu$  of  $\mu$ . Equivalently,  $\beta$  is  $\mu$ -independent with  $\bigvee_{k < 0} T^k(\beta) \vee \mathcal{P}_\beta$  relative to  $\mathcal{I}_T$ . Our assumption on  $\mathcal{F}$  and Corollary 4.7 implies that  $\mathcal{I}_T \subseteq \mathcal{F}$ , so  $\beta$  is in fact  $\mu$ -independent with  $\bigvee_{k < 0} T^k(\beta) \vee \mathcal{P}_\beta$  relative to  $\mathcal{F}$ . Noting the  $G$ -invariance of  $\mu$  and  $\mathcal{F}$ , we conclude that for every  $g \in G$ ,  $g \cdot \beta$  is  $\mu$ -independent with  $g \cdot (\bigvee_{k < 0} T^k(\beta) \vee \mathcal{P}_\beta)$  relative to  $\mathcal{F}$ .

Let  $Y_0$  be the set of  $y \in Y$  such that for all  $g \in G$  the partition  $g \cdot \beta$  is  $\mu_y$ -independent with  $g \cdot (\bigvee_{k < 0} T^k(\beta) \vee \mathcal{P}_\beta)$ . By the previous paragraph  $\nu(Y_0) = 1$ . Now fix  $y \in Y_0$  and fix a finite set  $W \subseteq G$  that maps injectively to  $W^{-1} \cdot y$ . Recall the quasi-ordering map  $x \mapsto \leq_x$  described above. This map is  $\mathcal{F}$ -measurable and hence constant  $\mu_y$ -almost-everywhere. Let's denote this constant by  $\leq_y$ . Since  $|W^{-1} \cdot y| = |W| < \infty$ ,  $\leq_y$  restricts to a total order on  $W$ , and we may enumerate the elements of  $W$  as  $w_1, \dots, w_n$  where  $w_j <_y w_i$  for all  $j < i$ . Set  $P_i = \{w_j^{-1} : j < i\}$ . By Lemma 8.3 we have, modulo  $\mu_y$ -null sets,

$$\beta^{P_i} \subseteq w_i \cdot \left( \bigvee_{k < 0} T^k(\beta) \vee \mathcal{P}_\beta \right).$$

The  $\sigma$ -algebra on the right is  $\mu_y$ -independent with  $w_i \cdot \beta$ . So  $w_i \cdot \beta$  is  $\mu_y$ -independent with  $\beta^{P_i}$ . This holds for every  $i$ , so the  $W$ -translates of  $\beta$  are mutually  $\mu_y$ -independent.  $\square$

Finally, we present a crucial lemma relating the entropies of the  $G$ -action and the  $T$ -action. In the author's opinion, this lemma represents the primary way that Rokhlin entropy is invoked in the proof of the main theorem.

**Lemma 8.5.** *If  $\xi$  is a countable partition with  $H(\xi | \mathcal{F}) < \infty$ , then*

$$h_G(\xi | \mathcal{F}) \leq h_T^{\text{KS}}(\xi | \mathcal{P}_\xi).$$

*Proof.* Pick a finite ordered partition  $\mathcal{Q} = \{Q_i : 1 \leq i \leq n\}$  of  $[0, 1]$  into sub-intervals with rational endpoints and with  $q < q'$  whenever  $j < i$ ,  $q \in Q_j$ ,  $q' \in Q_i$ . Set  $Y_i = \bigcup_{j < i} Q_j$ . Define

$$\mathcal{P}_\xi^\mathcal{Q} = \mathcal{F} \vee \bigvee_{1 \leq i \leq n} \sigma\text{-alg}_G(\xi_{Y_i}) \upharpoonright f^{-1}(Q_i).$$

Notice that the sets  $f^{-1}(Q_i)$  are  $T$ -invariant. Let  $\mu_i$  denote the normalized restriction of  $\mu$  to  $f^{-1}(Q_i)$ . Since Kolmogorov–Sinai entropy is an affine function on the space of  $T$ -invariant probability measures, we have

$$\begin{aligned} \sum_{i=1}^n h_{T,\mu}^{\text{KS}}(\xi_{Q_i} | \mathcal{F} \vee \sigma\text{-alg}_G(\xi_{Y_i})) &= \sum_{i=1}^n \sum_{j=1}^n \mu(f^{-1}(Q_j)) \cdot h_{T,\mu_j}^{\text{KS}}(\xi_{Q_i} | \mathcal{F} \vee \sigma\text{-alg}_G(\xi_{Y_i})) \\ &= \sum_{i=1}^n \mu(f^{-1}(Q_i)) \cdot h_{T,\mu_i}^{\text{KS}}(\xi_{Q_i} | \mathcal{F} \vee \sigma\text{-alg}_G(\xi_{Y_i})) \\ &= \sum_{i=1}^n \mu(f^{-1}(Q_i)) \cdot h_{T,\mu_i}^{\text{KS}}(\xi | \mathcal{P}_\xi^\mathcal{Q}) \\ &= h_{T,\mu}^{\text{KS}}(\xi | \mathcal{P}_\xi^\mathcal{Q}). \end{aligned}$$

The  $\sigma$ -algebras  $\mathcal{F} \vee \sigma\text{-alg}_G(\xi_{Y_i})$  are  $G$ -invariant. So by Lemma 4.6 and sub-additivity of Rokhlin entropy, we have

$$\begin{aligned} h_{G,\mu}(\xi | \mathcal{F}) &\leq \sum_{i=1}^n h_{G,\mu}(\xi_{Q_i} | \mathcal{F} \vee \sigma\text{-alg}_G(\xi_{Y_i})) \\ &\leq \sum_{i=1}^n h_{T,\mu}^{\text{KS}}(\xi_{Q_i} | \mathcal{F} \vee \sigma\text{-alg}_G(\xi_{Y_i})) = h_{T,\mu}^{\text{KS}}(\xi | \mathcal{P}_\xi^\mathcal{Q}). \end{aligned}$$

Since  $\mathcal{P}_\xi$  is the join of the  $\sigma$ -algebras  $\mathcal{P}_\xi^\mathcal{Q}$  as  $\mathcal{Q}$  varies, we conclude  $h_G(\xi | \mathcal{F}) \leq \inf_{\mathcal{Q}} h_{T,\mu}^{\text{KS}}(\xi | \mathcal{P}_\xi^\mathcal{Q}) = h_T^{\text{KS}}(\xi | \mathcal{P}_\xi)$ .  $\square$

## 9. THE FACTOR THEOREM

Now we present the proof of the main theorem.

**Theorem 9.1.** *For every countably infinite group  $G$ , every aperiodic ergodic p.m.p. action  $G \curvearrowright (X, \mu)$ , every  $G$ -invariant sub- $\sigma$ -algebra  $\Sigma$ , and every probability vector  $\bar{p}$ , if  $H(\bar{p}) \leq h_G(X, \mu | \Sigma)$  then there is a partition  $\alpha$  with  $\text{dist}(\alpha) = \bar{p}$  which is  $G$ -Bernoulli over  $\Sigma$ .*

*In fact, every countably infinite group satisfies the relative perturbative factor theorem. Specifically, there is a function  $\delta : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$  such that, for all  $G \curvearrowright (X, \mu)$ ,  $\Sigma$ , and  $\bar{p}$  satisfying the assumptions above, if  $H(\bar{p}) \leq M$  and  $\xi$  is a partition satisfying*

$$|\text{dist}(\xi) - \bar{p}| + |H(\xi) - H(\bar{p})| + |h_G(\xi | \Sigma) - H(\bar{p})| < \delta(M, \epsilon)$$

then the partition  $\alpha$  above may be chosen so that  $d_\mu^{\text{alg}}(\alpha, \xi) < \epsilon$ .

*Proof.* The proof of the full theorem has many parts which we break into individual claims. We assume throughout that  $G \curvearrowright (X, \mu)$ ,  $\Sigma$ , and  $\bar{p}$  have all of the properties assumed in the statement of the theorem.

**Claim 1.** *If  $\mathcal{F}$  is a  $G$ -invariant class-bijective sub- $\sigma$ -algebra,  $G \curvearrowright (X, \mu)$  has completely positive outer Rokhlin entropy relative to  $\mathcal{F}$ , and  $\xi$  is a partition satisfying  $H(\xi | \mathcal{F}) < \infty$  and  $H(\bar{p}) \leq h_G(\xi | \mathcal{F})$ , then there is a partition  $\alpha \subseteq \sigma\text{-alg}_G(\xi) \vee \mathcal{F}$  with  $\text{dist}(\alpha) = \bar{p}$  such that  $\alpha$  is  $G$ -Bernoulli over  $\mathcal{F}$ .*

*Proof of Claim.* Let  $T \in [E_G^X]$  and  $f : X \rightarrow [0, 1]$  be  $\mathcal{F}$ -expressible and  $\mathcal{F}$ -measurable, respectively, as given by Corollary 7.4. Note that  $\mathcal{I}_T \subseteq \mathcal{F} \subseteq \mathcal{P}_\xi$  by Corollary 4.7. By working with the  $T$ -action on the factor associated to  $\sigma\text{-alg}_T(\xi) \vee \mathcal{P}_\xi$  and by applying the non-ergodic, relative version of Sinai's factor theorem (specifically Corollary 6.5), we obtain a partition  $\beta \subseteq \sigma\text{-alg}_T(\xi) \vee \mathcal{P}_\xi$  such that  $H(\beta | \mathcal{I}_T) = h_T^{\text{KS}}(\xi | \mathcal{P}_\xi)$ ,  $\beta$  is  $T$ -weakly Bernoulli over  $\mathcal{I}_T$ , and  $\sigma\text{-alg}_T(\beta)$  is independent with  $\mathcal{P}_\xi$  relative to  $\mathcal{I}_T$ . Lemma 8.4.(2) tells us that  $\beta$  is  $G$ -weakly Bernoulli over  $\mathcal{F}$ .

Notice that  $H(\beta | \mathcal{F}) = H(\beta | \mathcal{I}_T)$  since  $\beta$  is independent with  $\mathcal{P}_\xi$  relative to  $\mathcal{I}_T$  and  $\mathcal{I}_T \subseteq \mathcal{F} \subseteq \mathcal{P}_\xi$ . Also notice that Lemma 8.2.(iii) gives  $\beta \subseteq \sigma\text{-alg}_T(\xi) \vee \mathcal{P}_\xi \subseteq \sigma\text{-alg}_G(\xi) \vee \mathcal{F}$ .

Apply Corollary 4.8 to get an aperiodic ergodic  $\mathcal{F}$ -expressible  $S \in [E_G^X]$ . By Lemma 4.5,  $\beta$  is  $S$ -weakly Bernoulli over  $\mathcal{F}$  and hence  $h_S^{\text{KS}}(\beta | \mathcal{F}) = H(\beta | \mathcal{F})$ . In particular, applying Lemma 8.5 we obtain

$$h_S^{\text{KS}}(\beta | \mathcal{F}) = H(\beta | \mathcal{F}) = H(\beta | \mathcal{I}_T) = h_T^{\text{KS}}(\xi | \mathcal{P}_\xi) \geq h_G(\xi | \mathcal{F}) \geq H(\bar{p}).$$

Again, we work with the ergodic  $S$ -action on the factor associated to  $\sigma\text{-alg}_S(\beta) \vee \mathcal{F}$  and apply the relative Sinai factor theorem (Lemma 6.1) in order to obtain a partition  $\alpha \subseteq \sigma\text{-alg}_S(\beta) \vee \mathcal{F}$  such that  $\text{dist}(\alpha) = \bar{p}$  and such that  $\alpha$  is  $S$ -Bernoulli over  $\mathcal{F}$ . By Lemma 4.5  $\alpha$  is  $G$ -Bernoulli over  $\mathcal{F}$ . Furthermore, Lemma 4.2 gives  $\sigma\text{-alg}_S(\beta) \vee \mathcal{F} \subseteq \sigma\text{-alg}_G(\beta) \vee \mathcal{F}$  and hence

$$\alpha \subseteq \sigma\text{-alg}_S(\beta) \vee \mathcal{F} \subseteq \sigma\text{-alg}_G(\beta) \vee \mathcal{F} \subseteq \sigma\text{-alg}_G(\xi) \vee \mathcal{F}. \quad \square$$

**Claim 2.** *If  $H(\bar{p}) < h_G(X, \mu | \Sigma) < \infty$ , then there is a partition  $\alpha$  with  $\text{dist}(\alpha) = \bar{p}$  which is  $G$ -Bernoulli over  $\Sigma$ .*

*Proof.* By definition of Rokhlin entropy, we can find a partition  $\xi$  satisfying  $H(\xi | \Sigma) < \infty$  and  $\sigma\text{-alg}_G(\xi) \vee \Sigma = \mathcal{B}(X)$ . It follows that  $H(\bar{p}) < h_G(X, \mu | \Sigma) = h_G(\xi | \Sigma)$ . Next, apply Lemma 2.3 to obtain a class-bijective  $G$ -invariant sub- $\sigma$ -algebra  $\mathcal{F} \supseteq \Sigma$  such that  $G \curvearrowright (X, \mu)$  has completely positive outer Rokhlin entropy relative to  $\mathcal{F}$  and such that  $H(\bar{p}) < h_G(\xi | \mathcal{F})$ . By applying Claim 1 we obtain a partition  $\alpha$  with  $\text{dist}(\alpha) = \bar{p}$  such that  $\alpha$  is  $G$ -Bernoulli over  $\mathcal{F}$ . In particular,  $\alpha$  is  $G$ -Bernoulli over  $\Sigma$ .  $\square$

**Claim 3.** *If  $h_G(X, \mu | \Sigma) = \infty$ , then there is a partition  $\alpha$  with  $\text{dist}(\alpha) = \bar{p}$  which is  $G$ -Bernoulli over  $\Sigma$ .*

*Proof of Claim.* Apply Lemma 2.3 to obtain a  $G$ -invariant class-bijective sub- $\sigma$ -algebra  $\mathcal{F} \supseteq \Sigma$  with  $h_G(X, \mu | \mathcal{F}) = \infty$ . A difficulty in the infinite entropy case stems from a potential, but unconfirmed, possible defect of Rokhlin entropy: it may be that  $\sup\{h_G(\xi | \mathcal{F}) : H(\xi | \mathcal{F}) < \infty\}$  is finite even though  $h_G(X, \mu | \mathcal{F}) = \infty$ . However, the inverse-limit formula for Rokhlin entropy obtained in [45, 46] shows

that there is  $c > 0$  and an increasing sequence  $(\xi_n)_{n \in \mathbb{N}}$  of finite partitions with  $\xi_0 = \{X\}$  and with  $h_G(\xi_n | \sigma\text{-alg}_G(\xi_{n-1}) \vee \mathcal{F}) > c$  for all  $n \geq 1$ .

Pick a probability vector  $\bar{q}$  with  $0 < H(\bar{q}) < c$ . For each  $n \in \mathbb{N}$  set  $\mathcal{F}_n = \Pi(\sigma\text{-alg}_G(\xi_n) \vee \mathcal{F})$  and note that Lemma 2.2 gives (for  $n \geq 1$ )

$$h_G(\xi_n | \mathcal{F}_{n-1}) = h_G(\xi_n | \sigma\text{-alg}_G(\xi_{n-1}) \vee \mathcal{F}) > c.$$

For every  $n \geq 1$ , apply Claim 1 to  $\bar{q}$ ,  $\xi_n$ , and  $\mathcal{F}_{n-1}$  to obtain a partition  $\alpha_n \subseteq \sigma\text{-alg}_G(\xi_n) \vee \mathcal{F}_{n-1} \subseteq \mathcal{F}_n$  such that  $\text{dist}(\alpha_n) = \bar{q}$  and  $\alpha_n$  is  $G$ -Bernoulli over  $\mathcal{F}_{n-1}$ . Since  $\Sigma \subseteq \mathcal{F}_n$  for all  $n \in \mathbb{N}$ , it follows that  $\bigvee_{n \geq 1} \alpha_n$  is  $G$ -Bernoulli over  $\Sigma$ . Since the  $\alpha_n$ 's are mutually independent and have the same distribution, every class in  $\bigvee_{n \geq 1} \alpha_n$  has measure 0. Therefore there is a coarser partition  $\alpha \leq \bigvee_{n \geq 1} \alpha_n$  with  $\text{dist}(\alpha) = \bar{p}$ . Of course,  $\alpha$  is  $G$ -Bernoulli over  $\Sigma$  as well.  $\square$

Let  $\delta_{\mathbb{Z}} : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$  be as described in Lemma 6.1. For  $M, \epsilon > 0$  set  $\epsilon' = (1/2) \min((1/5)\delta_{\mathbb{Z}}(M, \epsilon/4), \epsilon/4)$  and let  $v = v(M+1, \epsilon', \mathbb{Z})$  be as described in Corollary 6.5. Finally, define  $\delta(M, \epsilon) = (1/2) \min(v/6, \epsilon', 1/3)$ .

**Claim 4.** *Suppose that  $\mathcal{F}$  is a  $G$ -invariant class-bijective sub- $\sigma$ -algebra such that  $G \curvearrowright (X, \mu)$  has completely positive outer Rokhlin entropy relative to  $\mathcal{F}$ . Also suppose that  $T \in [E_G^X]$  and  $f : X \rightarrow [0, 1]$  are  $\mathcal{F}$ -expressible and  $\mathcal{F}$ -measurable, respectively, as given by Corollary 7.4. Let  $M, \epsilon > 0$  and let  $\xi$  be a countable partition of  $X$ . If  $H(\bar{p}) \leq M$ ,  $H(\bar{p}) < h_T^{\text{KS}}(\xi | \mathcal{P}_\xi) + h_G(X, \mu | \sigma\text{-alg}_G(\xi) \vee \mathcal{F})$ , and*

$$|\text{dist}(\xi) - \bar{p}| + |H(\xi) - H(\bar{p})| + |h_T^{\text{KS}}(\xi | \mathcal{P}_\xi) - H(\bar{p})| < 3\delta(M, \epsilon),$$

*then there is a partition  $\alpha$  with  $d_\mu^{\text{alg}}(\alpha, \xi) < \epsilon/2$  and  $\text{dist}(\alpha) = \bar{p}$  such that  $\alpha$  is  $G$ -Bernoulli over  $\mathcal{F}$ .*

*Proof of Claim.* We first prepare a partition  $\zeta$  which will be needed a bit later in the argument. If  $H(\bar{p}) \leq h_T^{\text{KS}}(\xi | \mathcal{P}_\xi)$  then we set  $\zeta = \{X\}$ , and otherwise we apply Claim 2 or Claim 3 to obtain a partition  $\zeta$  which is  $G$ -Bernoulli over  $\sigma\text{-alg}_G(\xi) \vee \mathcal{F}$  and satisfies

$$H(\zeta) = H(\bar{p}) - h_T^{\text{KS}}(\xi | \mathcal{P}_\xi) < h_G(X, \mu | \sigma\text{-alg}_G(\xi) \vee \mathcal{F}).$$

In either case, we have that  $\zeta$  is  $G$ -Bernoulli over  $\sigma\text{-alg}_G(\xi) \vee \mathcal{F}$  and  $H(\bar{p}) \leq h_T^{\text{KS}}(\xi | \mathcal{P}_\xi) + H(\zeta)$ .

Observe that  $H(\xi) \leq H(\bar{p}) + 3\delta(M, \epsilon) \leq M + 1$  and

$$h_T^{\text{KS}}(\xi | \mathcal{P}_\xi) > H(\bar{p}) - 3\delta(M, \epsilon) > H(\xi) - 6\delta(M, \epsilon) > H(\xi) - v.$$

Also, recall that  $\mathcal{I}_T \subseteq \mathcal{F} \subseteq \mathcal{P}_\xi$  by Corollary 4.7. So, by working with the  $T$ -action on the factor associated with  $\sigma\text{-alg}_T(\xi) \vee \mathcal{P}_\xi$ , we may apply the strongest statement in Corollary 6.5 to obtain a partition  $\beta \subseteq \sigma\text{-alg}_T(\xi) \vee \mathcal{P}_\xi$  such that  $\beta$  is  $T$ -weakly Bernoulli over  $\mathcal{I}_T$ ,  $\sigma\text{-alg}_T(\beta)$  is independent with  $\mathcal{P}_\xi$  relative to  $\mathcal{I}_T$ ,  $H(\beta | \mathcal{I}_T) = h_T^{\text{KS}}(\xi | \mathcal{P}_\xi)$ ,  $d_\mu^{\text{alg}}(\beta, \xi) < \epsilon'$ , and  $|H(\beta) - H(\xi)| < \epsilon'$ . In particular,

$$(9.1) \quad |\text{dist}(\beta) - \bar{p}| + |H(\beta) - H(\bar{p})| < 2\epsilon' + |\text{dist}(\xi) - \bar{p}| + |H(\xi) - H(\bar{p})|.$$

Lemma 8.4.(2) tells us that  $\beta$  is  $G$ -weakly Bernoulli over  $\mathcal{F}$ . Also,  $\beta \subseteq \sigma\text{-alg}_G(\xi) \vee \mathcal{F}$  by Lemma 8.2.(iii), and therefore  $\zeta$  is  $G$ -Bernoulli over  $\sigma\text{-alg}_G(\beta) \vee \mathcal{F}$ .

Apply Corollary 4.8 to get an aperiodic ergodic  $\mathcal{F}$ -expressible  $S \in [E_G^X]$ . By Lemma 4.5,  $\beta$  is  $S$ -weakly Bernoulli over  $\mathcal{F}$ . As in Claim 1 we have

$$h_S^{\text{KS}}(\beta | \mathcal{F}) = H(\beta | \mathcal{F}) = H(\beta | \mathcal{I}_T) = h_T^{\text{KS}}(\xi | \mathcal{P}_\xi).$$

Combining the above equation, (9.1), and our assumption we obtain

$$|\text{dist}(\beta) - \bar{p}| + |\mathbf{H}(\beta) - \mathbf{H}(\bar{p})| + |h_S^{\text{KS}}(\beta | \mathcal{F}) - \mathbf{H}(\bar{p})| < 3\delta(M, \epsilon) + 2\epsilon' < \delta_{\mathbb{Z}}(M, \epsilon/4).$$

Since  $\zeta$  is  $G$ -Bernoulli over  $\sigma\text{-alg}_G(\beta) \vee \mathcal{F}$ , Lemma 4.5 implies that  $\zeta$  is  $S$ -Bernoulli over  $\sigma\text{-alg}_G(\beta) \vee \mathcal{F}$ . So we have

$$h_S^{\text{KS}}(\beta \vee \zeta | \mathcal{F}) = h_S^{\text{KS}}(\beta | \mathcal{F}) + \mathbf{H}(\zeta) = h_T^{\text{KS}}(\xi | \mathcal{P}_\xi) + \mathbf{H}(\zeta) \geq \mathbf{H}(\bar{p}).$$

Working within the ergodic  $S$ -action on the factor associated to  $\sigma\text{-alg}_S(\beta \vee \zeta) \vee \mathcal{F}$ , the two previous inequalities allow us to apply Lemma 6.1 to the partition  $\beta$ . This results in a partition  $\alpha \subseteq \sigma\text{-alg}_S(\beta \vee \zeta) \vee \mathcal{F}$  which is  $S$ -Bernoulli over  $\mathcal{F}$  and satisfies  $\text{dist}(\alpha) = \bar{p}$  and  $d_\mu^{\text{alg}}(\alpha, \beta) < \epsilon/4$ . Lemma 4.5 implies that  $\alpha$  is  $G$ -Bernoulli over  $\mathcal{F}$ . Also,  $d_\mu^{\text{alg}}(\alpha, \xi) \leq d_\mu^{\text{alg}}(\alpha, \beta) + d_\mu^{\text{alg}}(\beta, \xi) < \epsilon/2$ .  $\square$

**Claim 5.** *If  $\mathbf{H}(\bar{p}) \leq M$  and  $\xi$  is a partition of  $X$  with*

$$|\text{dist}(\xi) - \bar{p}| + |\mathbf{H}(\xi) - \mathbf{H}(\bar{p})| + |h_G(\xi | \Sigma) - \mathbf{H}(\bar{p})| < \delta(M, \epsilon),$$

*then there is a partition  $\alpha$  with  $d_\mu^{\text{alg}}(\alpha, \xi) < \epsilon$  and  $\text{dist}(\alpha) = \bar{p}$  such that  $\alpha$  is  $G$ -Bernoulli over  $\Sigma$ .*

*Proof of Claim.* Fix a sequence of probability vectors  $\bar{p}^n$  satisfying  $\mathbf{H}(\bar{p}^n) < \mathbf{H}(\bar{p})$  and

$$|\bar{p}^n - \bar{p}| + 2|\mathbf{H}(\bar{p}^n) - \mathbf{H}(\bar{p})| < \min_{m \leq n} (1/2)\delta(M, 2^{-m}).$$

Then  $\bar{p}^n \rightarrow \bar{p}$  as  $n \rightarrow \infty$  and we have

$$(9.2) \quad |\bar{p}^n - \bar{p}^{n+1}| + 2|\mathbf{H}(\bar{p}^n) - \mathbf{H}(\bar{p}^{n+1})| < \delta(M, 2^{-n}).$$

Apply Lemma 2.3 to obtain a class-bijective  $G$ -invariant sub- $\sigma$ -algebra  $\Sigma_1 \supseteq \Sigma$  with  $h_G(\Sigma_1 | \Sigma) < \mathbf{H}(\bar{p}) - \mathbf{H}(\bar{p}^1)$ . Next, by working within the factor associated to  $\Sigma_1$ , we can apply Lemma 2.3 to obtain a class-bijective  $G$ -invariant sub- $\sigma$ -algebra  $\Sigma_2$  with  $\Sigma_1 \supseteq \Sigma_2 \supseteq \Sigma$  and  $h_G(\Sigma_2 | \Sigma) < \mathbf{H}(\bar{p}) - \mathbf{H}(\bar{p}^2)$ . By repeating this process, we obtain a decreasing sequence of class-bijective  $G$ -invariant sub- $\sigma$ -algebras  $\Sigma_n$  such that each one contains  $\Sigma$  and  $h_G(\Sigma_n | \Sigma) < \mathbf{H}(\bar{p}) - \mathbf{H}(\bar{p}^n)$ . Set  $\mathcal{F}_n = \Pi(\Sigma_n)$  and note that the  $\mathcal{F}_n$ 's are decreasing and satisfy  $h_G(\mathcal{F}_n | \Sigma) < \mathbf{H}(\bar{p}) - \mathbf{H}(\bar{p}^n)$ . For each  $n$ , let  $T_n \in [E_G^X]$  and  $f_n : X \rightarrow [0, 1]$  be  $\mathcal{F}_n$ -expressible and  $\mathcal{F}_n$ -measurable, respectively, as given by Corollary 7.4. We will write  $\mathcal{P}_\xi^n$  for the external past of  $\xi$  obtained from  $T_n, f_n, \mathcal{F}_n$ .

The sub-additivity of entropy implies that for any  $\mathcal{C} \subseteq \mathcal{B}(X)$

$$0 \leq h_G(\mathcal{C} | \Sigma) - h_G(\mathcal{C} | \mathcal{F}_n) \leq h_G(\mathcal{F}_n | \Sigma) < \mathbf{H}(\bar{p}) - \mathbf{H}(\bar{p}^n).$$

In particular,  $h_G(\xi | \mathcal{F}_n)$  converges to  $h_G(\xi | \Sigma)$ , and using  $\mathcal{C} = \mathcal{B}(X)$  above we get

$$(9.3) \quad \mathbf{H}(\bar{p}^n) < \mathbf{H}(\bar{p}) - h_G(X, \mu | \Sigma) + h_G(X, \mu | \mathcal{F}_n) \leq h_G(X, \mu | \mathcal{F}_n).$$

Pick  $n$  sufficiently large with  $2^{-n} < \epsilon/2$  and

$$|\text{dist}(\xi) - \bar{p}^n| + |\mathbf{H}(\xi) - \mathbf{H}(\bar{p}^n)| + |h_G(\xi | \mathcal{F}_n) - \mathbf{H}(\bar{p}^n)| < \delta(M, \epsilon).$$

By our assumptions  $\mathbf{H}(\xi | \mathcal{F}_n) \leq \mathbf{H}(\xi) < \infty$ , so from Lemma 8.5 we obtain

$$h_G(\xi | \mathcal{F}_n) \leq h_{T_n}^{\text{KS}}(\xi | \mathcal{P}_\xi^n) \leq \mathbf{H}(\xi).$$

As  $|h_G(\xi | \mathcal{F}_n) - \mathbf{H}(\xi)| < 2\delta(M, \epsilon)$ , we deduce that

$$|\text{dist}(\xi) - \bar{p}^n| + |\mathbf{H}(\xi) - \mathbf{H}(\bar{p}^n)| + |h_{T_n}^{\text{KS}}(\xi | \mathcal{P}_\xi^n) - \mathbf{H}(\bar{p}^n)| < 3\delta(M, \epsilon).$$



Also, from (9.3) we obtain

$$\begin{aligned} \mathbb{H}(\bar{p}^n) &< h_G(X, \mu | \mathcal{F}_n) \leq h_G(\xi | \mathcal{F}_n) + h_G(X, \mu | \sigma\text{-alg}_G(\xi) \vee \mathcal{F}_n) \\ &\leq h_{T_n}^{\text{KS}}(\xi | \mathcal{P}_\xi^n) + h_G(X, \mu | \sigma\text{-alg}_G(\xi) \vee \mathcal{F}_n). \end{aligned}$$

By Claim 4, there is a partition  $\alpha_n$  with  $\text{dist}(\alpha_n) = \bar{p}^n$  and  $d_\mu^{\text{alg}}(\alpha_n, \xi) < \epsilon/2$  such that  $\alpha_n$  is  $G$ -Bernoulli over  $\mathcal{F}_n$ .

Next we construct a sequence of partitions  $\alpha_k$ ,  $k \geq n$ , such that  $d_\mu^{\text{alg}}(\alpha_{k+1}, \alpha_k) < 2^{-k-1}$ ,  $\text{dist}(\alpha_k) = \bar{p}^k$ , and such that  $\alpha_k$  is  $G$ -Bernoulli over  $\mathcal{F}_k$ . Let  $k \geq n$  and inductively assume that  $\alpha_k$  has been constructed. As  $\mathcal{F}_{k+1} \subseteq \mathcal{F}_k$ , we see that  $\alpha_k$  is  $G$ -Bernoulli over  $\mathcal{F}_{k+1}$  and hence  $T_{k+1}$ -Bernoulli over  $\mathcal{P}_{\alpha_k}^{k+1}$  by Lemma 8.4.(1). Consequently,  $h_{T_{k+1}}^{\text{KS}}(\alpha_k | \mathcal{P}_{\alpha_k}^{k+1}) = \mathbb{H}(\alpha_k)$  and therefore (9.2) gives

$$|\text{dist}(\alpha_k) - \bar{p}^{k+1}| + |\mathbb{H}(\alpha_k) - \mathbb{H}(\bar{p}^{k+1})| + |h_{T_{k+1}}^{\text{KS}}(\alpha_k | \mathcal{P}_{\alpha_k}^{k+1}) - \mathbb{H}(\bar{p}^{k+1})| < \delta(M, 2^{-k}).$$

Additionally, from (9.3), sub-additivity, and Lemma 8.5 we see that

$$\begin{aligned} \mathbb{H}(\bar{p}^{k+1}) &< h_G(X, \mu | \mathcal{F}_{k+1}) \leq h_G(\alpha_k | \mathcal{F}_{k+1}) + h_G(X, \mu | \sigma\text{-alg}_G(\alpha_k) \vee \mathcal{F}_{k+1}) \\ &\leq h_{T_{k+1}}^{\text{KS}}(\alpha_k | \mathcal{P}_{\alpha_k}^{k+1}) + h_G(X, \mu | \sigma\text{-alg}_G(\alpha_k) \vee \mathcal{F}_{k+1}). \end{aligned}$$

Therefore the assumptions of Claim 4 are met. So there is a partition  $\alpha_{k+1}$  with  $d_\mu^{\text{alg}}(\alpha_{k+1}, \alpha_k) < 2^{-k-1}$  and  $\text{dist}(\alpha_{k+1}) = \bar{p}^{k+1}$  and such that  $\alpha_{k+1}$  is  $G$ -Bernoulli over  $\mathcal{F}_{k+1}$ . This completes the construction of the  $\alpha_k$ 's.

Let  $\alpha$  be the limit of the  $\alpha_k$ 's. Then  $\text{dist}(\alpha) = \bar{p}$  and

$$d_\mu^{\text{alg}}(\alpha, \xi) \leq \sum_{k \geq n} d_\mu^{\text{alg}}(\alpha_{k+1}, \alpha_k) + d_\mu^{\text{alg}}(\alpha_n, \xi) < 2^{-n} + \epsilon/2 < \epsilon.$$

Finally, since  $\Sigma$  is contained in every  $\mathcal{F}_k$ ,  $\alpha$  is  $G$ -Bernoulli over  $\Sigma$ .  $\square$

**Claim 6.** *If  $\mathbb{H}(\bar{p}) \leq h_G(X, \mu | \Sigma)$  then there is a partition  $\alpha$  which is  $G$ -Bernoulli over  $\Sigma$  and satisfies  $\text{dist}(\alpha) = \bar{p}$ .*

*Proof of Claim.* If  $\mathbb{H}(\bar{p}) < h_G(X, \mu | \Sigma)$  or if  $h_G(X, \mu | \Sigma) = \infty$ , then we are done by Claims 2 and 3. So assume  $\mathbb{H}(\bar{p}) = h_G(X, \mu | \Sigma) < \infty$ . Pick a probability vector  $\bar{q}$  with  $\mathbb{H}(\bar{q}) > \mathbb{H}(\bar{p})$  and

$$|\bar{q} - \bar{p}| + |\mathbb{H}(\bar{q}) - \mathbb{H}(\bar{p})| < \delta(\mathbb{H}(\bar{p}), 1).$$

By the relative finite generator theorem [44], there is a partition  $\xi$  with  $\text{dist}(\xi) = \bar{q}$  and  $\sigma\text{-alg}_G(\xi) \vee \Sigma = \mathcal{B}(X)$ . Then  $h_G(\xi | \Sigma) = h_G(X, \mu | \Sigma) = \mathbb{H}(\bar{p})$ , so

$$|\text{dist}(\xi) - \bar{p}| + |\mathbb{H}(\xi) - \mathbb{H}(\bar{p})| + |h_G(\xi | \Sigma) - \mathbb{H}(\bar{p})| = |\bar{q} - \bar{p}| + |\mathbb{H}(\bar{q}) - \mathbb{H}(\bar{p})| < \delta(\mathbb{H}(\bar{p}), 1).$$

Therefore Claim 5 implies that there is a partition  $\alpha$  which is  $G$ -Bernoulli over  $\Sigma$  with  $\text{dist}(\alpha) = \bar{p}$ .  $\square$

This now completes the proof, as the theorem follows from the final two claims.  $\square$

Note that our work in Section 5 implies that the non-ergodic versions of the factor theorem hold for all countably infinite groups.

To close this section, we point out the peculiar fact that the residual factor theorem can fail for non-amenable groups. This fact was discovered by Lewis Bowen. We thank him for permission to include it here.

**Proposition 9.2** (Lewis Bowen). *The residual factor theorem is false for non-amenable groups.*

*Proof.* Bowen’s original argument was based on the f-invariant (an invariant introduced by Bowen in [3]). His argument, in brief, was that if you fix a Bernoulli shift over the rank 2 free group, then in the space of ergodic self-joinings the generic joining will have f-invariant minus infinity, but if the residual factor theorem were true the generic joining would have f-invariant equal to the entropy of the original Bernoulli shift.

Here we give a different argument that requires less preparation to explain in detail. Let  $G = \langle a, b \rangle$  be the rank 2 free group. Let  $(X_i, \mu_i)$  be a copy of  $(2^G, u_2^G)$  for  $1 \leq i \leq 4$ , and let  $(X_L, \mu_L)$  be the product of  $(X_1, \mu_1)$  with  $(X_2, \mu_2)$  and let  $(X_R, \mu_R)$  be the product of  $(X_3, \mu_3)$  with  $(X_4, \mu_4)$ . Consider the Ornstein–Weiss factor map  $\psi : X_3 \rightarrow X_L$  defined by

$$\psi(x_3)(g) = \left( x_3(g) + x_3(ga), x_3(g) + x_3(gb) \right) \pmod{2}.$$

It is well known, and a simple exercise to verify, that  $\psi$  is a 2-to-1 surjection that pushes the Bernoulli measure  $\mu_3$  forward to  $\mu_L$ . Let  $\pi_3 : X_R \rightarrow X_3$  be the projection map and define  $\phi : (X_R, \mu_R) \rightarrow (X_L, \mu_L)$  by  $\phi = \psi \circ \pi_3$ . Now set  $\lambda = (\phi \times \text{id})_*(\mu_R)$ . Then  $\lambda$  is a joining between the equal entropy Bernoulli shifts  $(X_L, \mu_L)$  and  $(X_R, \mu_R)$ . For each  $i$  view  $\mathcal{B}(X_i) \subseteq \mathcal{B}(X_L \times X_R)$  in the natural way. By construction we have that  $\mathcal{F} = \mathcal{B}(X_1) \vee \mathcal{B}(X_2) \vee \mathcal{B}(X_3)$  coincides with  $\mathcal{B}(X_3) \pmod{\lambda}$ -null sets. Thus

$$h_G(\mathcal{F}, \lambda) = h_G(\mathcal{B}(X_3), \lambda) = \log(2) < \log(4).$$

Rokhlin entropy is upper-semicontinuous in the weak\*-topology [2], so there is an open neighborhood  $U$  of  $\lambda$  with  $h_G(\mathcal{F}, \nu) < \log(4)$  for all  $\nu \in U$ . However, if  $f : (X_L, \mu_L) \rightarrow (X_R, \mu_R)$  is a factor map and  $\nu = (\text{id} \times f)_*(\mu_L)$  the factor joining, then  $\mathcal{B}(X_L) = \mathcal{F} = \mathcal{B}(X_L \times X_R) \pmod{\nu}$ -null sets and hence

$$h_G(\mathcal{F}, \nu) = h_G(X_L \times X_R, \nu) = h_G(X_L, \mu_L) = \log(4).$$

Therefore there are no factor joinings from  $(X_L, \mu_L)$  to  $(X_R, \mu_R)$  lying in the open neighborhood  $U$  of  $\lambda$ . Thus the factor joinings from the Bernoulli shift  $(X_L, \mu_L)$  to the equal-entropy Bernoulli shift  $(X_R, \mu_R)$  do not form a dense set in the space of all joinings of  $\mu_L$  with  $\mu_R$ .  $\square$

## 10. FURTHER RESULTS AND DISCUSSION

A well-known property of classical entropy is that it is additive, meaning if  $G$  is amenable,  $G \curvearrowright (X, \mu)$  is a p.m.p. action,  $\mathcal{F}$  is a  $G$ -invariant sub- $\sigma$ -algebra, and  $G \curvearrowright (Y, \nu)$  is the factor associated to  $\mathcal{F}$ , then

$$h_G^{\text{KS}}(X, \mu) = h_G^{\text{KS}}(Y, \nu) + h_G^{\text{KS}}(X, \mu | \mathcal{F}).$$

When the group is non-amenable and Kolmogorov–Sinai entropy is replaced with Rokhlin entropy, additivity in general fails and only sub-additivity remains true.

A natural question in regard to the factor theorem is to ask that the Bernoulli factors preserve additivity. In particular, if  $G \curvearrowright (X, \mu)$  is free and ergodic and  $(L, \lambda)$  is a probability space with  $H(L, \lambda) = h_G(X, \mu)$ , is there a factor map from  $(X, \mu)$  onto  $(L^G, \lambda^G)$  that has 0 relative entropy? Our methods do not seem to answer either of these questions. However, we mention that the answers are positive if arbitrarily small errors are allowed.

**Theorem 10.1.** *Let  $G \curvearrowright (X, \mu)$  be an aperiodic ergodic p.m.p. action, let  $\Sigma$  be a  $G$ -invariant sub- $\sigma$ -algebra, and let  $\bar{p}$  be a finite probability vector with  $H(\bar{p}) \leq h_G(X, \mu | \Sigma) < \infty$ . For every  $\epsilon > 0$  there is a partition  $\alpha$  with  $\text{dist}(\alpha) = \bar{p}$  that is  $G$ -Bernoulli over  $\Sigma$  and satisfies*

$$H(\alpha) + h_G(X, \mu | \sigma\text{-alg}_G(\alpha) \vee \Sigma) < h_G(X, \mu | \Sigma) + \epsilon.$$

*Proof.* Recall the metric  $d_\mu^{\text{Rok}}$  introduced in Section 5. On the space of partitions of cardinality  $|\bar{p}|$  the metrics  $d_\mu^{\text{alg}}$  and  $d_\mu^{\text{Rok}}$  are uniformly equivalent [13, Fact 1.7.7]. So there is  $\epsilon' > 0$  with  $d_\mu^{\text{Rok}}(\alpha, \beta) < \epsilon/2$  whenever  $|\alpha| = |\beta| = |\bar{p}|$  and  $d_\mu^{\text{alg}}(\alpha, \beta) < \epsilon'$ . Let  $\delta : \mathbb{R}_+^2 \rightarrow \mathbb{R}$  be as in Theorem 9.1 and pick  $0 < \kappa < \min(\delta(H(\bar{p}), \epsilon'), \epsilon/2)$ . Pick a probability vector  $\bar{q}$  that refines  $\bar{p}$  and satisfies  $h_G(X, \mu | \Sigma) < H(\bar{q}) < h_G(X, \mu | \Sigma) + \kappa$ . By the generalized Krieger generator theorem [44] we can find a partition  $\mathcal{Q}$  with  $\sigma\text{-alg}_G(\mathcal{Q}) \vee \Sigma = \mathcal{B}(X)$  and  $\text{dist}(\mathcal{Q}) = \bar{q}$ . Let  $\mathcal{P}$  be the coarsening of  $\mathcal{Q}$  associated with the coarsening  $\bar{p}$  of  $\bar{q}$ . Sub-additivity and the inequality  $h_G(X, \mu | \sigma\text{-alg}_G(\mathcal{P}) \vee \Sigma) \leq H(\mathcal{Q} | \mathcal{P})$  give

$$H(\mathcal{P}) + H(\mathcal{Q} | \mathcal{P}) - \kappa = H(\mathcal{Q}) - \kappa < h_G(X, \mu | \Sigma) \leq h_G(\mathcal{P} | \Sigma) + H(\mathcal{Q} | \mathcal{P}),$$

hence  $h_G(\mathcal{P} | \Sigma) > H(\mathcal{P}) - \kappa$ . It follows that

$$|\text{dist}(\mathcal{P}) - \bar{p}| + |H(\mathcal{P}) - H(\bar{p})| + |h_G(\mathcal{P} | \Sigma) - H(\bar{p})| < \delta(H(\bar{p}), \epsilon').$$

By our main theorem, there is a partition  $\alpha$  with  $\text{dist}(\alpha) = \bar{p}$  and  $d_\mu^{\text{alg}}(\alpha, \mathcal{P}) < \epsilon'$  which is  $G$ -Bernoulli over  $\Sigma$ . Finally, by [2, Lem. 6.2] we have

$$\begin{aligned} h_G(X, \mu | \sigma\text{-alg}_G(\alpha) \vee \Sigma) &\leq d_\mu^{\text{Rok}}(\alpha, \mathcal{P}) + h_G(X, \mu | \sigma\text{-alg}_G(\mathcal{P}) \vee \Sigma) \\ &< \frac{\epsilon}{2} + H(\mathcal{Q} | \mathcal{P}) \\ &= \frac{\epsilon}{2} + H(\mathcal{Q}) - H(\bar{p}) \\ &< \epsilon + h_G(X, \mu | \Sigma) - H(\alpha). \end{aligned} \quad \square$$

**Corollary 10.2.** *Let  $G \curvearrowright (X, \mu)$  be an aperiodic ergodic p.m.p. action, and let  $\Sigma$  be a  $G$ -invariant sub- $\sigma$ -algebra. For every  $\epsilon > 0$  there is a partition  $\alpha$  that is  $G$ -Bernoulli over  $\Sigma$  and satisfies  $h_G(X, \mu | \sigma\text{-alg}_G(\alpha) \vee \Sigma) = 0$  and  $H(\alpha) \leq h_G(X, \mu | \Sigma) + \epsilon$ .*

*Proof.* First suppose that  $h_G(X, \mu | \Sigma) < \infty$ . By Theorem 10.1 there is a partition  $\alpha_1$  with  $H(\alpha_1) = h_G(X, \mu | \Sigma)$  that is  $G$ -Bernoulli over  $\Sigma$  and satisfies  $h_G(X, \mu | \sigma\text{-alg}_G(\alpha_1) \vee \Sigma) < \epsilon/2$ . Inductively assume that  $\alpha_n$  has been constructed such that  $\alpha_n$  is  $G$ -Bernoulli over  $\Sigma$ ,  $h_G(X, \mu | \sigma\text{-alg}_G(\alpha_n) \vee \Sigma) < \epsilon/2^n$ , and

$$H(\alpha_n) \leq h_G(X, \mu | \Sigma) + (1 - 2^{-n+1})\epsilon.$$

Apply Theorem 10.1 to get  $\beta$  with  $H(\beta) = h_G(X, \mu | \sigma\text{-alg}_G(\alpha_n) \vee \Sigma) < \epsilon/2^n$  such that  $\beta$  is  $G$ -Bernoulli over  $\sigma\text{-alg}_G(\alpha_n) \vee \Sigma$  and satisfies  $h_G(X, \mu | \sigma\text{-alg}_G(\beta \vee \alpha_n) \vee \Sigma) < \epsilon/2^{n+1}$ . Now set  $\alpha_{n+1} = \beta \vee \alpha_n$ . This completes the construction of the  $\alpha_n$ 's. Setting  $\alpha = \bigvee_{n \in \mathbb{N}} \alpha_n$  completes the proof in this case.

Now suppose that  $h_G(X, \mu | \Sigma) = \infty$ . Fix an increasing sequence of finite partitions  $(\xi_n)_{n \in \mathbb{N}}$  with  $\xi_0 = \{X\}$  and  $\bigvee_n \sigma\text{-alg}(\xi_n) = \mathcal{B}(X)$ . We may use Lemma 2.3 to choose  $\xi_1$  so that  $\sigma\text{-alg}_G(\xi_1)$  is class-bijective. Let  $G \curvearrowright (X_n, \mu_n)$  be the factor of  $(X, \mu)$  associated with  $\sigma\text{-alg}_G(\xi_n) \vee \Sigma$ , say via  $\phi_n : (X, \mu) \rightarrow (X_n, \mu_n)$ , and let  $\mathcal{F}_n$  be the image of  $\sigma\text{-alg}_G(\xi_{n-1}) \vee \Sigma$  for  $n \geq 1$ . For each  $n \geq 1$  we can apply the previous paragraph to obtain a partition  $\beta_n$  of  $X_n$  that is  $G$ -Bernoulli over  $\mathcal{F}_n$  and

satisfies  $h_G(X_n, \mu_n | \sigma\text{-alg}_G(\beta_n) \vee \mathcal{F}_n) = 0$ . Then  $\alpha = \bigvee_{n \geq 1} \phi_n^{-1}(\beta_n)$  is  $G$ -Bernoulli over  $\Sigma$  and

$$\begin{aligned} h_G(X, \mu | \sigma\text{-alg}_G(\alpha) \vee \Sigma) &\leq \sum_{n \geq 1} h_G(\xi_n | \sigma\text{-alg}_G(\alpha \vee \xi_{n-1}) \vee \Sigma) \\ &\leq \sum_{n \geq 1} h_G(X_n, \mu_n | \sigma\text{-alg}_G(\beta_n) \vee \mathcal{F}_n) = 0. \quad \square \end{aligned}$$

While in general Rokhlin entropy is not additive for factor maps (indeed factor actions can have greater entropy than their source), the above corollary does suggest an entropy-style structure theory of class-bijective factor maps. Specifically, for two actions  $G \curvearrowright (X, \mu)$  and  $G \curvearrowright (Y, \nu)$  and a factor map  $\phi : (X, \mu) \rightarrow (Y, \nu)$ , call  $\phi$  *entropy-additive*<sup>‡</sup> if

$$h_G(X, \mu) = h_G(Y, \nu) + h_G(X, \mu | \phi^{-1}(\mathcal{B}(Y))),$$

and say  $\phi$  has 0 relative entropy if  $h_G(X, \mu | \phi^{-1}(\mathcal{B}(Y))) = 0$ . The above corollary suggests that for every class-bijective factor  $\phi : (X, \mu) \rightarrow (Y, \nu)$  there is an action  $G \curvearrowright (Z, \eta)$  and maps  $(X, \mu) \xrightarrow{r} (Z, \eta) \xrightarrow{a} (Y, \nu)$  such that  $a \circ r = \phi$ ,  $r$  has 0 relative entropy, and  $a$  is entropy-additive. This will be true provided that entropy is additive for class-bijective Bernoulli extensions.

**Corollary 10.3.** *Let  $G \curvearrowright (X, \mu)$  be an aperiodic ergodic p.m.p. action with stabilizer type  $\theta$  and let  $G \curvearrowright (Y, \nu)$  be a class-bijective factor via  $\phi : (X, \mu) \rightarrow (Y, \nu)$ . Let  $\nu = \int_{\text{Sub}(G)} \nu_\Gamma d\theta(\Gamma)$  be the disintegration of  $\nu$  over  $\theta$ . Then there exists a class-bijective Bernoulli extension of  $(Y, \nu)$*

$$G \curvearrowright (Z, \eta) = G \curvearrowright \left( L^G \times Y, \int_{\text{Sub}(G)} \lambda^{\Gamma \setminus G} \times \nu_\Gamma d\theta(\Gamma) \right)$$

and maps  $(X, \mu) \xrightarrow{r} (Z, \eta) \xrightarrow{a} (Y, \nu)$  such that  $a \circ r = \phi$ ,  $r$  has 0 relative entropy, and  $a : Z = L^G \times Y \rightarrow Y$  is the projection map.

*Proof.* This is immediate from applying Corollary 10.2 with  $\Sigma = \phi^{-1}(\mathcal{B}(Y))$  and noting Lemma 3.3.  $\square$

A well-known component of classical Ornstein theory is that Bernoulli measures are precisely those measures which are finitely determined. As a consequence of the perturbative factor theorem, part of this statement extends to the non-amenable realm. Recall that for a finite set  $K$  and two measures  $\mu, \nu \in \mathfrak{M}_G(K^G)$ , the  $\bar{d}$ -distance between  $\mu$  and  $\nu$ , denoted  $\bar{d}(\mu, \nu)$ , is the infimum of

$$\lambda(\{(x, y) \in K^G \times K^G : x(1_G) \neq y(1_G)\})$$

as  $\lambda \in \mathfrak{M}_G(K^G \times K^G)$  varies over all joinings of  $\mu$  with  $\nu$ . A measure  $\mu \in \mathfrak{M}_G(K^G)$  is *finitely determined* if for every  $\epsilon > 0$  there is  $\delta > 0$  and a weak\*-open neighborhood  $U$  of  $\mu$  such that for every  $\nu \in \mathfrak{M}_G(K^G) \cap U$  with  $|h_G(L^G, \nu) - h_G(L^G, \mu)| < \delta$  and  $\text{Stab}_*(\nu) = \text{Stab}_*(\mu)$  we have  $\bar{d}(\mu, \nu) < \epsilon$ .

**Theorem 10.4.** *Let  $G$  be a countably infinite group, let  $(K, \kappa)$  be a finite probability space, and let  $\theta$  be an IRS supported on infinite-index subgroups. If  $h_G(K^G, \kappa^{\theta \setminus G}) = \mathbb{H}(K, \kappa)$  then  $\kappa^{\theta \setminus G}$  is finitely determined.*

<sup>‡</sup>Recall that entropy-additive maps must be class-bijective when  $h_G(X, \mu) > 0$  [2, Thm. 10.2].

*Proof.* Let  $\delta : \mathbb{R}_+^2 \rightarrow \mathbb{R}$  be as in Theorem 9.1. Identify  $\text{Prob}(K)$  with the set of  $K$ -indexed probability vectors, and fix a probability measure  $\omega$  on  $K$  satisfying  $H(\omega) < H(\kappa)$  and

$$(10.1) \quad |\omega - \kappa| + 2|H(\omega) - H(\kappa)| < (1/2)\delta(H(\kappa), \epsilon/2).$$

Let  $\beta = \{B_k : k \in K\}$  be the canonical partition of  $K^G$ , where  $B_k = \{x \in K^G : x(1_G) = k\}$ . Since  $K$  is finite, there is a weak\*-open neighborhood  $U$  of  $\kappa^{\theta \setminus G}$  such that

$$(10.2) \quad \forall \nu \in U \quad |\text{dist}_\nu(\beta) - \kappa| + |H_\nu(\beta) - H(\kappa)| < (1/3)\delta(H(\kappa), \epsilon/2).$$

Now consider a measure  $\nu \in \mathfrak{M}_G(K^G) \cap U$  satisfying  $\text{Stab}_*(\nu) = \theta$  and

$$(10.3) \quad |h_G(K^G, \nu) - h_G(K^G, \kappa^{\theta \setminus G})| < \min(|H(\omega) - H(\kappa)|, (1/6)\delta(H(\kappa), \epsilon/2)).$$

Since  $h_G(K^G, \kappa^{\theta \setminus G}) = H(\kappa)$  we have  $h_G(\beta, \nu) = h_G(K^G, \nu) > H(\omega)$ . Also (10.1), (10.2), (10.3) imply

$$|\text{dist}_\nu(\beta) - \omega| + |H_\nu(\beta) - H(\omega)| + |h_G(\beta, \nu) - H(\omega)| < \delta(H(\kappa), \epsilon/2).$$

By Theorem 9.1 there is a partition  $\beta' = \{B'_k : k \in K\}$  with  $\text{dist}(\beta') = \omega$  that is  $(G, \nu)$ -Bernoulli and satisfies  $d_\nu^{\text{alg}}(\beta, \beta') < \epsilon/2$ . Define  $\phi : K^G \rightarrow K^G$  by  $\phi(x)(g) = k \Leftrightarrow g^{-1} \cdot x \in B'_k$  and set  $\lambda = (\text{id} \times \phi)_*(\nu)$ . Then  $\lambda$  is a joining of  $\nu$  with  $\omega^{\theta \setminus G}$  and

$$\begin{aligned} \lambda(\{(x, y) \in K^G \times K^G : x(1_G) \neq y(1_G)\}) &= \nu(\{x \in K^G : x(1_G) \neq \phi(x)(1_G)\}) \\ &= d_\nu^{\text{alg}}(\beta, \beta') < \epsilon/2. \end{aligned}$$

Thus  $\bar{d}(\nu, \omega^{\theta \setminus G}) < \epsilon/2$ . Of course,  $\kappa^{\theta \setminus G}$  satisfies all of the assumptions on  $\nu$ , so we similarly have  $\bar{d}(\kappa^{\theta \setminus G}, \omega^{\theta \setminus G}) < \epsilon/2$ . Now its a simple exercise in measure theory to see that  $\bar{d}$  is indeed a metric and satisfies the triangle identity. Hence  $\bar{d}(\nu, \kappa^{\theta \setminus G}) < \epsilon$ .  $\square$

Finally, we mention that many delicate constructions are performed on Bernoulli shifts due to the concrete, combinatorial nature of these actions. By our theorem, these constructed objects can be pulled back to actions of positive entropy. We mention one explicit example here that we think is quite intriguing. The *measurable von Neumann–Day conjecture* asserts that for every aperiodic p.m.p. action  $G \curvearrowright (X, \mu)$  of a non-amenable group  $G$ , there is a p.m.p. free action  $F_2 \curvearrowright (X, \mu)$ , where  $F_2$  is the rank 2 free group, such that almost-every  $F_2$ -orbit is contained in a  $G$ -orbit. This conjecture was raised by Gaboriau in [17, p. 24, Question 5.16]. In [18] Gaboriau and Lyons proved that every non-amenable group admits a (free) Bernoulli shift action satisfying the conjecture, and in [7] Bowen proves that all (free) Bernoulli shifts over non-amenable groups satisfy the conjecture. For non-free actions, Bowen, Hoff, and Ioana have proved that for every non-amenable group  $G$  and every IRS  $\theta$  supported on infinite-index subgroups,  $G$  admits a type- $\theta$  Bernoulli shift satisfying conjecture [8].

**Corollary 10.5.** *Let  $G$  be non-amenable and let  $G \curvearrowright (X, \mu)$  be an aperiodic p.m.p. action. Assume either: (i) the action is free and  $h_G(X, \mu) > 0$ , or (ii)  $h_G(X, \mu) = \infty$ . Then  $G \curvearrowright (X, \mu)$  satisfies the measurable von Neumann–Day conjecture.*

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