FORCING CONSTRUCTIONS AND COUNTABLE BOREL EQUIVALENCE RELATIONS

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Abstract. We prove a number of results about countable Borel equivalence relations with forcing constructions and arguments. These results reveal hidden regularity properties of Borel complete sections on certain orbits. As consequences they imply the nonexistence of Borel complete sections with certain features.

1. Introduction

This paper is a contribution to the study of countable Borel equivalence relations. We consider Borel actions of countable groups on Polish spaces and study the orbit equivalence relations which they generate. Properties such as hyperfiniteness, treeability, chromatic numbers, matchings, etc. have received much interest both in ergodic theory and descriptive set theory. Typically, investigations into these properties begin with the construction of Borel complete sections possessing special properties. In this paper we introduce new methods based on forcing techniques for studying Borel complete sections. We use forcing constructions to prove the existence of certain regularity phenomena in complete sections. This of course prevents the existence of complete sections with certain features. We remark that our work here is entirely in the Borel setting, as our results generally fail if null sets are ignored.

Recall that a set S is a *complete section* for an equivalence relation E if S meets every E-class. A classic result on complete sections is the Slaman–Steel lemma which states that every aperiodic countable Borel equivalence relation E admits a decreasing sequence of Borel complete sections S_n with empty intersection (this result is stated explicitly as Lemma 6.7 of [4], where they attribute it to Slaman-Steel; the proof is implicit in Lemma 1 of [13]). This result played an important role in their proof that every equivalence relation generated by a Borel action of $\mathbb{Z}$ is hyperfinite. A long standing open problem asks if every equivalence relation generated by a Borel action of a countable amenable group must be hyperfinite, and progress on this problem is in some ways connected to strengthening the Slaman–Steel lemma. In particular, constructing sequences of complete sections ("marker sets") with certain geometric properties is central to the proofs of [6, 12] that every equivalence relation generated by the Borel action of an abelian, or even locally nilpotent, group is hyperfinite. In particular, the constructions in [6, 12] build complete sections B_n (facial boundaries) which are sequentially orthogonal, or repel one another, so that the sequence $S_n = \bigcup_{i>n} B_i$ is decreasing and vanishes.

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Thus, the question of what kinds of marker sets various equivalence relations can admit is an important one.

Our first theorem unveils a curious property which limits how quickly a sequence of complete sections can vanish. In fact, this theorem says that if a sequence of complete sections vanishes, then it must do so arbitrarily slowly.

Theorem 1.1. *Let Γ be a countable group, X a compact Polish space, and $\Gamma \curvearrowright X$ a continuous action giving rise to the orbit equivalence relation E. Let $(S_n)_{n\in\mathbb{N}}$ be a sequence of Borel complete sections of E. If $(A_n)_{n\in\mathbb{N}}$ is any sequence of finite subsets of Γ such that every finite subset of Γ is contained in some A_n, then there is an $x \in X$ such that for infinitely many n we have $A_n \cdot x \cap S_n \neq \varnothing$.*

We remark that the above result is easily seen to be inherited from subspaces, so one can instead simply require that X contain a compact invariant subset. In particular, by results in [7, 8] the above result holds when $X = F(2^\Gamma)$, where $F(2^\Gamma)$ is the set of points in 2^Γ having trivial stabilizer.

This theorem was motivated by a similar result for the case when each S_n is clopen, the proof of which is a straightforward topological argument without forcing. We will define a forcing notion, called *orbit forcing*, that will allow us to give a proof of Theorem 1.1 that is essentially a generalization of the topological proof. It will turn out that forcing can be removed and a pure topological proof is possible (in fact we will give such a proof), but the forcing proof is shorter and more intuitive.

We remark that, after learning of the above theorem, C. Conley and A. Marks obtained another interesting result on the behavior of distances to sequences of complete sections [1].

The orbit forcing can be used to obtain more results, the following being an example.

Theorem 1.2. *If $B \subseteq F(2^\Gamma)$ is a Borel complete section then B meets some orbit recurrently, i.e., there is $x \in F(2^\Gamma)$ and finite $T \subseteq \Gamma$ such that for any $y \in [x]$, $T \cdot y \cap B \neq \varnothing$.*

Again, if B is assumed to be clopen then the result follows from the fact that minimal elements form a dense set in $F(2^\Gamma)$ [8, Theorem 5.3.6]. We find that the most direct way to obtain this "Borel result" is to mimic the topological proof but use forcing.

The above result can be strengthened in various ways. For example, in the case of $\Gamma = \mathbb{Z}^d$ we can require that the recurrences occur at odd distances (distance here refers to the taxi-cab metric induced by the ℓ_1 norm $\|(g_1, \ldots, g_d)\| = |g_1| + \cdots + |g_d|$).

Theorem 1.3. *Let $d \geq 1$. If $B \subseteq F(2^{\mathbb{Z}^d})$ is a Borel complete section then B meets some orbit recurrently with odd distances, i.e., there is $x \in F(2^{\mathbb{Z}^d})$ and finite $T \subseteq \{g \in \mathbb{Z}^d \colon \|g\| \text{ is odd }\}$ such that for any $y \in [x]$, $T \cdot y \cap B \neq \varnothing$.*

It is worth noting that Theorem 1.3, and in fact all of the forcing results in this paper, can be proved using the orbit forcing method. However, we believe that forcing arguments in general may provide a new path for studying countable Borel equivalence relations. Thus, in order to demonstrate the flexibility of forcing arguments in this setting, we define and use other forcing notions beyond the orbit forcing. We choose to prove the above theorem by using a notion of an odd minimal 2-coloring forcing.

Another forcing notion we introduce is that of a grid periodicity forcing. Using this forcing, we obtain the following result which reveals a surprising amount of regularity in complete sections.

Theorem 1.4. *Let $d \geq 1$. If $B \subseteq F(2^{\mathbb{Z}^d})$ is a Borel complete section then there is an $x \in F(2^{\mathbb{Z}^d})$ and a lattice $L \subseteq \mathbb{Z}^d$ such that $L \cdot x \subseteq B$.*

If $B \subseteq F(2^{\mathbb{Z}^d})$ is a Borel set but not a complete section, then there is x with $\mathbb{Z}^d \cdot x \cap B = \varnothing$. Thus we have the following immediate corollary.

Corollary 1.5. *Let $d \geq 1$. If $B \subseteq F(2^{\mathbb{Z}^d})$ is Borel then there is an $x \in F(2^{\mathbb{Z}^d})$ and a lattice $L \subseteq \mathbb{Z}^d$ such that either $L \cdot x \subseteq B$ or $L \cdot x \cap B = \varnothing$.*

A. Marks [9] has proved a similar result for free groups using Borel determinacy. Also, after discussing Theorem 1.4 with him, he generalized Theorem 1.4 to all countable residually finite groups [11]. His proof also uses forcing, though it uses none of the forcing notions we introduce here. This again suggests that the flexibility in choosing a forcing notion may be important for future applications to Borel equivalence relations.

The above results can be viewed as ruling out certain Borel complete sections (marker sets) with strong regularity properties. Alternatively, they can be viewed as saying that Borel marker sets must, on some equivalence classes, exhibit regular structure. In general, regular marker sets and structures are desirable in hyperfiniteness proofs or Borel combinatorial results (e.g., in the study of Borel chromatic numbers). The negative results stated below unveil a fine line between what is possible and what is not possible.

In [6], the first two authors proved that all equivalence relations generated by Borel actions of countable abelian groups are hyperfinite (this has since been extended to locally nilpotent groups [12]). For the finite equivalence relations they construct, the shapes of the classes at a sufficiently large scale look like rectangles. However, at finer and finer scales the shapes appear to be increasingly fractal-like. We use forcing to prove a claim made in [6] stating that this fractal-like behavior is necessary. This fact indicates that hyperfiniteness results of this type have a necessary degree of complexity. The theorem below is stated for rectangles but the proof works for any reasonable polygon.

Theorem 1.6. *There does not exist a sequence $\mathcal{R}_n$ of Borel finite subequivalence relations on $F(2^{\mathbb{Z}^2})$ satisfying all the following:*

(1) (regular shape) *For each n, each marker region R of $\mathcal{R}_n$ is a rectangle.*
(2) (bounded size) *For each n, there is an upper bound $w(n)$ on the size of the edge lengths of the marker regions R in $\mathcal{R}_n$.*
(3) (increasing size) *Letting $v(n)$ denote the smallest edge length of a marker region R of $\mathcal{R}_n$, we have $\lim_n v(n) = +\infty$.*
(4) (vanishing boundary) *For each $x \in F(2^{\mathbb{Z}^2})$ we have that $\lim_n \rho(x, \partial\mathcal{R}_n) = +\infty$.*

Our last two negative results touch upon the theory of Borel chromatic numbers. It is not difficult to show that $F(2^{\mathbb{Z}^2})$ has Borel chromatic number strictly greater than 2. By using the odd minimal 2-coloring forcing, we show that in fact there cannot exist any Borel chromatic coloring of $F(2^{\mathbb{Z}^2})$ which uses two colors on arbitrarily large regions.

Theorem 1.7. *There does not exist a Borel chromatic coloring* $f\colon F(2^{\mathbb{Z}^2}) \to \{0, 1, \ldots, k\}$ *such that for all* $x \in F(2^{\mathbb{Z}^2})$ *there are arbitrarily large rectangles* R *in* $\mathbb{Z}^2$ *such that* $f(R \cdot x)$ *consists of only two elements of* $\{0, 1, \ldots, k\}$.

A useful structure for the study of Borel graphs and chromatic numbers is the notion of *toast* or "barrier" as named in [2]. For example, in [2] C. Conley and B. Miller used barriers to prove that for a large class of Borel graphs G, the Baire-measurable and μ-measurable chromatic numbers of G are at most twice the standard chromatic number of G minus one. In a similar fashion, the existence of a toast structure on $F(2^{\mathbb{Z}^2})$ would easily imply the existence of a Borel chromatic 3-coloring. As a consequence of Theorem 1.1, we deduce that there is no toast structure which is layered.

Corollary 1.8. *There is no Borel layered toast on* $F(2^{\mathbb{Z}^d})$*, i.e., there is no sequence of finite subequivalence relations* $\{T_n\}$ *of* $E_{\mathbb{Z}^d}$ *on some subsets* $\mathrm{dom}(T_n) \subseteq F(2^{\mathbb{Z}^d})$ *such that*

(0) $\bigcup_n \mathrm{dom}(T_n) = F(2^{\mathbb{Z}^d})$;
(1) *For each* T_n*-equivalence class* C*, and each* T_m*-equivalence class* C' *where* $m > n$*, if* $C \cap C' \neq \varnothing$ *then* $C \subseteq C'$*; and*
(2) *For each* T_n*-equivalence class* C *there is a* T_{n+1}*-equivalence class* C' *such that* $C \subseteq C' \setminus \partial C'$.

We mention that unlayered toast (defined in Section 4) does exist and thus $F(2^{\mathbb{Z}^2})$ does have Borel chromatic number 3. This result will appear in an upcoming paper.

It is our opinion that this is only the beginning of nontrivial results about countable Borel equivalence relations that can be proved using forcing. It is curious to note the tension that the first five "positive" results all state the existence of points with certain regularity properties, whereas the last three "negative" results state the nonexistence of regular structure on orbits. Of course, the positive results are all obtained by generically building such elements in the generic extension (and then asserting their existence in the ground model by absoluteness), and it is known that the results do not hold for comeager or conull sets of reals. Thus what we are using is some method that goes beyond the usual measure and category arguments.

One of the central notions of the theory of countable Borel equivalence relations is that of Borel reducibility, which is entirely missing in the narrative of this paper, but is in fact an important motivation. All known methods to prove nonreducibility results for countable Borel equivalence relations have been measure-theoretic (it is well known that category arguments would not work). But measure-theoretic arguments have their limitations. There have been persistent attempts by researchers to invent new methods that are not measure-theoretic. For instance, recent work of S. Thomas [14] and A. Marks [10] explore the use of Martin's ultrafilter and its generalizations as a largeness notion (see also [9] for other recent uses of determinacy in the study of Borel equivalence relations). The forcing methods presented in this paper can also be viewed as an attempt in this direction.

2. Preliminaries

In this section we present some preliminaries that will be used throughout the rest of the paper. More background terminology and results will be recalled as needed in subsequent sections.

2.1. **Countable Borel equivalence relations and group actions.** In this paper we will be concerned mainly with countable Borel equivalence relations. Let X be a Polish space and E an equivalence relation on X. E is *Borel* if it is a Borel subset of $X \times X$. E is *countable* if each E-equivalence class is countable. For $x \in X$, we let $[x]_E$ denote the E-equivalence class of x, i.e.,

$$[x]_E = \{y \in X \,:\, xEy\}.$$

When there is no ambiguity we will omit the subscript and only write $[x]$.

Countable Borel equivalence relations typically arise from orbit equivalence relations of countable group actions. If Γ is a countable discrete group and $\Gamma \curvearrowright X$ is a Borel action of Γ on a Polish space X, then the *orbit equivalence relation* E^X_Γ defined by

$$E^X_\Gamma = \{(x, y) \in X \times X \,:\, \exists g \in \Gamma\, (g \cdot x = y)\}$$

is obviously a countable Borel equivalence relation. Conversely, by a well-known theorem of Feldman–Moore, every countable Borel equivalence relation is of the form E^X_Γ for some Borel action $\Gamma \curvearrowright X$ of a countable group Γ. For this reason, whenever we speak of a countable Borel equivalence relation E we assume that there has been fixed a Borel action of a countable group $\Gamma \curvearrowright X$ so that $E = E^X_\Gamma$. For any $x \in X$, note that $[x] = \Gamma \cdot x$; we also refer to $[x]$ as the *orbit* of x.

A particularly important case for this paper is the action of $\Gamma = \mathbb{Z}^d$ on $X = 2^{\mathbb{Z}^d}$. In this case we write $E_{\mathbb{Z}^d}$ for the orbit equivalence relation E^X_Γ. We will frequently restrict the action to the free part $F(2^{\mathbb{Z}^d})$ (defined formally below), and we will also write $E_{\mathbb{Z}^d}$ for the restriction of the orbit equivalence relation to the free part (which is also a Polish space). The precise meaning will be clear from the context.

If $\Gamma \curvearrowright X$ and $\Gamma \curvearrowright Y$ are two actions of Γ on Polish spaces X and Y, respectively, a Γ-*map*, or an *equivariant map*, from X to Y is a map $\varphi : X \to Y$ such that for all $g \in \Gamma$ and $x \in X$,

$$\varphi(g \cdot x) = g \cdot \varphi(x).$$

If in addition φ is injective, it will be called a Γ-*embedding* or an *equivariant embedding*.

For a countable group Γ, the *Bernoulli shift* of Γ is the action $\Gamma \curvearrowright 2^\Gamma$ defined by

$$(g \cdot x)(h) = x(g^{-1}h)$$

for $x \in 2^\Gamma$ and $g, h \in \Gamma$. A closely related action $\Gamma \curvearrowright 2^{\Gamma\times\omega}$ is defined by

$$(g \cdot x)(h, n) = x(g^{-1}h, n)$$

for $x \in 2^{\Gamma\times\omega}$, $g, h \in \Gamma$, and $n \in \omega$. A theorem of Becker–Kechris states that this latter action is a *universal* Borel Γ-action. That is, for any Borel action $\Gamma \curvearrowright X$ of Γ on a Polish space X, there is a Borel Γ-embedding from X into $2^{\Gamma\times\omega}$. In view of this, any Γ-action on a Polish space X is Borel isomorphic to the action of Γ restricted to an invariant Borel subset of $2^{\Gamma\times\omega}$.

2.2. **Aperiodicity, hyperaperiodicity, and minimality.** Let Γ be a countable group, X a Polish space, and $\Gamma \curvearrowright X$ a Borel action. An element $x \in X$ is *aperiodic* if for any nonidentity $g \in \Gamma$, $g \cdot x \neq x$. The set of all aperiodic elements of X is called the *free part* of X, and is denoted as $F(X)$. When $F(X) = X$ we say that the action is *free*.

An element $x \in X$ is *hyperaperiodic* if the closure of its orbit is contained in the free part of X, i.e., $\overline{[x]} \subseteq F(X)$. A hyperaperiodic element of 2^Γ is sometimes also called a 2-*coloring* on Γ for the following reason.

Lemma 2.1 ([7])**.** *A point $x \in 2^\Gamma$ is hyperaperiodic if and only if for all $e_\Gamma \neq s \in \Gamma$, there is a finite $T \subseteq \Gamma$ such that*

$$\forall g \in \Gamma \ \exists t \in T \quad x(gst) \neq x(gt).$$

The above combinatorial property emphasizes x as a function assigning two colors $0, 1$ to elements of Γ in a way that for any shift s, the pair of elements g and gs might not have different colors, but it only takes a "small" perturbation t to take the pair to a new pair gt and gst with different colors. When the underlying space is 2^Γ, we will use the two terms hyperaperiodic and 2-coloring interchangeably.

Unfortunately, in this paper we will also consider graph colorings in the usual sense that adjacent vertices have different colors. If k many colors are used, we will refer to such colorings as *graph k-colorings* or *chromatic k-colorings*.

The action $\Gamma \curvearrowright X$ is *minimal* if every orbit is dense, i.e., $\overline{[x]} = X$ for every $x \in X$. In general, we call an element $x \in X$ *minimal* if the induced action of Γ on $\overline{[x]}$ is minimal. When X is a compact Polish space and the action is continuous, an application of Zorn's lemma shows that there always exist minimal elements. In fact, when X is compact Polish (or even compact with a wellordered base) we can prove this in ZF (i.e., we don't need any form of AC to prove this).

Fact 2.2 (ZF)**.** Let X be a compact T_2 topological space with a wellordered base $\{U_\eta\}_{\eta<\lambda}$. Let Γ be a group and $(g, x) \mapsto g \cdot x \in X$ a continuous action of Γ on X. Then there is an $x \in X$ which is minimal.

If the action of Γ is only Borel, then the same conclusion holds if X is compact and Polish [8, Lemma 2.4.4].

Proof. Let $K_0 = X$. We define by transfinite recursion on the ordinals a sequence of non-empty compact sets $K_\alpha \subseteq X$ which are decreasing, in that if $\alpha < \beta$ then $K_\beta \subseteq K_\alpha$, and also invariant, in that if $x \in K_\alpha$ then $[x] = \{g \cdot x : g \in \Gamma\} \subseteq K_\alpha$. For α limit we set $K_\alpha = \bigcap_{\beta<\alpha} K_\beta$. For the successor case, suppose K_α is defined, and is non-empty, compact, and invariant. If K_α is minimal, we are done. Otherwise there is a least $\eta < \lambda$ such that $U_\eta \cap K_\alpha \neq \emptyset$ and $K_\alpha - \mathrm{sat}(U_\eta) \neq \emptyset$ (here $\mathrm{sat}(U) = \{y : \exists x \in U \exists g \in \Gamma \ y = g \cdot x\}$ is the saturation of U under the equivalence relation on X generated by Γ). This exists since we are assuming there is a non-empty, compact (hence closed), invariant $K \subsetneq K_\alpha$. Let $K_{\alpha+1} = K_\alpha - \mathrm{sat}(U_\eta)$. Since the action of Γ is continuous, $\mathrm{sat}(U_\eta)$ is open, so $K_{\alpha+1}$ is non-empty and compact. It is also invariant (the difference of two invariant sets), and properly contained in K_α. As the sets K_α are decreasing, the sequence must terminate in some K_α which is minimal. $\square$

Corollary 2.3 (ZF)**.** *A continuous action of a Polish group Γ on a compact Polish space X has a minimal element.*

In the case of 2^Γ, minimality is captured by the following combinatorial condition. We will use the following fact repeatedly.

Lemma 2.4. *A point $x \in 2^\Gamma$ is minimal if and only if for every finite $A \subseteq \Gamma$ there is a finite $T \subseteq \Gamma$ such that*

$$\forall g \in \Gamma \ \exists t \in T \ \forall a \in A \quad x(gta) = x(a).$$

Proof. This is well known and follows from a simple compactness argument. A proof can be found, for example, in [8]. □

It was proved in [8] that minimal 2-colorings exist on every countable group Γ.

2.3. **Borel complete sections, Borel graphs, and geometry on orbits.** For a Polish space X with a countable Borel equivalence relation E, a *complete section* S is a subset of X that meets every orbit of X, i.e., for any $x \in X$, $S \cap [x] \neq \varnothing$. A useful fact in the theory of countable Borel equivalence relations is the theorem of Slaman–Steel that for any countable Borel equivalence relation E with only infinite equivalence classes, there is a decreasing sequence of Borel complete sections

$$S_0 \supseteq S_1 \supseteq S_2 \supseteq \cdots\cdots$$

such that $\bigcap_n S_n = \varnothing$ [13].

A particularly interesting collection of examples is given by the Bernoulli shifts of $\mathbb{Z}^d$ on $F(2^{\mathbb{Z}^d})$ for $d \geq 1$. The Borel complete sections given by the Slaman-Steel theorem lead to a quick proof that the orbit equivalence relation in the case $d = 1$ is *hyperfinite*. Recall that a countable equivalence relation on a standard Borel space X is hyperfinite if there is a an increasing sequence of Borel equivalence relations

$$R_0 \subseteq R_1 \subseteq R_2 \subseteq \cdots\cdots$$

on X with all R_n-equivalence classes finite, such that $E = \bigcup_n R_n$. Weiss later showed that this holds for all d (from which it follows that any Borel action of $\mathbb{Z}^d$ is hyperfinite). In these examples we rely on the geometric structure of the Cayley graph of the group to understand the behavior of the orbit equivalence relation. Complete sections are frequently built to posses properties of geometric significance and for this reason are informally called sets of *markers*.

The following notions and terminology are tools to study the geometric structures. Fix $d \geq 1$. For an element $g = (g_1, \ldots, g_d) \in \mathbb{Z}^d$, let

$$\|g\| = \sum_{i=1}^{d} |g_i|.$$

The metric induced by this norm is often called the *taxi-cab metric*. If $x, y \in F(2^{\mathbb{Z}^d})$ are in the same orbit, then there is a unique $g_{x,y} \in \mathbb{Z}^d$ with $g_{x,y} \cdot x = y$, and we define $\rho(x, y) = \|g_{x,y}\|$. If $y \notin [x]$, we just let $\rho(x, y) = +\infty$. This ρ is thus a distance function that is a metric on each orbit.

For $A \subseteq F(2^{\mathbb{Z}^d})$ we also define $\rho(x, A) = \min\{\rho(x, y) : y \in A\}$. If A is a complete section, then $\rho(x, A) < +\infty$ for any x. If $\{S_n\}$ is a decreasing sequence of Borel complete sections with $\bigcap_n S_n = \varnothing$ as in the Slaman–Steel theorem, then for any $x \in F(2^{\mathbb{Z}^d})$, we have $\lim_n \rho(x, S_n) = +\infty$. In fact, the function $\varphi_x(n) = \rho(x, S_n)$ is a monotone increasing function diverging to infinity for each x. Our first theorem in this paper, which is presented in the next section, implies that this function grows arbitrarily slowly. More precisely, given any function $f(n)$ with $\limsup_n f(n) = +\infty$, we show that there is x with $\varphi_x(n) < f(n)$ for infinitely many values of n.

In general, for any finitely generated group Γ with a finite symmetric generating set S (meaning $\gamma^{-1} \in S$ for every $\gamma \in S$), a number of standard objects can be associated with the space 2^Γ. First, there is the *Cayley graph* $C_S(\Gamma) = (\Gamma, D)$ with Γ as the vertex set and with the edge relation D defined by

$$(g, h) \in D \iff \exists \gamma \in S \ (g = \gamma h).$$

The Cayley graph induces a Borel graph $C_S(2^\Gamma) = (2^\Gamma, \tilde{D})$ on 2^Γ, where the edge relation $\tilde{D}$ is defined as

$$(x, y) \in \tilde{D} \iff \exists \gamma \in S \ (x = \gamma \cdot y).$$

The geodesics in $C_S(2^\Gamma)$ give a distance function ρ_Γ, i.e., $\rho_\Gamma(x, y)$ is the length of the shortest path from x to y in $C_S(2^\Gamma)$. Note that the distance function ρ defined above for $2^{\mathbb{Z}^d}$ is an example of the more general ρ_Γ, with S being the set of standard generators for $2^{\mathbb{Z}^d}$.

If $A \subseteq 2^\Gamma$, the *boundary* of A, denoted ∂A, is the set

$$\partial A = \{x \in A : \exists \gamma \in S \ \gamma \cdot x \notin A\}.$$

3. Orbit Forcing

We start with a very general forcing construction in which one generically builds an element in a Polish space with a countable Borel equivalence relation.

Definition 3.1. Let E be a countable Borel equivalence relation on a Polish space X, and let $x \in X$. The *orbit forcing* $\mathbb{P}_x = \mathbb{P}_x^E$ is defined by

$$\mathbb{P}_x = \{U \subseteq X : U \text{ is open and } U \cap [x]_E \neq \varnothing\}$$

with its elements ordered by inclusion, that is, $U \leq U'$ iff $U \subseteq U'$.

Since $U \cap \overline{[x]} \neq \varnothing$ iff $U \cap [x] \neq \varnothing$, we can view the sets $U \cap \overline{[x]}$ as the objects in the forcing notion, in which case the forcing notion $\mathbb{P}_x$ can simply be viewed as ordinary Cohen forcing on the closed subspace $Y = \overline{[x]}$ of X. Thus, as with usual Cohen forcing, we can regard forcing arguments using $\mathbb{P}_x$ as category arguments on the space $\overline{[x]}$. Nevertheless, we will see that the forcing proofs can be more intuitive than category arguments.

As remarked in the previous section, we can view the countable Borel equivalence relation E as coming from a Borel action of a countable group Γ, and view the space X as a certain invariant Borel subset of $2^{\Gamma \times \omega}$. Now if G is $\mathbb{P}_x$-generic over V, the space $X^{V[G]}$ continues to be a standard Borel space and $E^{V[G]}$ continues to be a countable Borel equivalence relation. Moreover, the generic G can be identified with an element $x_G \in X^{V[G]}$. With a slight abuse of terminology we will refer to x_G as a generic element of X for the orbit forcing $\mathbb{P}_x$. Note that we always have that $x_G \in \overline{[x]_E}$, where both the orbit and the closure are computed in $V[G]$.

As a first application of the orbit-forcing we present a general result about sequences of complete sections in equivalence relations generated by continuous actions of countable groups on compact spaces. This includes the case of Bernoulli shift actions on 2^Γ and $F(2^\Gamma)$ since there exist compact invariant sets $X \subseteq F(2^\Gamma)$ [7, 8].

Theorem 3.2. *Let Γ be a countable group, X a compact Polish space, and $\Gamma \curvearrowright X$ a continuous action giving rise to the orbit equivalence relation E. Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of finite subsets of Γ such that every finite subset of Γ is contained in some A_n. Let $(S_n)_{n\in\mathbb{N}}$ be a sequence of Borel complete sections of E. Then there is an $x \in X$ such that for infinitely many n we have $A_n \cdot x \cap S_n \neq \varnothing$.*

Proof. Since X is compact, we may fix an $x \in X$ which is minimal. Let $\mathbb{P} = \mathbb{P}_x$ be the the corresponding orbit forcing. Let κ be a large enough regular cardinal and $Z \prec V_\kappa$ a countable elementary substructure containing Γ and all the real parameters used in the definitions of X (as a Borel subspace of $2^{\Gamma\times\omega}$), the action $\Gamma \curvearrowright X$, and the sequences $\{A_n\}$ and $\{S_n\}$. Let $\pi\colon M \to Z$ be the inverse of the transitive collapse. Let $\pi(\mathbb{P}') = \mathbb{P}$. Fix an arbitrary G which is $\mathbb{P}'$-generic over M, and let x_G be the unique element of $\bigcap G$. Let $\dot{x}_G$ be the canonical $\mathbb{P}'$-name for x_G. Note that for any $\gamma \in \Gamma$, $\gamma \cdot G$ is also $\mathbb{P}'$-generic over M, and we have $M[G] = M[\gamma \cdot G]$ and $x_{\gamma\cdot G} = \gamma \cdot x_G$.

Working in V, we will define a sequence U_n of conditions in $\mathbb{P}'$ such that the filter H generated by $\{U_n\}$ is $\mathbb{P}'$-generic over M, and an increasing sequence i_n of integers such that

$$U_n \Vdash (A_{i_n} \cdot \dot{x}_G \cap S_{i_n} \neq \varnothing).$$

To see this suffices, let x_H be the unique element of $\bigcap_n U_n$. By assumption, x_H is $\mathbb{P}'$-generic over M. From the forcing theorem, we have in $M[H]$ that $A_{i_n} \cdot x_H \cap S_{i_n} \neq \varnothing$. Therefore, the statement $\exists y \forall n\, (A_{i_n} \cdot y \cap S_{i_n} \neq \varnothing)$ is true in $M[H]$. Since this is a $\boldsymbol{\Sigma}^1_1$ statement, by absoluteness it is also true in V, which gives the desired result.

Let $D_0, D_1, \dots$ enumerate the dense subsets of $\mathbb{P}'$ which lie in M. To begin, let i_0 be such that the identity $e_\Gamma \in A_{i_0}$. The statement that S_{i_0} is a Borel complete section is $\boldsymbol{\Pi}^1_1$, and hence by absoluteness it continues to be a Borel complete section in $M[G]$. Fix any $y \in [x_G] \cap S_{i_0}$. There is a certain $\gamma \in \Gamma$ such that $y = \gamma \cdot x_G$. Since y is also $\mathbb{P}'$-generic over M, there is a $U_0' \in \mathbb{P}'$ such that $U_0' \Vdash (\dot{x}_G \in S_{i_0})$. This gives $U_0' \Vdash (A_{i_0} \cdot \dot{x}_G \cap S_{i_0} \neq \varnothing)$. Let $U_0 \subseteq U_0'$ be in $\mathbb{P}' \cap D_0$.

In general, suppose $U_n \in \mathbb{P}'$ and i_n are given so that $U_n \Vdash (A_{i_n} \cdot \dot{x}_G \cap S_{i_n} \neq \varnothing)$. By minimality of x, there is a finite $T \subseteq \Gamma$ such that for all $z \in \overline{[x]}$ there is a $t \in T$ with $t \cdot z \in U_n$. Fix such a T. The statement $\forall z \in \overline{[x]}\ (T \cdot z \cap U_n \neq \varnothing)$ is $\boldsymbol{\Pi}^1_1$ and continues to be true in $M[G]$. Since $x_G \in \overline{[x]}$ in $M[G]$, we have that for any $z \in [x_G]$, $T \cdot z \cap U_n \neq \varnothing$. Let i_{n+1} be such that $T^{-1} \subseteq A_{i_{n+1}}$. Without loss of generality we may assume that $i_{n+1} > i_n$. Now fix any $y \in [x_G] \cap S_{i_{n+1}}$. Fix $t \in T$ such that $t \cdot y \in U_n$. So, $t^{-1} \cdot (t \cdot y) \in S_{i_{n+1}}$. As $t \cdot y$ is generic and $t^{-1} \in A_{i_{n+1}}$ we have that there is a $U_{n+1}' \subseteq U_n$ in $\mathbb{P}'$ such that $U_{n+1}' \Vdash (A_{i_{n+1}} \cdot \dot{x}_G \in S_{i_{n+1}})$. Let $U_{n+1} \subseteq U_{n+1}'$ be in $\mathbb{P}' \cap D_{n+1}$. This completes the construction of the U_n and finishes the proof of the theorem. □

We have the following immediate corollary concerning complete sections in $F(2^{\mathbb{Z}^n})$.

Corollary 3.3. *Let $f\colon \mathbb{N} \to \mathbb{N}$ be such that $\limsup_n f(n) = +\infty$. Let $\{S_n\}$ be a sequence of Borel complete sections of $F(2^{\mathbb{Z}^d})$. Then there is an $x \in F(2^{\mathbb{Z}^d})$ such that for infinitely many n we have $\rho(x, S_n) < f(n)$.*

Proof. Let x be any 2-coloring (or hyperaperiodic element) in $2^{\mathbb{Z}^d}$. Then $X = \overline{[x]}$ is a compact invariant subspace. Apply Theorem 3.2 to X with $A_n = \{\gamma \in \mathbb{Z}^d : \|\gamma\| < f(n)\}$. □

Remark 3.4. (1) The proof of Theorem 3.2 still works if each S_n is just assumed to be absolutely $\boldsymbol{\Delta}^1_2$, instead of Borel. By this we mean there are $\boldsymbol{\Sigma}^1_2$ and $\boldsymbol{\Pi}^1_2$ statements φ and ψ respectively which define S_n, and such that φ, ψ continue to define complimentary sets in all forcing extension $V[G]$ of V. There are two applications of absoluteness regarding the S_n in the proof of Theorem 3.2. In the first application (getting the S_n to be complete sections in $M[G]$), the S_n needs to be $\mathbf{\Pi}^1_2$, and in the second application (lifting the property of y from $M[H]$ to V), it needs to be $\mathbf{\Sigma}^1_2$.

(2) The proof of Theorem 3.2 shows that we may weaken the hypothesis of Corollary 3.3 that the S_n are complete sections to the statement that for each $x \in F(2^{\mathbb{Z}^d})$ and for each n there is an $m \geq n$ such that $S_m \cap [x] \neq \varnothing$. However, we need to assume now that $\liminf_n f(n) = +\infty$.

As we mentioned above, a forcing argument using $\mathbb{P}_x$ is essentially the same as a category argument on the subspace $\overline{[x]}$ of the Polish space X. In particular, the above proof can be given in purely topological terms. We feel it is worth presenting this alternative argument explicitly. We first recall a concept and record a simple lemma.

Definition 3.5. Consider a Borel action of a countable group Γ on a Polish space X. A point $x \in X$ is *recurrent* if for every open set $U \subseteq X$ with $U \cap [x] \neq \varnothing$, there is a finite $T \subseteq \Gamma$ so that for all $y \in [x]$ there is $t \in T$ with $t \cdot y \in U$.

Lemma 3.6. *Let Γ be a countable group, X a compact Polish space, and $\Gamma \curvearrowright X$ a continuous, minimal action. Then every $x \in X$ is recurrent.*

Proof. Fix a point $x \in X$ and a non-empty open set $U \subseteq X$. Enumerate Γ as $\gamma_1, \gamma_2, \ldots$ and set $T_n = \{\gamma_i : 1 \leq i \leq n\}$. Towards a contradiction, suppose that for every n there is $x_n \in [x]$ with $T_n \cdot x_n \cap U = \varnothing$. Since X is compact, there is an accumulation point y of the sequence x_n. Now for any $i \in \mathbb{N}$ we have $\gamma_i \cdot x_n \notin U$ for every $n \geq i$. Since Γ acts continuously and U is open, it follows that $\gamma_i \cdot y \notin U$. Thus the orbit of y does not meet U and hence is not dense, a contradiction to the minimality of the action. $\square$

To give a topological proof of Theorem 3.2, we will make use of the strong Choquet game. Let us recall this game. The Strong Choquet game on a topological space X consists of two players (I and II) who play in alternating turns. On each of player I's turns, player I plays a pair (U, x) consisting of an open set U and a point $x \in U$. Player II plays an open set on each of her turns. A play of the game is illustrated below.

$$\begin{array}{lccccccccc} \mathrm{I} & (U_0, x_0) & & (U_1, x_1) & & \cdots & & \cdots & \\ \mathrm{II} & & V_0 & & V_1 & & \cdots & & \cdots \end{array}$$

The game requires that $U_{n+1} \subseteq V_n$ and $x_n \in V_n \subseteq U_n$ for all n. The first player breaking these rules loses. Player II wins the game if and only if $\bigcap_n U_n = \bigcap_n V_n \neq \varnothing$. A space is called *strong Choquet* if player II has a winning strategy for this game. It is an easy fact that every completely metrizable space is strong Choquet (cf., e.g., [5, §4.1]).

We now present the alternative argument for Theorem 3.2. By considering $\overline{[x]}$ where $x \in X$ is minimal, we may assume that Γ acts continuously and minimally on X. As Borel sets in a Polish space have the Baire property, we can find a Γ-invariant dense G_δ set $X' \subseteq X$ such that each set $S_n \cap X'$ is relatively open in X'. By Lemma 3.6, every $x \in X$ is recurrent and it is easy to see that this implies that every $x \in X'$ is recurrent with respect to the relative topology of X'. We will consider the strong Choquet game on X'. Since X' is G_δ in X, X' is completely metrizable, and thus player II has a winning strategy. Fix a winning strategy τ for player II. From this point forward we will only work with X', not X. Let i_0 be such that the identity $e_\Gamma \in A_{i_0}$. To begin the game, fix any $x_0 \in S_{i_0} \cap X' \subseteq A_{i_0}^{-1} \cdot S_{i_0} \cap X'$ and have player I play the open set $U_0 = A_{i_0}^{-1} \cdot S_{i_0} \cap X'$ and the point x_0. Such an x_0 exists since X' is Γ-invariant and S_{i_0} is a complete section. Let $V_0 \subseteq X'$ be the open set played by player II according to τ. As x_0 is recurrent, there is a finite $T_0 \subseteq \Gamma$ so that for all $y \in [x_0]$ we have $T_0 \cdot y \cap V_0 \neq \varnothing$. The largeness assumption on the A_n's implies that we can find $i_1 > i_0$ with $T_0^{-1} \subseteq A_{i_1}$.

The game now proceeds inductively. Assume that player I has played the pairs $(U_0, x_0), (U_1, x_1), \ldots, (U_n, x_n)$, player II has played the open sets $V_0, V_1, \ldots, V_n$ according to her winning strategy τ, and that i_{n+1} has been defined and the following two conditions are satisfied:

(i) for every $y \in [x_n]$ we have $A_{i_{n+1}}^{-1} \cdot y \cap V_n \neq \varnothing$;
(ii) for all $y \in V_n$ and all $1 \leq m \leq n$ we have $A_{i_m} \cdot y \cap S_{i_m} \neq \varnothing$.

Fix a point $x'_{n+1} \in S_{i_{n+1}} \cap [x_n]$. Such x'_{n+1} exists since $S_{i_{n+1}}$ is a complete section. By clause (i) we may pick a point $x_{n+1} \in A_{i_{n+1}}^{-1} \cdot x'_{n+1} \cap V_n$. Have player I play the point x_{n+1} and the open set

$$U_{n+1} = V_n \cap A_{i_{n+1}}^{-1} \cdot S_{i_{n+1}}.$$

Note that for every $y \in U_{n+1}$ and every $1 \leq m \leq n+1$ we have $A_{i_m} \cdot y \cap S_{i_m} \neq \varnothing$. Let V_{n+1} be the open set played by player II according to τ. Then $x_{n+1} \in V_{n+1}$ and by recurrence there is a finite $T_{n+1} \subseteq \Gamma$ so that for all $y \in [x_{n+1}]$ we have $T_{n+1} \cdot y \cap V_{n+1} \neq \varnothing$. Now we can find a number $i_{n+2} > i_{n+1}$ with $T_{n+1}^{-1} \subseteq A_{i_{n+2}}$. This completes the induction.

Since player II has followed τ, we have that there is $x \in \bigcap_n V_n$. Thus it follows from clause (ii) that for every $m \in \mathbb{N}$ we have $A_{i_m} \cdot x \cap S_{i_m} \neq \varnothing$. This completes the alternative proof of Theorem 3.2. □

As a consequence of the methods of Theorem 3.2 we get the following result on the existence of recurrent points in the range of factor maps.

Theorem 3.7. *Let Γ be a countable group, X a compact Polish space, $\Gamma \curvearrowright X$ a continuous action , Y a Polish space, and $\Gamma \curvearrowright Y$ a Borel action. Let $\varphi\colon X \to Y$ be a Borel equivariant map. Then there is an $x \in X$ such that $\varphi(x)$ is a recurrent point of Y.*

Proof. Let $x \in X$ be minimal, and let $\mathbb{P} = \mathbb{P}_x$ and $\mathbb{P}'$ be as in the proof of Theorem 3.2 (along with M, π, etc., with codes for X, Y and φ in M). Let G be $\mathbb{P}'_x$-generic over M. Note that $x_G \in \overline{[x]}$, and so x_G is also a minimal point. We claim that $\varphi(x_G)$ is recurrent. Let $V_0, V_1, \ldots$ enumerate the basic open sets of Y which intersect $[\varphi(x_G)]$. Let $S_n = \varphi^{-1}(V_n)$, so S_n is a Borel subset of X. For each n, let $\gamma_n \in \Gamma$ and $U_n \in \mathbb{P}'_x$ be such that $U_n \Vdash (\gamma_n \cdot \dot{x}_G \in S_n)$. Since x_G is in the open set U_n, and since x_G is minimal, it follows that there is a finite $T_n \subseteq \Gamma$ such that for any $y = \gamma \cdot x_G \in [x_G]$ there is a $t \in T_n$ such that $t \cdot y \in U_n$. Since $t \cdot y$ is also generic, if follows that $\gamma_n \cdot (t \cdot y) \in S_n$. By equivariance of φ we have $(\gamma_n t \gamma) \cdot \varphi(x_G) \in V_n$. So, for all $\gamma \in \Gamma$ there exists $h \in \gamma_n T_n$ such that $h \cdot (\gamma \cdot \varphi(x_G)) \in V_n$. This shows that $\varphi(x_G)$ is recurrent. □

Corollary 3.8. *For any countable group Γ, any Borel action $\Gamma \curvearrowright Y$ of Γ on a Polish space Y, and any Borel equivariant map $\varphi\colon F(2^\Gamma) \to Y$, there is an $x \in F(2^\Gamma)$ such that $\varphi(x)$ is recurrent.*

Proof. By [7, 8], there is an invariant compact set $X \subseteq F(2^\Gamma)$. Now apply Theorem 3.7 to X. $\square$

Corollary 3.9. *Let τ be a Polish topology on $F(2^\Gamma)$ having the same Borel sets as the standard topology. Then there is a τ-recurrent point.*

Corollary 3.10. *If $B \subseteq F(2^\Gamma)$ is a Borel complete section then B meets some orbit recurrently, i.e., there is $x \in F(2^\Gamma)$ and finite $T \subseteq \Gamma$ such that for any $y \in [x]$, $T \cdot y \cap B \neq \varnothing$.*

Proof. Let τ be a Polish topology on $F(2^\Gamma)$ with $B \in \tau$ and having the same Borel sets as the standard topology (cf., e.g., [5, §4.2]). Apply Corollary 3.9. $\square$

Corollary 3.10 rules out the existence of Borel complete sections with certain geometric properties. The following is an example.

Corollary 3.11. *There does not exist $B \subseteq F(2^{\mathbb{Z}^2})$ with the following properties:*

(i) *Both B and $F(2^{\mathbb{Z}^2}) \setminus B$ are Borel complete sections, and*
(ii) *For any $x \in B$ and $(\gamma, \eta) \in \mathbb{N}^2$, $(\gamma, \eta) \cdot x \in B$.*

Proof. Assume toward a contradiction that B satisfies both (i) and (ii). By Corollary 3.10 there is $x \in F(2^{\mathbb{Z}^2})$ and finite $T \subseteq \mathbb{Z}^2$ such that for any $y \in [x]$, $T \cdot y \cap B \neq \varnothing$. Let $z \in [x] \setminus B$. Such a z exists since $F(2^{\mathbb{Z}^2}) \setminus B$ is also a complete section. By (ii), for any $(\gamma, \eta) \in \mathbb{N}^2$, $(-\gamma, -\eta) \cdot z \notin B$. Now let $n > \|t\|$ for all $t \in T$, and consider $y = (-n, -n) \cdot z$. The property of z implies that $T \cdot y \cap B = \varnothing$, a contradiction. $\square$

We note that it is also possible to give a non-forcing proof of Corollary 3.11. Namely, if such a B existed, then the subequivalence relation of $F(2^{\mathbb{Z}^2})$ generated by the group element $g = (1, 1)$ would be smooth (that is, have a Borel selector). However, a simple category argument shows that there cannot be such a Borel selector for this relation.

In all forcing arguments in the rest of this paper we will skip the metamathematical details we presented in the proofs of Theorems 3.2 and 3.7. Specifically, instead of using the countable elementary substructure M and the forcing version $\mathbb{P}'$ in M (in the case of Cohen forcing we in fact have $\mathbb{P}' = \mathbb{P}$), we will just use $\mathbb{P}$ and pretend that we can find a generic x for $\mathbb{P}$ over V. In reality, we should take x to be M-generic for $\mathbb{P}'$ and then use absoluteness between $M[x]$ and V as we did in Theorem 3.2. Since these details do not vary in the arguments, we shall henceforth omit them.

4. Borel Layered Toast

In this short section we present another application of Corollary 3.3 on the non-existence of certain types of *strong marker structure* on $F(2^{\mathbb{Z}^d})$. The name "toast" for the type of structure defined below was coined by B. Miller. We will define two versions of this notion, the general or "unlayered" toast structure, and the more restrictive notion of "layered" toast. These are both strong types of marker structures to impose on the orbits of $F(2^{\mathbb{Z}^d})$. We can consider both the Borel as well as the clopen versions of these notions, which leads to four separate questions concerning the existence of these structures. It turns out that a Borel unlayered toast structure does exist, but the answers are no for all the other existence questions. We present the proof for the nonexistence of Borel layered toast here; the other results will be presented in a forthcoming paper.

We note that the notion of toast arose naturally through its connections with interesting problems in Borel combinatorics. For example, the existence of Borel unlayered toast, which will be proved in an upcoming paper, gives a proof that there is a Borel chromatic 3-coloring of $F(2^{\mathbb{Z}^d})$ (and so $F(2^{\mathbb{Z}^d})$ has Borel chromatic number 3). These toast structures have been constructed modulo meager sets and modulo μ-null sets by C. Conley and B. Miller and used to bound the Baire-measurable and μ-measurable chromatic numbers of many Borel graphs [2].

First we make precise the notion of a toast marker structure.

Definition 4.1. Let $\{T_n\}$ be a sequence of subequivalence relations of $E_{\mathbb{Z}^d}$ on some subsets $\mathrm{dom}(T_n) \subseteq F(2^{\mathbb{Z}^d})$ with each T_n-equivalence class finite. Assume $\bigcup_n \mathrm{dom}(T_n) = F(2^{\mathbb{Z}^d})$. We say $\{T_n\}$ is a *(unlayered) toast* if:

(1) For each T_n-equivalence class C, and each T_m-equivalence class C' where $m > n$, if $C \cap C' \neq \varnothing$ then $C \subseteq C'$.
(2) For each T_n-equivalence class C there is $m > n$ and a T_m-equivalence class C' such that $C \subseteq C' \setminus \partial C'$.

We say $\{T_n\}$ is a *layered toast* if, instead of (2) above, we have

(2') For each T_n-equivalence class C there is a T_{n+1}-equivalence class C' such that $C \subseteq C' \setminus \partial C'$.

Figure 1 illustrates the definitions of layered and unlayered toast.

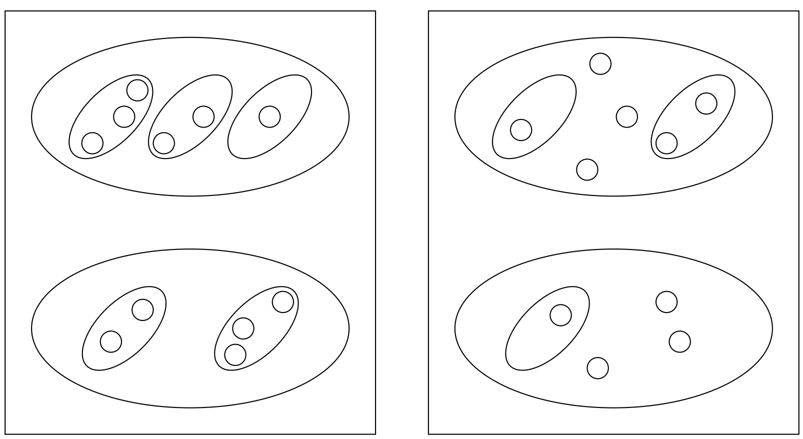

FIGURE 1. (a) layered toast (b) general toast

Theorem 4.2. *There is no Borel layered toast on $F(2^{\mathbb{Z}^d})$.*

Proof. Suppose $\{T_n\}$ was a sequence of Borel subequivalence relations of $E_{\mathbb{Z}^d}$ on some subsets $\mathrm{dom}(T_n) \subseteq F(2^{\mathbb{Z}^d})$ forming a layered toast structure on $F(2^{\mathbb{Z}^d})$. For each n let ∂T_n be the the union of all boundaries of the T_n-equivalence classes. For $x \in F(2^{\mathbb{Z}^d})$, let $f_x \colon \mathbb{N} \to \mathbb{N}$ be defined by $f_x(n) = \rho(x, \partial T_n)$ if $x \in \mathrm{dom}(T_n)$ and $f_x(n) = 0$ otherwise. This is well-defined as each T_n-equivalence class is finite.

Since $\{T_n\}$ is a layered toast, by (2') we have $\mathrm{dom}(T_n) \subseteq \mathrm{dom}(T_{n+1})$ for all n. For $x \in F(2^{\mathbb{Z}^d})$, let n_0 be large enough so that $x \in \mathrm{dom}(T_{n_0})$. We claim that for $n \geq n_0$ that $f_x(n) < f_x(n+1)$. To see this, let $n \geq n_0$, and let $a = f_x(n)$. Let $g \in \mathbb{Z}^n$ with $\|g\| \leq a$. It follows easily from the definitions of a and ∂T_n that $g \cdot x$ is T_n-equivalent to x (if we choose a path p from $\vec{0}$ to g of length a, then by an easy induction along the path we have that $g' \cdot x$ is T_n-equivalent to x for all g' in p). So, from property (2') we have that $g \cdot x \notin \partial T_{n+1}$. Thus, $\rho(x, \partial T_{n+1}) > a$. So, for all $x \in F(2^{\mathbb{Z}^d})$ and all sufficiently large n (which may depend on x) we have $f_x(n) < f_x(n+1)$.

If we let $f \colon \mathbb{N} \to \mathbb{N}$ be the function $f(n) = \sqrt{n}$, then for all $x \in F(2^{\mathbb{Z}^n})$ we have that for all but finitely many $n \in \mathbb{N}$ that $\rho(x, \partial T_n) > f(n)$. This violates Corollary 3.3. □

5. Cohen Forcing and Bounded Geometry of Marker Regions

In this section we use forcing to prove a nonexistence theorem for marker regions in $F(2^{\mathbb{Z}^2})$ that are of regular shape. A version of this theorem was stated without proof as Theorem 3.5 of [6]. We will have to recall a good amount of terminology and results from [6]. But the forcing used is going to be just Cohen forcing on a countable group Γ.

Given a countable group Γ and $k \leq \omega$, the Cohen forcing $\mathbb{P}_\Gamma(k)$ is defined by

$$\mathbb{P}_\Gamma(k) = \{p \in k^{\mathrm{dom}(p)} \,:\, \mathrm{dom}(p) \subseteq \Gamma \text{ is finite}\}$$

with the order of inverse inclusion, that is, $p \leq q$ iff $p \supseteq q$.

If G is $\mathbb{P}_\Gamma(k)$-generic over V, and $x_G = \bigcup G$, then $x_G \in F(k^\Gamma)$. This is because, for any $\gamma \in \Gamma$, the set

$$D_\gamma = \{p \in \mathbb{P}_\Gamma(k) \,:\, \exists g \in \mathrm{dom}(p) \cap \gamma^{-1} \cdot \mathrm{dom}(p) \ [\, p(g) \neq p(\gamma \cdot g)\,]\}$$

is dense in $\mathbb{P}_\Gamma(k)$. Thus the generic real is always an aperiodic element of k^Γ.

We recall some facts about the orthogonal marker construction of [6]. The construction was done on $F(2^{\mathbb{Z}^d})$ for any $d \geq 1$. Here, we focus on $d = 2$ for simplicity. We continue to use $E_{\mathbb{Z}^2}$ to denote the orbit equivalence relation on $F(2^{\mathbb{Z}^2})$ given by the Bernoulli shift action of $\mathbb{Z}^2$. By a *finite subequivalence relation* on $F(2^{\mathbb{Z}^2})$ we mean an equivalence relation $\mathcal{R} \subseteq E_{\mathbb{Z}^2}$ with all equivalence classes finite. If R is an equivalence class of $\mathcal{R}$ and $x \in R$, then we consider the finite subset of $\mathbb{Z}^2$ defined by

$$S_x = \{(\gamma, \eta) \in \mathbb{Z}^2 \ \ (\gamma, \eta) \cdot x \in R\}.$$

We can speak of the *shape* of S_x, e.g., S_x is a *rectangle* if it is of the form $[a, b] \times [c, d] \subseteq \mathbb{Z}^2$. It is obvious that the shape of S_x does not depend on the choice of x, since these sets are translates of each other with different choices of reference points. Thus we often abuse the terminology and just speak of the *shape* of an equivalence class R.

In the orthogonal marker construction one produces a sequence $\mathcal{R}_n$ of relatively clopen finite subequivalence relations on $F(2^{\mathbb{Z}^2})$. Here the term *relatively clopen* means that for every $g \in \mathbb{Z}^2$, the set of $x \in F(2^{\mathbb{Z}^2})$ with $x\mathcal{R}_n g \cdot x$ is clopen. The $\mathcal{R}_n$ equivalence classes are also called *marker regions*. There is a *scale* $d_n \in \mathbb{N}$ associated to each $\mathcal{R}_n$, in the sense that each equivalence class R of $\mathcal{R}_n$ restricted to some orbit $[x]$ is *roughly a rectangle on the scale* d_n, that is, there is a rectangle R' such that the ρ-Hausdorff distance between R and R' is less than αd_{n-1} for some fixed constant $0 < \alpha < 1$. Here we assume that $d_{n-1} \lll d_n$ for all n. In the construction of [6], each $R \in \mathcal{R}_n$ is obtained from a true rectangle R' by modifying the boundary in $n-1$ stages. At stage k the adjustments are on the order of scale d_k. Thus, the boundaries of the $R \in \mathcal{R}_n$ become increasingly fractal-like as n increases. The key property the marker regions of $\mathcal{R}_n$ have is that for any $x \in F(2^{\mathbb{Z}^2})$, we have that $\rho(x, \partial\mathcal{R}_n) \to \infty$, where $\partial\mathcal{R}_n$ denotes the union of the boundaries of the regions R in $\mathcal{R}_n$. This construction, which results in marker regions with the above *vanishing boundary property*, is the main ingredient of the hyperfiniteness proof of [6].

Theorem 3.2 gives us some additional information about this construction. Namely, for $x \in F(2^{\mathbb{Z}^2})$, let $\varphi_x \colon \mathbb{N} \to \mathbb{N}$ be given by $\varphi_x(n) = \rho(x, \partial\mathcal{R}_n)$. The orthogonal marker construction says that each φ_x tends to infinity with n whereas Theorem 3.2 says that the growth rate of the φ_x can be arbitrarily slow. More precisely, given any $f \colon \mathbb{N} \to \mathbb{N}$ with $\limsup_n f(n) = +\infty$, there is an $x \in F(2^{\mathbb{Z}^2})$ such that $\varphi_x(n) < f(n)$ for infinitely many n. Thus, we cannot improve the orthogonal marker theorem by prescribing a growth rate for the functions φ_x, not even if we seek Borel, instead of clopen, finite subequivalence relations.

It is natural also to ask whether the fractal-like nature of the $\mathcal{R}_n$ is also necessary. Could we have a sequence $\mathcal{R}_n$ of Borel finite subequivalence relations, with vanishing boundary, where the regions $R \in \mathcal{R}_n$ have a regular geometry? For instance, can we have all marker regions in $\mathcal{R}_n$ to be rectangles with edge lengths between $v(n)$ and $w(n)$ where $\lim_n v(n) = \infty$. The next result shows that this potential improvement to the orthogonal marker construction is also impossible. For simplicity we state the result only for rectangles, but the proof works for reasonably regular polygons.

Theorem 5.1. *There does not exist a sequence $\mathcal{R}_n$ of Borel finite subequivalence relations on $F(2^{\mathbb{Z}^2})$ satisfying all the following:*

(1) (regular shape) *For each n, each marker region R of $\mathcal{R}_n$ is a rectangle.*
(2) (bounded size) *For each n, there is an upper bound $w(n)$ on the size of the edge lengths of the marker regions R in $\mathcal{R}_n$.*
(3) (increasing size) *Letting $v(n)$ denote the smallest edge length of a marker region R of $\mathcal{R}_n$, we have* $\lim_n v(n) = +\infty$.
(4) (vanishing boundary) *For each $x \in F(2^{\mathbb{Z}^2})$ we have that* $\lim_n \rho(x, \partial\mathcal{R}_n) = +\infty$.

Proof. Assume $\mathcal{R}_n$ were Borel finite subequivalence relations with all the stated properties. We view conditions p for our Cohen forcing $\mathbb{P} = \mathbb{P}_{\mathbb{Z}^2}(2)$ as being partial functions $p \colon [a,b] \times [c,d] \to \{0,1\}$ for some rectangle $A = [a,b] \times [c,d]$ in $\mathbb{Z}^2$. We will produce an $x \in F(2^{\mathbb{Z}^2})$ which will lie on the boundary of a rectangle R of $\mathcal{R}_n$ for infinitely many n. This will contradict property (4). Given $p \in \mathbb{P}$ and $k \in \mathbb{N}$, we will produce a $q \leq p$ and an $n > k$ such that $q \Vdash (\dot{x}_G \in \partial\mathcal{R}_n)$. Any $\mathbb{P}$-generic x will then be as desired.

So, fix $p \in \mathbb{P}$ and $k \in \mathbb{N}$. We may assume that $A = \mathrm{dom}(p) = [-a,a] \times [a,a]$. By (3) we may choose $n > k$ large enough so that the minimum edge length $v(n)$ of any R of $\mathcal{R}_n$ is greater than $2(2a+1)^2$. Let $w = w(n)$ be the maximum edge length of any R of $\mathcal{R}_n$, which is well-defined by (2). Let $b > 3w$, and let $B = [0,b] \times [0,b] \subseteq \mathbb{Z}^2$. Let $r \in \mathbb{P}$ be the condition with domain B obtained by restricting to B the following function r':

$$r'(i,j) = p(i'-a, j'-a),$$

where

$$i' = i \mod (2a+1)$$
$$j' = j + \frac{i-i'}{2a+1} \mod (2a+1).$$

The function r' is obtained by tiling $\mathbb{Z}^2$ with copies of p as follows. First put a copy of p at $[0,2a]\times[0,2a]$ and vertically stack copies of p to tile the column $[0,2a]\times\mathbb{Z}$. Then, on the vertical column $[2a+1,4a+1]\times\mathbb{Z}$ immediately to the right we shift this stack down by one. We continue this, so on the vertical stack which is c columns to the right, that is, on $[c(2a+1), c(2a+1)+2a]\times\mathbb{Z}$, we shift down by c mod $(2a+1)$. This is illustrated in Figure 2.

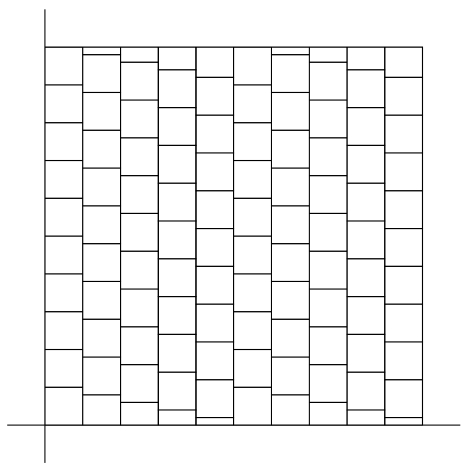

FIGURE 2. The construction of the condition r.

Let x_G be a $\mathbb{P}$-generic real extending r. Let $\mathcal{R}_n(x_G)$ denote the set of rectangles in $\mathbb{Z}^2$ induced by the subequivalence relation $\mathcal{R}_n$ on the class $[x_G]$. Since $b>3w$, there is a rectangle R in $\mathcal{R}_n(x_G)$ lying entirely in $B=[0,b]\times[0,b]$. Let e be a horizontal edge of R. Then e also lies entirely in B. Since e has length greater than $2(2a+1)^2$, the offsets of the columns in the definition of r gives that there is a translate $A' = (i_0,j_0)+([-a,a]\times[-a,a])$ of $\mathrm{dom}(p)$ such that $r\restriction A' = p$ and such that e passes through the center point (i_0,j_0) of A'. By genericity, there is a condition $s\le r$ in G such that

$$s \Vdash (\exists R\in\mathcal{R}_n(\dot{x}_G)\ (i_0,j_0)\in\partial R)\wedge(\dot{x}_G\restriction A' = p)$$

Let π be the automorphism of $\mathbb{P}$ obtained by translating by $(-i_0,-j_0)$. Let $q=\pi(s)$. Then

$$q \Vdash (\exists R\in\mathcal{R}_n(\pi(\dot{x}_G))\ (i_0,j_0)\in\partial R)\wedge(\pi(\dot{x}_G)\restriction A' = p)$$

From the invariance of the $\mathcal{R}_n$ this gives

$$q \Vdash (\exists R \in \mathcal{R}_n(\dot{x}_G)\ (0,0) \in \partial R) \wedge (\dot{x}_G \upharpoonright A = p)$$

So, $q \leq p$ and $q \Vdash \dot{x}_G \in \partial\mathcal{R}_n$. ◻

6. Minimal 2-coloring Forcing on $F(2^{\mathbb{Z}^d})$

For any countable group Γ, there is an $x \in F(2^\Gamma)$ which is a minimal 2-coloring (cf. [7], [8]). With the corresponding orbit forcing $\mathbb{P}_x$, a generic real x_G continues to be a minimal 2-coloring in the generic extension $V[G]$. On the other hand, it is possible to directly define a forcing notion which generically adds minimal 2-colorings in 2^Γ. The advantage of this approach is that we will be able to take advantage of some extra properties of the generic reals that are not obviously available from a general 2-coloring x without other features. In this section we will describe such a forcing notion, which we call *minimal 2-coloring forcing*, and some variations of it, and use these forcing notions to prove some new results about Borel complete sections.

It is possible to define the minimal 2-coloring forcing for a general group Γ, but such a definition is cumbersome to describe, as it necessarily embodies the construction of a 2-coloring, which is not easy. For the case $\Gamma = \mathbb{Z}^d$, this forcing has a simple and natural definition. Since our remaining applications concern $F(2^{\mathbb{Z}^d})$, we will present the definition of the minimal 2-coloring forcing in this case. It will be clear that a part of the definition is designed for producing a generic 2-coloring, and another part for producing a generic minimal element. In applications, we sometimes need just the 2-coloring property of the generic real, sometimes we need just the minimality, and frequently we need some additional properties which requires us to modify the forcing.

We begin with a description of the basic forcing $\mathbb{P}_{mt}$ for adding a minimal 2-coloring in $2^{\mathbb{Z}^d}$. Since the definition is essentially the same for all $\mathbb{Z}^d$ for $d \geq 1$, to ease notation we consider the case $d = 2$.

Definition 6.1. The *basic minimal* 2*-coloring forcing* $\mathbb{P}_{mt}$ on $\mathbb{Z}^2$ consists of conditions

$$\mathfrak{p} = (p, n, t_1, \ldots, t_n, T_1, \ldots, T_n, m, f_1, \ldots, f_m, F_1, \ldots, F_m)$$

where

(1) $p \in 2^{<\mathbb{Z}^2}$ with $\mathrm{dom}(p) = [a,b] \times [c,d]$ for some $a < b$, $c < d$, $a,b,c,d \in \mathbb{Z}$;
(2) $n, m \in \mathbb{N}$;
(3) $t_1, \ldots, t_n \in \mathbb{Z}^2 - \{(0,0)\}$;
(4) $f_1, \ldots, f_m \in 2^{<\mathbb{Z}^2}$;
(5) $T_1, \ldots, T_n$, $F_1, \ldots, F_m$ are finite subsets of $\mathbb{Z}^2$;

such that the following conditions are satisfied:

(a) (2-coloring property) For any $1 \leq i \leq n$ and $g \in \mathrm{dom}(p)$ there is $\tau \in T_i$ such that $g + \tau, g + t_i + \tau \in \mathrm{dom}(p)$ and $p(g + \tau) \neq p(g + t_i + \tau)$;
(b1) (minimality) For any $1 \leq j \leq m$ and $g \in \mathrm{dom}(p)$ there is $\sigma \in F_j$ such that $g + \sigma + \mathrm{dom}(f_j) \subseteq \mathrm{dom}(p)$ and for all $u \in \mathrm{dom}(f_j)$, $p(g + \sigma + u) = f_j(u)$.
(b2) (minimality with flips) For any $1 \leq j \leq m$ and $g \in \mathrm{dom}(p)$ there is $\sigma \in F_j$ such that $g + \sigma + \mathrm{dom}(f_j) \subseteq \mathrm{dom}(p)$ and for all $u \in \mathrm{dom}(f_j)$, $p(g + \sigma + u) = 1 - f_j(u)$.

When we need to speak of another forcing condition $\mathfrak{q}$ we will denote the main part of the forcing condition, namely the partial function, as q, and the rest of the terms in the tuple as $n(\mathfrak{q}), \vec{t}(\mathfrak{q}), \vec{T}(\mathfrak{q}), m(\mathfrak{q}), \vec{f}(\mathfrak{q})$, and $\vec{F}(\mathfrak{q})$, respectively.

If $\mathfrak{p}, \mathfrak{q} \in \mathbb{P}$, we define the extension relation $\mathfrak{p} \leq \mathfrak{q}$ by

(i) $p \supseteq q$,
(ii) $n(\mathfrak{p}) \geq n(\mathfrak{q})$,
(iii) for all $1 \leq i \leq n(\mathfrak{q})$, $t_i(\mathfrak{p}) = t_i(\mathfrak{q})$ and $T_i(\mathfrak{p}) = T_i(\mathfrak{q})$,
(iv) $m(\mathfrak{p}) \geq m(\mathfrak{q})$,
(v) for all $1 \leq j \leq m(\mathfrak{q})$, $f_j(\mathfrak{p}) = f_j(\mathfrak{q})$ and $F_j(\mathfrak{p}) = F_j(\mathfrak{q})$.

For $h \in \mathbb{Z}^2$ and $p \in 2^{<\mathbb{Z}^2}$, we define $h \cdot p$ by letting $\mathrm{dom}(h \cdot p) = h + \mathrm{dom}(p)$ and for all $g \in \mathrm{dom}(p)$, $(h \cdot p)(h + g) = p(g)$. For

$$\mathfrak{p} = (p, n, t_1, \ldots, t_n, T_1, \ldots, T_n, m, f_1, \ldots, f_m, F_1, \ldots, F_m),$$

define

$$h \cdot \mathfrak{p} = (h \cdot p, n, t_1, \ldots, t_n, T_1, \ldots, T_n, m, f_1, \ldots, f_m, F_1, \ldots, F_m).$$

The invariance of the forcing notion is easy to check.

We now prove a few simple lemmas which guarantee that $\mathbb{P}_{mt}$ does indeed add a minimal 2-coloring in $2^{\mathbb{Z}^2}$.

Lemma 6.2. *For any $g \in \mathbb{Z}^2$, the set $D_g = \{\mathfrak{p} \in \mathbb{P}_{mt} \colon g \in \mathrm{dom}(p)\}$ is dense in $\mathbb{P}_{mt}$.*

Proof. Let $\mathfrak{q} \in \mathbb{P}_{mt}$ and $g \in \mathbb{Z}^2$. We need to find $\mathfrak{p} \leq \mathfrak{q}$ with $g \in \mathrm{dom}(p)$. Intuitively, we use q as a "building block" and construct p by a "tiling" of q until the domain of p covers the element g. More precisely, suppose $\mathrm{dom}(q) = [a, b] \times [c, d]$. If $g = (g_1, g_2) \in [a, b] \times [c, d]$ we just take $\mathfrak{p} = \mathfrak{q}$. Without loss of generality assume $g_1 > b$, $g_2 > d$ (the other cases being similar). Let $w = b - a + 1$, $h = d - c + 1$. Let

$$i_0 = \left\lfloor \frac{g_1 - a}{w} \right\rfloor, \quad j_0 = \left\lfloor \frac{g_2 - c}{h} \right\rfloor.$$

We define p with $\mathrm{dom}(p) = [a, b+i_0 w] \times [c, d+j_0 h]$ by letting $p(a+iw+i', c+jh+j') = q(a+i', c+j')$ for all $0 \leq i \leq i_0$, $0 \leq j \leq j_0$, $0 \leq i' < w$, and $0 \leq j' < h$.

Then define $n(\mathfrak{p}) = n(\mathfrak{q})$, $\vec{t}(\mathfrak{p}) = \vec{t}(\mathfrak{q})$, $\vec{T}(\mathfrak{p}) = \vec{T}(\mathfrak{q})$, $m(\mathfrak{p}) = m(\mathfrak{q})$, $\vec{f}(\mathfrak{p}) = \vec{f}(\mathfrak{q})$, and $\vec{F}(\mathfrak{p}) = \vec{F}(\mathfrak{q})$.

We need to verify that $\mathfrak{p} \in \mathbb{P}_{mt}$. It suffices to verify the conditions (a), (b1) and (b2). For (a) let $0 \leq i \leq n(\mathfrak{p})$ and let $h = (h_1, h_2) \in \mathrm{dom}(p)$. Let $k = (k_1, k_2) \in [a, b] \times [c, d]$ be the unique element such that $w \mid (h_1 - k_1)$, $h \mid (h_2 - k_2)$. Let $\tau \in T_i$ be such that $k + \tau, k + t_i + \tau \in [a, b] \times [c, d]$ and $q(k + \tau) \neq q(k + t_i + \tau)$. Then by commutativity we have $h + \tau, h + t_i + \tau \in \mathrm{dom}(p)$ and

$$p(h + \tau) = q(k + \tau) \neq q(k + t_i + \tau) = p(h + t_i + \tau).$$

For (b1) let $1 \leq j \leq m(\mathfrak{p})$ and $h \in \mathrm{dom}(p)$. Again let $k \in [a, b] \times [c, d]$ be the unique element such that $w \mid (h_1 - k_1)$, $h \mid (h_2 - k_2)$. Then there is $\sigma \in F_j$ such that $k + \sigma + \mathrm{dom}(f_j) \subseteq [a, b] \times [c, d]$ and for all $u \in \mathrm{dom}(f_j)$, $q(k + \sigma + u) = f_j(u)$. Now $h+\sigma+\mathrm{dom}(f_j) \subseteq \mathrm{dom}(p)$ and for all $u \in \mathrm{dom}(f_j)$, $p(h+\sigma+u) = q(k+\sigma+u) = f_j(u)$. This proves (b1). The proof of (b2) is similar. $\square$

Note that in the construction of p from q in the above proof, p is a "tiling" by q. Denote $\overline{q} = 1 - q$ and call it the *flip* of q. Then p could also be constructed as a "tiling" by both q and $\overline{q}$, using an arbitrary combination of these two kinds of "tiles". The use of the flip tile was not necessary in the above proof, but will be necessary in the following lemmas.

Lemma 6.3. *For any* $t \in \mathbb{Z}^2 - \{(0,0)\}$ *the set*

$$E_t = \{\mathfrak{p} \in \mathbb{P}_{mt} \colon \exists 1 \leq i \leq n(\mathfrak{p})\ t_i(\mathfrak{p}) = t\}$$

is dense in $\mathbb{P}_{mt}$.

Proof. Let $t = (t_1, t_2) \in \mathbb{Z}^2 - \{(0,0)\}$ and $\mathfrak{q} \in \mathbb{P}_{mt}$. We will find $\mathfrak{p} \leq \mathfrak{q}$ with $t = t_i(\mathfrak{p})$ for some $1 \leq i \leq n(\mathfrak{p})$. If $t = t_i(\mathfrak{q})$ for some $1 \leq i \leq n(\mathfrak{q})$ we just take $\mathfrak{p} = \mathfrak{q}$. Otherwise, we will define $\mathfrak{p}$ so that $n(\mathfrak{p}) = n(\mathfrak{q}) + 1$ and $t = t_{n(\mathfrak{p})}(\mathfrak{p})$. For notational simplicity let $n(\mathfrak{q}) = n - 1$ and assume $\mathrm{dom}(q) = [a,b] \times [c,d]$. Also, assume $t_1, t_2 > 0$ (the other cases being similar). Let $w = b-a+1$, $h = d-c+1$, and let $b+t_1 = a+i_0w+i_1$, $d+t_2 = c+j_0h+j_1$ where $0 \leq i_1 < w$, $0 \leq j_1 < h$. Note that at least one of i_0, j_0 is greater than 0. Define p with $\mathrm{dom}(p) = [a, b+i_0w] \times [c, d+j_0h]$ in a similar fashion as we did in the proof of the previous lemma. Specifically, let $p(a+iw+i', c+jh+j') = q(a+i', c+j')$ for all $0 \leq i \leq i_0$, $0 \leq j \leq j_0$, $0 \leq i' < w$, and $0 \leq j' < h$, except in the case $i = i_0$ and $j = j_0$. If $q(a+i_1, c+j_1) \neq q(b,d)$, then let $p(a+i_0w+i', c+j_0h+j') = q(a+i', c+j')$ for all $0 \leq i' < w$, $0 \leq j' < h$. Otherwise, if $q(a+i_1, c+j_1) = q(b,d)$, then let $p(a+i_0w+i', c+j_0h+j') = 1 - q(a+i', c+j')$ for all such i', j'. Intuitively, p is a tiling of q and $\overline{q}$, with only the last tile being possibly $\overline{q}$. The choice between q and $\overline{q}$ is made to ensure $p(b,d) \neq p((b,d)+t)$.

Let $T_n = [-i_0w, b-a] \times [-j_0h, d-c]$. Then $n(\mathfrak{p}) = n = n(\mathfrak{q}) + 1$, $\vec{t}(\mathfrak{p}) = \vec{t}(\mathfrak{q})^\frown t$, $\vec{T}(\mathfrak{p}) = \vec{T}(\mathfrak{q})^\frown T_n$, $m(\mathfrak{p}) = m(\mathfrak{q})$, $\vec{f}(\mathfrak{p}) = \vec{f}(\mathfrak{q})$, and $\vec{F}(\mathfrak{p}) = \vec{F}(\mathfrak{q})$.

The proof of (a) for all elements of $\vec{t}(\mathfrak{q})$ is the same as in the previous proof. We verify (a) only for t. Let $g \in \mathrm{dom}(p)$. Let $\tau = (b,d) - g$. Then $\tau \in T_n$, $g + \tau = (b,d)$ and $g + t + \tau = (b,d) + t$. Both $g + \tau, g + t + \tau \in \mathrm{dom}(p)$ and $p(g+\tau) = p(b,d) \neq p((b,d)+t) = p(g+t+\tau)$. Thus (a) holds.

We verify (b1) and (b2). For (b1) let $1 \leq j \leq m(\mathfrak{p})$ and $g = (g_1, g_2) \in \mathrm{dom}(p)$. If the building block containing g is a copy of q, then the proof is the same as in the proof of Lemma 6.2. So suppose that the block $(i \cdot w, j \cdot h) + [a,b] \times [c,d]$ containing g is a copy of $\overline{q}$ (by our construction, this necessitates $i = i_0$ and $j = j_0$). Let $k = (k_1, k_2) \in [a,b] \times [c,d]$ be the unique element such that $w \mid (g_1 - k_1)$, $h \mid (g_2 - k_2)$. Then by (b2) for $\mathfrak{q}$ there is $\sigma \in F_j$ such that $k + \sigma + \mathrm{dom}(f_j) \subseteq [a,b] \times [c,d]$ and for all $u \in \mathrm{dom}(f_j)$, $q(k+\sigma+u) = 1 - f_j(u)$. Now $g + \sigma + \mathrm{dom}(f_j) \subseteq \mathrm{dom}(p)$ and for all $u \in \mathrm{dom}(f_j)$, $p(g+\sigma+u) = 1 - q(k+\sigma+u) = f_j(u)$. This proves (b1) for $\mathfrak{p}$. The proof of (b2) is similar. □

Lemma 6.4. *For any finite set* $A \subseteq \mathbb{Z}^2$*, the set*

$$D_A = \{\mathfrak{p} \in \mathbb{P}_{mt} \colon \exists 1 \leq j \leq m(\mathfrak{p})\ A \subseteq \mathrm{dom}(f_j(\mathfrak{p}))\}$$

is dense in $\mathbb{P}_{mt}$.

Proof. Let $\mathfrak{q} \in \mathbb{P}$ and $A \subseteq \mathbb{Z}^2$ be finite. By repeated application of Lemma 6.2 we can obtain $\mathfrak{r} \leq \mathfrak{q}$ such that $A \subseteq \mathrm{dom}(r)$. Let $\mathrm{dom}(r) = [a,b] \times [c,d]$. We define p with $\mathrm{dom}(p) = [a, 2b-a+1] \times [c,d]$ to be a copy of r (on $[a,b] \times [c,d]$) and a copy of $\overline{r}$ immediately to the right (on $[b+1, 2b-a+1] \times [c,d]$).

Then define $n(\mathfrak{p}) = n(\mathfrak{r})$, $\vec{T}(\mathfrak{p}) = \vec{T}(\mathfrak{r})$, $m(\mathfrak{p}) = m(\mathfrak{r}) + 1$, $\vec{f}(\mathfrak{p}) = \vec{f}(\mathfrak{r})^\frown r$, and $\vec{F}(\mathfrak{p}) = \vec{F}(\mathfrak{r})^\frown([a-2b-1, b-2a+1] \times [-d, -c])$.

It suffices to verify that $\mathfrak{p} \in \mathbb{P}_{mt}$. (a) holds by a similar argument as in the proofs of previous two lemmas. For (b1) and (b2), the proof for $1 \le j \le m(\mathfrak{r})$ is similar to the proofs in the previous two lemmas. Finally, for $g = (g_1, g_2) \in \mathrm{dom}(p)$ there are obviously $\sigma, \sigma' \in [a-2b-1, b-2a+1] \times [-d, -c]$ such that $g + \sigma = (0,0)$ and $g + \sigma' = (b-a+1, 0)$. Then for all $u \in [a,b] \times [c,d]$, $p(g+\sigma+u) = r(u)$ and $p(g+\sigma'+u) = 1 - r(u)$. $\square$

In fact, the above proof gives the following lemma.

Lemma 6.5. *For any $\mathfrak{q} \in \mathbb{P}_{mt}$, the set*

$$D_q = \{\mathfrak{p} \in \mathbb{P}_{mt} \colon \exists 1 \le j \le m(\mathfrak{p})\ q \subseteq f_j(\mathfrak{p})\}$$

is dense below $\mathfrak{q}$ in $\mathbb{P}_{mt}$.

Proof. As in the previous proof, define p to be a tiling with one copy of q and a copy of $\overline{q}$ to the right. $\square$

Putting these lemmas together we have the following.

Lemma 6.6. *If x_G is generic for $\mathbb{P}_{mt}$, then x_G is a minimal 2-coloring.*

Proof. Lemma 6.2 gives that $x_G \in 2^{\mathbb{Z}^2}$. Lemma 6.3 gives that x_G is a 2-coloring. To see that x_G is minimal, let $A \subseteq \mathbb{Z}^2$ be finite, and let $f = x_G \restriction A$. Let $\mathfrak{q} \in G$ be such that $\mathrm{dom}(q) \supseteq A$. From Lemma 6.5 there is a $\mathfrak{p} \in G$ with $f \subseteq f_j$ for some $1 \le j \le m(\mathfrak{p})$. We then have that $F_j(\mathfrak{p})$ witnesses the minimality condition for A, that is, for all $g \in \mathbb{Z}^2$ there is a $t \in F_j(\mathfrak{p})$ such that $x_G(g+t+u) = f(u)$ for all $u \in \mathrm{dom}(f) = A$. $\square$

Using Lemma 6.6 and the proof of Theorem 3.2 we can get a direct proof of Corollary 3.3 which is self-contained and does not rely on the *a priori* construction of a minimal 2-coloring.

We next consider a relatively minor variation of $\mathbb{P}_{mt}$ which will turn out to have interesting consequences.

Definition 6.7. The *odd minimal* 2*-coloring forcing* $\mathbb{P}_{omt}$ is defined exactly as $\mathbb{P}_{mt}$ in Definition 6.1 except that we add the requirement that if $\mathrm{dom}(p) = [a,b] \times [c,d]$ then both $b-a+1$ and $d-c+1$ are odd. That is, the rectangle representing the domain of p must have odd numbers of vertices on each of the sides.

The following result intuitively states that any Borel complete section has an odd recurrence on some orbit.

Theorem 6.8. *Let $O = \{g \in \mathbb{Z}^2 : \|g\| \text{ is odd }\}$. If $B \subseteq F(2^{\mathbb{Z}^2})$ is a Borel complete section then there is $x \in F(2^{\mathbb{Z}^2})$ and finite $T \subseteq O$ such that for any $y \in [x]$, $T \cdot y \cap B \neq \varnothing$.*

Proof. Let x_G be a generic real for $\mathbb{P}_{omt}$. Since $B^{V[G]}$ continues to be a Borel complete section, there is $g_0 \in \mathbb{Z}^2$ such that $g_0 \cdot x_G \in B$. So there is $\mathfrak{p} \in G$ such that $\mathfrak{p} \Vdash g_0 \cdot \dot{x}_G \in B$. Now note that for any $\mathfrak{q} \le \mathfrak{p}$ there is an $\mathfrak{r} \le \mathfrak{q}$ such that r contains two disjoint copies of p at an odd distance apart. We can, in fact, get r by placing two copies of q next to each other, since $\mathrm{dom}(q)$ is a rectangle with an odd number of vertices on each side. This implies that there is a $\mathfrak{q} \in G$ with $\mathfrak{q} \le \mathfrak{p}$ and such that q contains two disjoint copies of p an odd distance apart. Let $\|q\|$ denote the sum of the side lengths of $\mathrm{dom}(q)$.

Since x_G is minimal, there is $N \in \mathbb{N}$ such that for all $g \in \mathbb{Z}^2$ there is a $\tau \in \mathbb{Z}^2$ with $\|\tau\| \leq N$ such that $\tau \cdot (g \cdot x_G) \in U_q$, where U_q is the basic open set in $2^{\mathbb{Z}^2}$ determined by q. Now let T be all the elements $g \in O$ with $\|g\| \leq N + \|q\| + \|g_0\|$. Fix any $y \in [x_G]$. Fix τ with $\|\tau\| \leq N$ such that $\tau \cdot y \in U_q$. In particular $\tau \cdot y \in U_p$. Let $h \in O$ be such that $\|h\| \leq \|q\|$ and $h \cdot (\tau \cdot y) \in U_p$. Since both $\tau \cdot y$ and $h \cdot (\tau \cdot y)$ are also generic and extend the condition $\mathfrak{p}$, we have that $g_0 \cdot (\tau \cdot y) \in B$ and $g_0 \cdot (h \cdot (\tau \cdot y)) \in B$. We have both $\|\tau + g_0\| \leq \|\tau\| + \|g_0\| \leq N + \|g_0\|$ and $\|\tau + h + g_0\| \leq N + \|q\| + \|g_0\|$. Since $h \in O$, one of $\tau + g_0$ and $\tau + h + g_0$ is an element of O, and therefore an element of T. Thus we have shown that $T \cdot y \cap B \neq \varnothing$ as required. □

The next theorem is about Borel chromatic k-colorings of $F(2^{\mathbb{Z}^2})$. Intuitively, it states that there does not exists a Borel chromatic k-coloring f of $F(2^{\mathbb{Z}^2})$ such that on every orbit there are arbitrarily large regions on which f induces a chromatic 2-coloring.

Theorem 6.9. *Suppose $f \colon F(2^{\mathbb{Z}^2}) \to \{0, 1, \ldots, k-1\}$ is a Borel function. Then there is an $x \in F(2^{\mathbb{Z}^2})$ and an $M \in \mathbb{N}$ such that if the map $t \mapsto f(t \cdot x)$ is a chromatic 2-coloring on $[a, b] \times [c, d]$, then $b - a, d - c \leq M$.*

Proof. Fix a Borel function $f \colon F(2^{\mathbb{Z}^2}) \to \{0, 1, \ldots, k-1\}$. Let x_G be a generic real for $\mathbb{P}_{omt}$. We claim that x_G is as required. Recall that x_G is a minimal 2-coloring. Suppose that on x_G the function f had arbitrarily large regions which were chromatic 2-colorings. Let $\mathfrak{p} \in G$ be such that $\mathfrak{p} \Vdash (f(\dot{x}_G) = i)$ for some fixed $i \in \{0, \ldots, k-1\}$. Let $\mathfrak{q} \in G$ with $\mathfrak{q} \leq \mathfrak{p}$ and such that q contains two disjoint copies of p an odd distance apart. Let again $\|q\|$ denote the sum of the side lengths of $\mathrm{dom}(q)$.

By minimality of x_G, let $N \in \mathbb{N}$ be such that for all $g \in \mathbb{Z}^2$, there is a $\tau \in \mathbb{Z}^2$ with $\|\tau\| \leq N$ such that $\tau \cdot (g \cdot x_G) \in U_q$. Since x_G is assumed to have arbitrarily large regions which are chromatically 2-colored by f, fix $\sigma \in \mathbb{Z}^2$ such that $\tau \mapsto f(\tau \cdot (\sigma \cdot x_G))$ is a chromatic 2-coloring of a square $[-a, a]^2$ in $\mathbb{Z}^2$ with $a > N + \|q\|$. Fix τ with $\|\tau\| \leq N$ such that $\tau \cdot (\sigma \cdot x_G) \in U_q$. In particular $\tau \cdot (\sigma \cdot x_G) \in U_p$. Let $h \in O$ be such that $\|h\| \leq \|q\|$ and $h \cdot (\tau \cdot (\sigma \cdot x_G)) \in U_p$. Now both $\tau \cdot (\sigma \cdot x_G)$ and $h \cdot (\tau \cdot (\sigma \cdot x_G))$ are generic and both extend the condition $\mathfrak{p}$. So, $f(\tau \cdot (\sigma \cdot x_G)) = i = f(h \cdot (\tau \cdot (\sigma \cdot x_G)))$. This is a contradiction as $\|\tau\| \leq N \leq a$ and $\|h + \tau\| \leq N + \|h\| \leq a$, and τ, $h + \tau$ are an odd distance apart in $\mathbb{Z}^2$. □

Remark 6.10. It is easy to construct Borel $f \colon F(2^{\mathbb{Z}^2}) \to \{0, 1\}$ such that for comeager many $x \in F(2^{\mathbb{Z}^2})$ we have that $t \mapsto f(t \cdot x)$ has arbitrarily large regions which are chromatically 2-colored. We can, in fact, take $f(x) = x(0, 0)$. Similarly, measure-one (in the natural product measure) many $x \in F(2^{\mathbb{Z}^2})$ have arbitrarily large regions which are chromatically 2-colored by f. This shows that ordinary category arguments (or, equivalently, Cohen forcing) and measure arguments are not sufficient to prove Theorem 6.9.

7. Grid Periodicity Forcing

In this last section we introduce another variation of the minimal 2-coloring forcing which (similar to Theorem 6.9) will show that Borel complete sections in $F(2^{\mathbb{Z}^d})$ must have orbits on which highly regular structure is exhibited. We will take $d = 2$ for the following arguments for simplicity, though the arguments in the general case are only notationally more complicated.

As before, we can describe our forcing either as a special case of orbit forcing (by first building a particular minimal 2-coloring x and then considering $\mathbb{P}_x$), or we can describe the forcing directly. Here we again describe the forcing directly.

Definition 7.1. Let n be a positive integer. The *grid periodicity forcing* $\mathbb{P}_{gp}(n)$ is defined as follows.

(1) A condition $p \in \mathbb{P}_{gp}(n)$ is a function

$$p \colon R \setminus \{u\} \to \{0, 1\}$$

where $R = [a, b] \times [c, d]$ is a rectangle in $\mathbb{Z}^2$ with $w = b - a + 1$, $h = d - c + 1$ both powers of n, and $u \in R$. We write $R(p)$, $w(p)$, $h(p)$, $u(p)$ for the corresponding objects and parameters.

(2) The conditions are ordered by $p \leq q$ iff
   (a) $R(p)$ is obtained by a rectangular tiling by $R(q)$, that is, $R(p)$ is the disjoint union $R(p) = \bigcup_{t \in A} t \cdot R(q)$ where A is a subset of $\mathbb{Z}^2$ of the form
   $$A = \{(iw(q), jh(q)) : \, i_0 \leq i \leq i_1, j_0 \leq j \leq j_1\}$$
   for some $i_0 \leq i_1, j_0 \leq j_1$;
   (b) If $c \in \mathrm{dom}(q)$ and $t \in A$, then $p(c + t) = q(c)$;
   (c) For some $t \in A$ we have $u(p) = u(q) + t$.

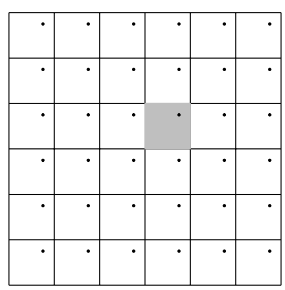

FIGURE 3. The extension relation in the grid periodicity forcing $\mathbb{P}_{gp}$.

When n is understood or does not matter, we will just write $\mathbb{P}_{gp}$ instead of $\mathbb{P}_{gp}(n)$. The conditions and extension relation for $\mathbb{P}_{gp}$ are illustrated in Figure 3. A $\mathbb{P}_{gp}$-generic G gives a real $x_G \in 2^{\mathbb{Z}^2}$. We next establish the properties of this real.

Lemma 7.2. *x_G is a minimal 2-coloring.*

Proof. Let $q \in \mathbb{P}_{gp}$ and let $s \in \mathbb{Z}^2 \setminus \{(0, 0)\}$. Let $p \leq q$ be defined as follows. First extend $R(q)$ to $R(p)$ by picking sufficiently large intervals $[i_0, i_1], [j_0, j_1]$ as in Definition 7.1(2a), and setting

$$A = \{(iw(q), jh(q)) : \, i_0 \leq i \leq i_1, j_0 \leq j \leq j_1\}$$

and $R(p) = \bigcup_{t \in A} t \cdot R(q)$, so that $u(q) + s \in R(p)$. For all $c \in R(q) \setminus \{u(q)\}$ and $t \in A$, we have $p(c + t) = q(c)$ by Definition 7.1 (2b). We then define p on the point $u(q)$, and possibly $u(q) + s$ if p is not already defined there, so that $p(u(q)) \neq p(u(q) + s)$.

We finally extend the domain so that p is a condition in $\mathbb{P}_{gp}$; in other words, we pick a point of the form $u(q)+t$ for some $t \in A$ to be the new point $u(p)$ at which p is undefined, and define p arbitrarily elsewhere. What we have just shown is the density of the set

$$D_s = \{p \in \mathbb{P}_{gp} : \exists g \in \mathrm{dom}(p)\ (g+s \in \mathrm{dom}(p) \text{ and } p(g) \neq p(g+s))\}.$$

Let $p \in D_s \cap G$ be arbitrary. Fix $g_0 \in \mathrm{dom}(p)$ so that $g_0 + s \in \mathrm{dom}(p)$ and $p(g_0) \neq p(g_0+s)$. Let $T = [-w(p), w(p)] \times [-h(p), h(p)]$. We claim that p forces the 2-coloring property for the shift s with witnessing set T. That is,

$$p \Vdash \forall g \in \mathbb{Z}^2\ \exists t \in T\ (\dot{x}_G(g+t) \neq \dot{x}_G(g+s+t))$$

To see this, note that from the definition of the extension relation we have that x_G is a tiling by p except at points of the form $u(p)+(iw(p), jh(p))$ for $(i,j) \in \mathbb{Z}^2$. That is, if $c \in \mathrm{dom}(p)$ then $x_G(c+(iw(p), jh(p))) = p(c)$ for all $(i,j) \in \mathbb{Z}^2$. Any $g \in \mathbb{Z}^2$ is in $R(p) + (iw(p), jh(p))$ for some $(i,j) \in \mathbb{Z}^2$, and so there is a $\tau \in T$ such that $g+\tau$ is of the form $g_0 + (iw, jh)$. So $x_G(g+\tau) = p(g_0) \neq p(g_0+s) = x_G(g+s+\tau)$. This shows that x_G is a 2-coloring.

To see that x_G is minimal, fix a finite $F \subseteq \mathbb{Z}^2$. Note that the set

$$S_F = \{p \in \mathbb{P}_{gp} : F \subseteq \mathrm{dom}(p)\}$$

is dense. Let $p \in S_F \cap G$, and again let $T = [-w(p), w(p)] \times [-h(p), h(p)]$. Let E be the set of all points of the form $u(p) + (iw(p), jh(p))$ for $(i,j) \in \mathbb{Z}^2$. Then $F \cap E = \varnothing$. Since x_G is a tiling by p off E, it follows that for any $g \in \mathbb{Z}^2$ there is a $\tau \in T$ such that $p(\sigma) = x_G(g+\tau+\sigma)$ for all $\sigma \in F$. Thus x_G is minimal. □

Since x_G is a minimal 2-coloring, we certainly have that x_G is not periodic, that is, $x_G \in F(2^{\mathbb{Z}^2})$. However, x_G satisfies some weak form of periodicity as the next lemma shows.

Lemma 7.3. *Let x_G be a generic real for $\mathbb{P}_{gp}(n)$.*

(i) *For any vertical or horizontal line ℓ in $\mathbb{Z}^2$, $x_G \restriction \ell$ is periodic with period a power of n.*

(ii) *For any finite $A \subseteq \mathbb{Z}^2$, there is a lattice $L = (w\mathbb{Z}) \times (h\mathbb{Z})$, with both w and h powers of n, and there is $u \in \mathbb{Z}^2 \setminus (A+L)$ such that x_G is constant on $k+L$ whenever $k+L \neq u+L$.*

Proof. (ii). The proof is similar to the proof that x_G is minimal in Lemma 7.2. Given A and $q \in \mathbb{P}_{gp}$, there is a $p \leq q$ with $A \subseteq R(p) \setminus \{u(p)\}$. There is such a p which forces that x_G has horizontal and vertical periods $w(p)$ and $h(p)$ (which are powers of n) off of the set $u(p) + L(p)$, where $L(p) = \{(iw(p), jh(p)) : (i,j) \in \mathbb{Z}^2\}$. It follows by genericity that x_G has the stated grid periodicity property.

(i). Given any vertical or horizontal line ℓ in $\mathbb{Z}^2$, the set of $p \in \mathbb{P}_{gp}$ with

$$\ell \cap (u(p) + L(p)) = \ell \cap \{u(p) + (iw(p), jh(p)) : (i,j) \cap \mathbb{Z}^2\} = \varnothing$$

is dense. This implies that $x_G \restriction \ell$ has a period n^k for some k. □

As an application of grid periodicity forcing we now have the following structure theorem for Borel complete sections of $F(2^{\mathbb{Z}^2})$.

Theorem 7.4. *Let $B \subseteq F(2^{\mathbb{Z}^2})$ be a Borel complete section. Then there is an $x \in F(2^{\mathbb{Z}^2})$ and a lattice $L = k + \{(iw, jh) \colon (i,j) \in \mathbb{Z}^2\}$ such that $L \cdot x \subseteq B$.*

Proof. Let x_G be a generic real for $\mathbb{P}_{gp}$. We claim that $B \cap [x_G]$ contains a lattice as required. Since $B \cap [x_G] \neq \varnothing$, we may fix $k \in \mathbb{Z}^2$ and $q \in G$ such that $q \Vdash (k \cdot \dot{x}_G \in B)$. For any $(i,j) \in \mathbb{Z}^2$, let $\pi_{i,j}$ be the translation defined by $\pi_{i,j}(g) = g + (iw(q), jh(q))$. Then $\pi_{i,j}$ induces an automorphism of $\mathbb{P}_{gp}$ and

$$\pi_{i,j}(q) \Vdash (\pi_{i,j}(k) \cdot \dot{x}_G \in B).$$

Note that $\pi_{i,j}(k) \cdot \dot{x}_G = (k + (iw(p), jh(p))) \cdot \dot{x}_G$. It suffices therefore to show that G contains the condition $\pi_{i,j}(q)$. By density, there is a $p \leq q$ in G with $R(p) \supseteq R(q) \cup R(\pi_{i,j}(q))$. It is clear, however, from the definition of the extension relation that $p \leq \pi_{i,j}(q)$. Thus, $\pi_{i,j}(q) \in G$ as well. □

We mention that while A. Marks was visiting the authors, he used forcing methods to generalize the above theorem to all countable residually finite groups Γ [11].

The proof of Theorem 7.4 also gives the following variation of Theorem 7.4.

Theorem 7.5. *Let $f \colon F(2^{\mathbb{Z}^2}) \to \mathbb{N}$ be Borel. Then there is an $x \in F(2^{\mathbb{Z}^2})$ and a lattice $L \subseteq \mathbb{Z}^2$ such that the map $s \mapsto f(s \cdot x)$ is constant on L.*

Considering the characteristic function of the Borel set B gives:

Corollary 7.6. *If $B \subseteq F(2^{\mathbb{Z}^2})$ is Borel, then there is an $x \in F(2^{\mathbb{Z}^2})$ such that either $\{s : \, s \cdot x \in B\}$ or $\{s : \, s \cdot x \in 2^{\mathbb{Z}^2} \setminus B\}$ contains a lattice L in $\mathbb{Z}^2$.*

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