Tea Time in Princeton

He said :"That's the form factor for the pair correlation of eigenvalues of random Hermitian matrices !".

This note is about who"He" is, what "That" is, and why you should never miss tea time.

Since the seminal work of Riemann, it is well-known that the distribution of prime numbers is closely related to the behavior of the ζ function. Most importantly, it was conjectured in [13] that all its (non-trivial) zeros are aligned [1], and Hilbert and Pólya put forward the idea of a spectral origin for this phenomenon.

> *I spent two years in Göttingen ending around the begin of 1914. I tried to learn analytic number theory from Landau. He asked me one day : "You know some physics. Do you know a physical reason that the Riemann hypothesis should be true ?" This would be the case, I answered, if the nontrivial zeros of the ξ-function were so connected with the physical problem that the Riemann hypothesis would be equivalent to the fact that all the eigenvalues of the physical problem are real.*
>
> George Pólya, correspondence with Andrew Odlyzko, 1982.

Despite the lack of progress concerning the horizontal distribution of the zeros (i.e. all their real parts being supposedly equal), some support for the Hilbert-Pólya idea came from the vertical distribution, i.e. the distribution of the gaps between the imaginary parts of the non-trivial zeros. Indeed, in 1972, the number theorist Hugh Montgomery evaluated the pair correlation of these zeros, and the mathematical physicist Freeman Dyson realized that they exhibit the same repulsion as the eigenvalues of typical large random Hermitian matrices. In this expository note, we aim at explaining Montgomery's result, placing emphasis on the common points with random matrices. These statistical connections have since been extended to many other L-functions (e.g. over function fields, cf. [12]) ; for the sake of brevity we only consider the Riemann zeta function, and refer for example to [8] for many other connections between analytic number theory and random matrices.

1 Independent random points

As a first step towards the repulsion between some particles, eigenvalues or zeros of the zeta function, we wish to understand what happens when there is *no* repulsion, in particular for *independent* random points. For this, consider the following

1. For a definition of the Riemann zeta function and the Riemann hypothesis, see the beginning of Section 2.

natural question.

Choose n independent and uniform points on the interval $[0,1]$. What is the typical spacing between two successive such points ?

A good way to make this question more precise is to assume that amongst these points $x_1,\dots,x_n$, we label one, say x_1, and we consider the probability that it has no right-neighbor up to distance δ. Denoting $\chi(I)$ the number of x_i's in an interval I, the probability of such an event is

$$\int_0^1 \mathbb{P}(\chi((y,y+\delta])=0 \mid x_1=y)\mathrm{d}y,$$

because x_1 is uniformly distributed. Now, as all the x_i's are independent, the integrand is also (when $y+\delta<1$)

$$\mathbb{P}(\cap_{i=2}^n\{x_i\notin(y,y+\delta]\})=\prod_{i=2}^n\mathbb{P}(x_i\notin(y,y+\delta])=(1-\delta)^{n-1}.$$

Choosing $\delta=\frac{u}{n}$ and considering the limit $n\to\infty$, we get that the probability that the gap between x_1 and its right neighbor is greater than $\frac{u}{n}$ converges to e^{-u}. More generally, denoting by Δ the gap between x_1 and its right-neighbor, we obtain that, for any $0<a<b$,

$$\mathbb{P}(n\Delta\in[a,b])\underset{n\to\infty}{\longrightarrow}\int_a^b e^{-u}\mathrm{d}u\,. \qquad (1)$$

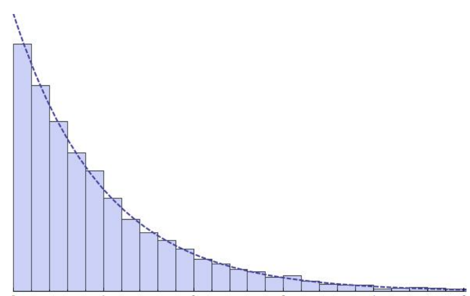

FIGURE 1 – Histogram of 10^5 nearest-neighbor spacings (i.e. $n\Delta$). Dashed : the rescaled e^{-u} curve.

Another way to quantify the microscopic structure of these independent points consists in looking at the following statistics, $r(f,n)=\frac{1}{n}\sum_{1\leq j,k\leq n,j\neq k}f(n(x_j-x_k))$, for a generic test function f. The reader will easily prove the following asymptotics :

$$\mathbb{E}\left(r(f,n)\right)\underset{n\to\infty}{\longrightarrow}\int_{\mathbb{R}}f(y)\mathrm{d}u. \qquad (2)$$

This limiting exponential distribution (1) and the pair correlation (2) appear universally, i.e. when the sampled points are sufficiently close to independence, no matter which distribution they have[2]. It is a natural question whether this remains valid for other random points, and we will explain what happens when considering the ζ zeros with large imaginary part or the eigenvalues of random matrices. The gaps statistics will be very different, both for the former (Section 2) and the latter (Section 3), for which a common type of correlations appears in the limit. The following sections are widely independent.

2. For example, the reader could consider independent points with strictly positive density with respect to the uniform measure on $[0,1]$, and he would obtain an exponential law in the limit as well.

2 The pair correlation of the ζ zeros.

In this section, we state some elementary properties of the Riemann zeta function, mentioning along the way a formal analogy between the ζ zeros and the eigenvalues of the Laplacian on some symmetric spaces. We then come to more quantitative estimates through Montgomery's result on the repulsion between the ζ zeros.

For $\sigma = \Re(s) > 1$, the Riemann zeta function can be defined as a Dirichlet series or an Euler product :

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \in \mathcal{P}} \frac{1}{1 - \frac{1}{p^s}},$$

where $\mathcal{P}$ is the set of all prime numbers. The second equality is a consequence of the expansion $(1 - p^{-s})^{-1} = \sum_{k \geq 0} p^{-ks}$ and uniqueness of factorization of integers into prime numbers. Remarkably, as proved in Riemann's original paper, ζ can be meromorphically extended to $\mathbb{C} - \{1\}$, and this extension satisfies a functional equation (see e.g. [15] for a proof) : writing $\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$, we have

$$\xi(s) = \xi(1 - s).$$

Consequently, the zeta function admits trivial zeros at $s = -2, -4, -6, \dots$ corresponding to the poles of $\Gamma(s/2)$. All the other zeros are confined in the critical strip $0 \leq \sigma \leq 1$, and they are symmetrically positioned about the real axis and the critical line $\sigma = 1/2$. The Riemann Hypothesis states that all of this *non-trivial* zeros are exactly on the line $\sigma = 1/2$.

Trace formulas. The first similarity between the zeta zeros and spectral properties of operators occurs when looking at linear statistics. Namely, we state the Weil explicit formula concerning the ζ zeros and Selberg's trace formula for the Laplacian on surfaces with constant negative curvature.

First consider the Riemann zeta function. For a function $f : (0, \infty) \to \mathbb{C}$, define its Mellin transform $F(s) = \int_0^{\infty} f(s) x^{s-1} \mathrm{d}x$. Then the inversion formula (where σ is chosen in the fundamental strip, i.e. where the image function F converges)

$$f(x) = \frac{1}{2\pi \mathrm{i}} \int_{\sigma - \mathrm{i}\infty}^{\sigma + \mathrm{i}\infty} F(s) x^{-s} \mathrm{d}s$$

holds under suitable smoothness assumptions, in a similar way as the inverse Fourier transform. Hence, for example,

$$\sum_{n=2}^{\infty} \Lambda(n) f(n) = \sum_{n=2}^{\infty} \Lambda(n) \frac{1}{2\pi \mathrm{i}} \int_{2 - \mathrm{i}\infty}^{2 + \mathrm{i}\infty} F(s) n^{-s} \mathrm{d}s = \frac{1}{2\pi \mathrm{i}} \int_{2 - \mathrm{i}\infty}^{2 + \mathrm{i}\infty} \left(-\frac{\zeta'}{\zeta} \right)(s) F(s) \mathrm{d}s,$$

where Λ is Van Mangoldt's function [3]. To derive the above formula, we use that $-\frac{\zeta'}{\zeta}(s) = \sum_{n \geq 2} \frac{\Lambda(n)}{n^s}$, which is obtained by deriving the formula $-\log \zeta(s) = \sum_{\mathcal{P}} \log(1 - p^{-s})$. Now, changing the line of integration from $\Re(s) = 2$ to $\Re(s) = -\infty$, all trivial and non-trivial poles (as well as $s = 1$) are crossed, leading to the following formula,

$$\sum_{\rho} F(\rho) + \sum_{n \geq 0} F(-2n) = F(1) + \sum_{p \in \mathcal{P}, m \in \mathbb{N}} (\log p) \, f(p^m),$$

3. $\Lambda(n) = \log p$ if $n = p^k$ for some prime p, 0 otherwise.

where the first sum is over non-trivial zeros counted with multiplicities. When replacing the Mellin transform by the Fourier transform, the above formula linking linear statistics of zeros and primes takes the following form, known as the Weil explicit formula.

Theorem. *Let h be even, analytic on $|\Im(z)| < 1/2+\delta$, bounded, and decreasing as $h(z) = \mathrm{O}(|z|^{-2-\delta})$ for some $\delta > 0$. Here, the sum is over all γ_n's such that $1/2 + \mathrm{i}\gamma_n$ is a non-trivial zero, and $\hat{h}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} h(y)e^{-\mathrm{i}xy}\mathrm{d}y$:*

$$\sum_{\gamma_n} h(\gamma_n) - 2h\left(\frac{i}{2}\right) = \frac{1}{2\pi}\int_{\mathbb{R}} h(r)\left(\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+\frac{i}{2}r\right) - \log\pi\right)\mathrm{d}r \\ - 2\sum_{p\in\mathcal{P} m\in\mathbb{N}} \frac{\log p}{p^{m/2}}\hat{h}(m\log p). \quad (3)$$

In a very distinct context holds a similar relation, the Selberg's trace formula. In one of its simplest manifestations, it can be stated as follows. Let $\Gamma\backslash\mathbb{H}$ be a quotient of the Poincaré half-plane, where Γ is a subgroup of $\mathrm{PSL}_2(\mathbb{R})$, the orientation-preserving isometries of $\mathbb{H} = \{x + \mathrm{i}y, y > 0\}$ endowed with the metric

$$(\mathrm{d}s)^2 = \frac{(\mathrm{d}x)^2 + (\mathrm{d}y)^2}{y^2}. \quad (4)$$

The Laplace-Beltrami operator $\Delta = -y^2(\partial_{xx} + \partial_{yy})$ is self-adjoint with respect to the invariant measure associated to (4), $\mathrm{d}\mu = \frac{\mathrm{d}x\mathrm{d}y}{y^2}$, i.e. $\int v(\Delta u)\mathrm{d}\mu = \int(\Delta v)u\mathrm{d}\mu$, so all eigenvalues of Δ are real and positive. If $\Gamma\backslash\mathbb{H}$ is compact, the spectrum of Δ restricted to a fundamental domain $\mathcal{D}$ of representatives of the conjugation classes is discrete, noted $0 \leq \lambda_0 < \lambda_1 < \dots$ To state Selberg's trace formula, we need, as previously, a function h analytic on $|\Im(z)| < 1/2+\delta$, even, bounded, and decreasing as $h(z) = \mathrm{O}(|z|^{-2-\delta})$, for some $\delta > 0$.

Theorem. *Under the above hypotheses, setting $\lambda_k = s_k(1-s_k)$, $s_k = 1/2 + \mathrm{i}r_k$, then*

$$\sum_{k=0}^{\infty} h(r_k) = \frac{\mu(\mathcal{D})}{2\pi}\int_{-\infty}^{\infty} rh(r)\tanh(\pi r)\mathrm{d}r + \sum_{p\in\mathcal{P}, m\in\mathbb{N}^*} \frac{\ell(p)}{2\sinh\left(\frac{m\ell(p)}{2}\right)}\hat{h}(m\ell(p)), \quad (5)$$

where $\hat{h}$ is the Fourier transform of h ($\hat{h}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} h(y)e^{-\mathrm{i}xy}\mathrm{d}y$), $\mathcal{P}$ is now the set of all primitive[4] periodic orbits[5] and ℓ is the geodesic distance corresponding to (4).

The similarity between (3) and (5) may make you wish that prime numbers would correspond to primitive orbits, with lengths $\log p$, $p \in \mathcal{P}$. No result in this direction is known however, and it seems safer not to think about this analogy as a conjecture, but rather just as a tool guiding intuition (as done e.g. in [3] to understand the pair correlations between the zeros of ζ). Nevertheless, the reader could prove that, as a consequence of Selberg's trace formula, the number of primitive orbits with length less than x is

$$|\{\ell(p) < x\}| \underset{x\to\infty}{\sim} \frac{e^x}{x}.$$

4. i.e. not the repetition of shorter periodic orbits
5. of the geodesic flow on $\Gamma\backslash\mathbb{H}$

Similarly, by the prime number theorem,

$$|\{\log(p) < x\}| \underset{x\to\infty}{\sim} \frac{e^x}{x}.$$

Montgomery's theorem. A more quantitative connection of analytic number theory with a spectral problems appeared in the early 70's thanks to a conversation, during tea time, in Princeton, about some research on the spacings between the ζ zeros. Here is a how the author of this work, Hugh Montgomery, relates this "serendipity" moment [6].

I took afternoon tea that day in Fuld Hall with Chowla. Freeman Dyson was standing across the room. I had spent the previous year at the Institute and I knew him perfectly well by sight, but I had never spoken to him. Chowla said : " Have you met Dyson ? " I said no, I hadn't. He said : "I'll introduce you." I said no, I didn't feel I had to meet Dyson. Chowla insisted, and so I was dragged reluctantly across the room to meet Dyson. He was very polite, and asked me what I was working on. I told him I was working on the differences between the non-trivial zeros of Riemann's zeta function, and that I had developed a conjecture that the distribution function for those differences had integrand $1-\left(\frac{\sin \pi u}{\pi u}\right)^2$. *He got very excited. He said : "That's the form factor for the pair correlation of eigenvalues of random Hermitian matrices !" I'd never heard the term "pair correlation." It really made the connection. The next day Atle (Selberg) had a note Dyson had written to me giving references to Mehta's book, places I should look, and so on. To this day I've had one conversation with Dyson and one letter from him. It was very fruitful. I suppose by this time the connection would have been made, but it was certainly fortuitous that the connection came so quickly, because then when I wrote the paper for the proceedings of the conference, I was able to use the appropriate terminology and give the references and give the interpretation. I was amused when, a few years later, Dyson published a paper called "Missed Opportunities." I'm sure there are lots of missed opportunities, but this was a counterexample. It was real serendipity that I was able to encounter him at this crucial juncture.*

So what was it exactly that Montgomery proved ? To state his result, we need to first introduce some notation. First by choosing for h an appropriate approximation of an indicator function, from the explicit formula (3) one can prove the following : the number of ζ zeros ρ counted with multiplicities in $0 < \Im(\rho) < t$ is asymptotically

$$\mathcal{N}(t) \underset{t\to\infty}{\sim} \frac{t}{2\pi}\log t. \qquad (6)$$

In particular, the mean spacing between ζ zeros at height t is $2\pi/\log t$. Now, we write as previously $1/2 \pm \mathrm{i}\gamma_n$ for the zeta zeros counted with multiplicity, assuming the Riemann hypothesis and the ordering $\gamma_1 \leq \gamma_2 \leq \dots$ Let $\omega_n = \frac{\gamma_n}{2\pi}\log\frac{\gamma_n}{2\pi}$. From (6) we know that $\delta_n = \omega_{n+1} - \omega_n$ has a mean value 1 as $n \to \infty$. A more precise understanding of the zeta zeros interactions relies on the study of the spacings distribution function below for $t \to \infty$,

$$\frac{1}{\mathcal{N}(t)}|\{(n,m) \in [\![1, \mathcal{N}(t)]\!]^2 : \alpha < \omega_n - \omega_m < \beta, n \neq m\}|,$$

and more generally on the operator

$$\tilde{r}(f,t) = \frac{1}{\mathcal{N}(t)} \sum_{1\leq j,k\leq \mathcal{N}(t), j\neq k} f(\omega_j - \omega_k).$$

As we saw in (2), if the ω_k's behaved as independent random variables (up to the ordering), $\tilde{r}(f,t)$ would converge to $\int_{\mathbb{R}} f(y)\mathrm{d}y$ as $t \to \infty$. The following result by Montgomery [10] proves that the zeros are actually not asymptotically independent, but present some statistical repulsion instead. We include an outline of a proof directly following the statement for the interested reader.

Theorem. *Assume the Riemann hypothesis. Suppose f is a test function with the following property : its Fourier transform*[6] *is $\mathscr{C}^\infty$ and supported in $(-1,1)$. Then*

$$\tilde{r}(f,t) \underset{t\to\infty}{\longrightarrow} \int_{\mathbb{R}} f(y)\tilde{r}(y)\mathrm{d}y,$$

where $\tilde{r}(y) = 1 - \left(\frac{\sin(\pi y)}{\pi y}\right)^2$.

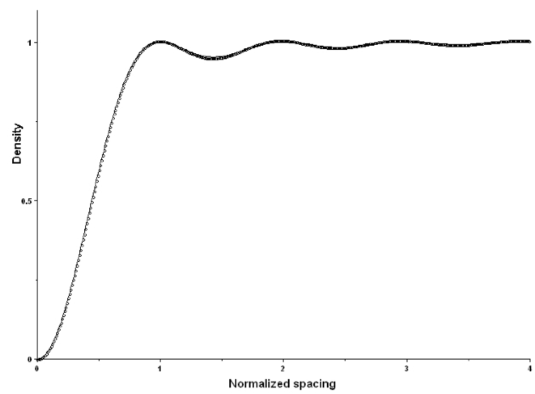

FIGURE 2 – The function $\tilde{r}(y)$ and the histogram of the normalized spacing between non-necessarily consecutive ζ zeros, at height 10^{13} (a number of 2×10^9 zeros have been used to compute the empirical density, represented as small circles). Source : Xavier Gourdon [7]

In fact an important conjecture due to Montgomery asserts that the above result holds with no condition on the support of the Fourier transform. However, weakening the restriction even to $\operatorname{supp}\hat{f} \subset (-1-\varepsilon, 1+\varepsilon)$ for some $\varepsilon > 0$ out of reach with known techniques. The Montgomery conjecture would have important consequences for example in terms of the statistics of gaps between the prime numbers $p_1 < p_2 < \dots$: for example, it would imply that $p_{n+1} - p_n \ll \sqrt{p_n \log p_n}$.

Sketch of proof of Montgomery's Theorem. Consider the function

$$F(\alpha,t) = \frac{1}{\frac{t}{2\pi}\log t} \sum_{0<\gamma,\gamma'<t} t^{\mathrm{i}\alpha(\gamma-\gamma')} \frac{4}{4+(\gamma-\gamma')^2},$$

where the γ's are the imaginary parts of the ζ zeros. This is the Fourier transform of the normalized spacings, up to the factor $4/(4+(\gamma-\gamma')^2)$, present here just for technical convergence reasons. This function naturally appears when counting the second order moments

$$\int_0^t |G(s,t^\alpha)|^2 \mathrm{d}s = F(\alpha,t)t\log t + \mathrm{O}(\log^3 t),\ G(s,x) = 2\sum_\gamma \frac{x^{\mathrm{i}\gamma}}{1+(s-\gamma)^2}. \quad (7)$$

6. Contrary to the Weil and Selberg formulas (3) and (5), the chosen normalization here is $\hat{f}(x) = \int_{-\infty}^{\infty} f(y)e^{-\mathrm{i}2\pi xy}\mathrm{d}y$

As G is a linear functional of the zeros, it can be written as a sum over primes by an appropriate explicit formula like (3) : Montgomery proved that

$$G(s,x)=-\sqrt{x}\left(\sum_{n\leq x}\Lambda(n)\left(\frac{x}{n}\right)^{-\frac{1}{2}+\mathrm{i}s}+\sum_{n>x}\Lambda(n)\left(\frac{x}{n}\right)^{\frac{3}{2}+\mathrm{i}s}\right)+\varepsilon(s,x),$$

where $\varepsilon(s,x)$ is an error term which, under the Riemann hypothesis, can be bounded efficiently and makes no contribution in the following asymptotics. The moment (7) can therefore be expanded as a sum over primes, and the Montgomery-Vaughan inequality (cf. the exercise hereafter) leads to

$$\int_0^t |G(s,t^\alpha)|^2\mathrm{d}s=(t^{-2\alpha}\log t+\alpha+\mathrm{o}(1))t\log t. \tag{8}$$

These asymptotics can be proved by the Montgomery Vaughan inequality, but only in the range $\alpha\in(0,1)$, which explains the support restriction in the hypotheses. Gathering both asymptotic expressions for the second moment of G yields $F(\alpha,t)=t^{-2\alpha}\log t+\alpha+\mathrm{o}(1)$. Finally, by the Fourier inversion formula,

$$\frac{1}{\frac{t}{2\pi}\log t}\sum_{0\leq\gamma,\gamma'\leq t}f\left((\gamma-\gamma')\frac{\log t}{2\pi}\right)\frac{4}{4+(\gamma-\gamma')^2}=\int_{\mathbb{R}}F(\alpha,t)\hat f(\alpha)\mathrm{d}\alpha.$$

If $\operatorname{supp}\hat f\subset(-1,1)$, this is approximately

$$\int_{\mathbb{R}}\hat f(\alpha)(t^{-2|\alpha|}+|\alpha|)\mathrm{d}\alpha=\int_{\mathbb{R}}e^{-2|\alpha|}\hat f(\alpha/\log t)\mathrm{d}\alpha+\int_{\mathbb{R}}|\alpha|\hat f(\alpha)\mathrm{d}\alpha$$
$$=\hat f(0)+f(0)-\int_{\mathbb{R}}(1-|\alpha|)\hat f(\alpha)\mathrm{d}\alpha+\mathrm{o}(1)=f(0)+\int_{\mathbb{R}}f(x)\left(1-\left(\frac{\sin\pi x}{\pi x}\right)^2\right)\mathrm{d}x+\mathrm{o}(1),$$

by the Plancherel formula. □

(Difficult) Exercise. *Let (a_r) be complex numbers, (λ_r) distinct real numbers and $\delta_r=\min_{s\neq r}|\lambda_r-\lambda_s|$. Then the Montgomery-Vaughan inequality asserts that*

$$\frac{1}{t}\int_0^t|\sum_r a_r e^{\mathrm{i}\lambda_r s}|^2\mathrm{d}s=\sum_r|a_r|^2\left(1+\frac{3\pi\theta}{t\delta_r}\right)$$

for some $|\theta|<1$. In particular,

$$\int_0^t\left|\sum_{n=1}^\infty\frac{a_n}{n^{\mathrm{i}s}}\right|^2\mathrm{d}s=t\sum_{n=1}^\infty|a_n|^2+\mathrm{O}\left(\sum_{n=1}^\infty n|a_n|^2\right).$$

Prove that the above result implies (8).

To numerically test Montgomery's conjecture, Odlyzko [11] computed the normalized gaps, $\omega_{i+1} - \omega_i$, and produced the joint histogram. In particular, note that the limiting density vanishes at 0, contrasting with Figure 1, and that this type of repulsion coincides remarkably with the shape of gaps for random matrices.

Moreover, Montgomery's result has been extended in the work by Rudnick and Sarnak [14], who proved that for some statistics depending on more than just one gap, the ζ zeros also present the same limit distribution as predicted by Random Matrix Theory. This urges us to explain in more details what we mean by *random matrices*.

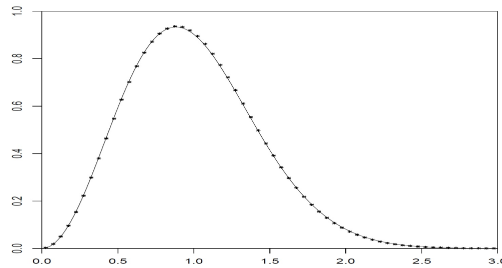

FIGURE 3 – The distribution function of asymptotic gaps between eigenvalues of random matrices compared with the histogram of gaps between successive normalized ζ zeros, based on a billion zeros near $\#1.3 \cdot 10^{16}$.

3 Eigenvalues repulsion for random matrices

Let χ be a point process, i.e. a random set of points $\{x_1, x_2, \dots\}$, in a metric space Λ, identified with the random punctual measure $\sum_i \delta_{x_i}$. The kth correlation function for this point process, ρ_k, is defined as the asymptotic (normalized) probability of having exactly one particle in respective neighborhoods of k fixed points. More precisely, if the u_i's are distinct in Λ,

$$\rho_k(u_1, \dots, u_k) = \lim_{\varepsilon \to 0} \frac{\mathbb{P}\left(\chi(B_{u_i,\varepsilon}) = 1, 1 \leq i \leq k\right)}{\prod_{j=1}^{k} \lambda(B_{u_i,\varepsilon})},$$

provided that the limit exists (here $B_{u_i,\varepsilon}$ denotes the ball with radius ε and center u, and the measure λ will be specified later). If χ consists almost surely of n points, the correlation functions satisfy the integration property

$$(n-k)\rho_k(u_1, \dots, u_k) = \int_\Lambda \rho_{k+1}(u_1, \dots, u_{k+1}) \mathrm{d}\lambda(u_{k+1}). \tag{9}$$

Interestingly, many properties about a point process are well-understood when the correlation functions are also determinants. More precisely, assume now that $\Lambda = \mathbb{C}$. If there exists a function $K : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ such that for all $k \geq 1$ and $(z_1, \dots, z_k) \in \mathbb{C}^k$

$$\rho_k(z_1, \dots, z_k) = \det\left(K(z_i, z_j)_{i,j=1}^{k}\right),$$

then χ is said to be a determinantal point process with respect to the underlying measure λ and with correlation kernel K.

The determinantal condition for *all* correlation functions is quite restrictive. Nevertheless, as stated in the following theorem, any bidimensional system of particles with quadratic interaction is determinantal (see [1] for a proof).

Theorem. *Let* $\mathrm{d}\lambda$ *be any*[7] *finite measure on* $\mathbb{C}$ *(eventually concentrated on a line).*

7. We just need a decreasing of the mass at infinity of type $\int_{|z|>t} \mathrm{d}\lambda(z) \ll t^{-k}$ for any $k > 0$.

Consider the probability distribution with density

$$c(n)\prod_{1\leq k<l\leq n}|z_l-z_k|^2$$

with respect to $\prod_{j=1}^n \mathrm{d}\lambda(z_j)$*, where* $c(n)$ *is the normalization constant. For this joint distribution,* $\{z_1,\dots,z_n\}$ *is a determinantal point process with the following explicit kernel,*

$$K(x,y)=\sum_{k=0}^{n-1}P_k(x)\overline{P_k(y)}$$

where P_k $(0\leq k\leq n-1)$ *is a polynomial with degree* k *and the* P_j*'s are orthonormal for the Hermitian product* $f,g\mapsto\int f\overline{g}\mathrm{d}\lambda$*.*

We apply the above result to the following examples, which are among the most studied random matrices. First, consider the so-called Gaussian unitary ensemble (GUE). This is the ensemble (or set) of random $n\times n$ Hermitian matrices with independent (up to symmetry) Gaussian entries : $M_{ij}^{(n)}=\overline{M_{ji}^{(n)}}=\frac{1}{\sqrt{n}}(X_{ij}+\mathrm{i}Y_{ij}), 1\leq i<j\leq n$, where the X_{ij}'s and Y_{ij}'s are independent centered real Gaussians entries with mean 0 and variance 1/2 and $M_{ii}^{(n)}=X_{ii}/\sqrt{n}$ with X_{ii} real centered Gaussians with variance 1, still independent. These random matrices are natural in the sense that they are uniquely characterized by the independence (up to symmetry) of their entries, and invariance by unitary conjugacy. A similar natural set of matrices, when the entries are now real Gaussian, called GOE (Gaussian orthogonal ensemble) will appear in the next section.

For the GUE, the distribution of the eigenvalues has an explicit density,

$$\frac{1}{Z_n}e^{-n\sum_{i=1}^n\lambda_i^2/2}\prod_{1\leq i<j\leq n}|\lambda_i-\lambda_j|^2 \tag{10}$$

with respect to Lebesgue measure (see e.g. [1] for a derivation of this result). We denote by (h_n) the Hermite polynomials, more precisely the successive monic polynomials orthogonal with respect to the Gaussian weight $e^{-x^2/2}\mathrm{d}x$, and consider the associated normalized functions

$$\psi_k(x)=\frac{e^{-x^2/4}}{\sqrt{\sqrt{2\pi}k!}}\ h_k(x).$$

Then from the previous Theorem, one can prove that the set of point $\{\lambda_1,\dots,\lambda_n\}$ with law (10) is a determinantal point process whose kernel (with respect to the Lebesgue measure on $\mathbb{R}$) is given by

$$K^{\mathrm{GUE}(n)}(x,y)=n\frac{\psi_n(x\sqrt{n})\psi_{n-1}(y\sqrt{n})-\psi_{n-1}(x\sqrt{n})\psi_n(y\sqrt{n})}{x-y},$$

extended by continuity when $x=y$. Here we used a simplification : the sum over all orthogonal polynomials can simplify as a sum over just two of them, this is the Christoffel-Darboux formula.

The Plancherel-Rotach asymptotics for the Hermite polynomials implies that, as $n\to\infty$, $K^{\mathrm{GUE}(n)}(x,x)/n$ has a non-trivial limit.

More precisely, the empirical spectral distribution $\frac{1}{n}\sum \delta_{\lambda_i}$ converges in probability to the semicircle law with density

$$\rho_{sc}(x) = \frac{1}{2\pi}\sqrt{(4-x^2)_+}$$

with respect to Lebesgue measure. This is the asymptotic behavior of the spectrum in the macroscopic regime. The microscopic interactions between eigenvalues also can be evaluated thanks to asymptotics of the Hermite orthogonal polynomials : for any $x \in (-2,2)$, $u \in \mathbb{R}$,

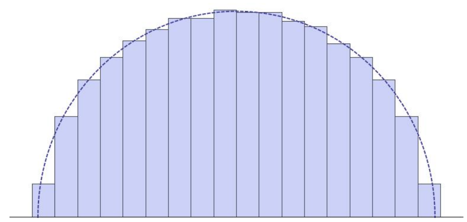

Figure 4 – Histogram of the eigenvalues from the Gaussian Unitary Ensemble in dimension 10^4. Dashed : the rescaled semicircle law.

$$\frac{1}{n\rho_{sc}(x)} K^{\mathrm{GUE}(n)}\left(x, x+\frac{u}{n\rho_{sc}(x)}\right) \underset{n\to\infty}{\longrightarrow} K(u) = \frac{\sin(\pi u)}{\pi u}.$$

This leads to a repulsive correlation structure for the eigenvalues at the scale of the average gap : for example the two-point correlation function asymptotics are

$$\left(\frac{1}{n\rho_{sc}(x)}\right)^2 \rho_2^{\mathrm{GUE}(n)}\left(x, x+\frac{u}{n\rho_{sc}(x)}\right) \underset{n\to\infty}{\longrightarrow} \tilde{r}(u) = 1-\left(\frac{\sin(\pi u)}{\pi u}\right)^2,$$

the strict analogue to Montogmery's result, an analogy identified by Dyson as mentioned in Section 2.

Figure 5 – Upper line : a sample of independent points distributed according to the semicircle law after zooming in the bulk. Middle line : a sample eigenvalues of the GUE after zooming in the bulk of the spectrum. Lower line : a sequence of imaginary parts of the ζ zeros, about height 10^5.

A remarkable fact about the above limiting sine kernel is that it appears universally in the limiting correlation functions of random Hermitian matrices with independent (up to symmetry) entries (not necessarily Gaussian) ; these deep universality results were achieved, still for the Hermitian symmetry class, in recent works by Erdős, Yau et al, or by Tao, Vu. In the case of other symmetry classes [8], the universality of the local eigenvalues statistics has also been proved by Erdős, Yau et al.

Finally we want to mention the following *structural* reason for the repulsion of the eigenvalues of typical matrices : as an exercise, the reader could prove that the space of Hermitian matrices with at least one repeated eigenvalue has codimension 3 in the space of all Hermitian matrices. Repeated eigenvalues therefore occur with very small probability compared to independent points (on a product space, the codimension of the subspace where two points coincide is 1). László Erdős asked me about a *structural*, heuristic, argument for the repulsion of the ζ zeros. Unable to answer it, I transmit the question to the readers.

8. i.e. for random symmetric matrices or random symplectic matrices

4 Eigenvalues repulsion for quantum billiards

To conclude this expository note, we wish to mention some conjectures about the asymptotic distribution of eigenvalues, for the Laplacian on compact spaces.

The examples we consider are two-dimensional quantum billiards[9]. For some billiards, the classical trajectories are integrable[10] and for others they are chaotic.

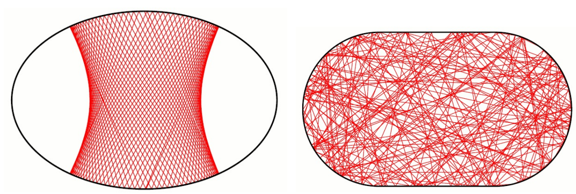

Figure 6 – An integrable billiard (ellipse) and a chaotic one (stadium)

On the quantum side, we consider the Helmholtz equation inside the billiard, describing the standing waves :

$$-\Delta\psi_n = \lambda_n \psi_n,$$

where the spectrum is discrete as the domain is compact, with ordered eigenvalues $0 \leq \lambda_1 \leq \lambda_2 \dots$, and appropriate Dirichlet or Neumann boundary conditions. The questions about quantum billiards we are interested here is about the asymptotic behavior of the λ_n's, i.e. whether they will present asymptotic independence or a Random Matrix Theory type of repulsion. The situation is still somehow mysterious : there is a conjectural dichotomy between the chaotic and integrable cases.

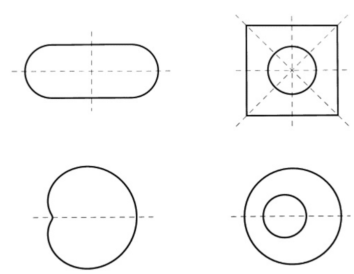

Figure 7 – Some chaotic billiards, from left to right, up to down : the stadium, Sinai's billard, the cardioid, and a billiard with no name.

First, in 1977, Berry and Tabor [4] put forward the conjecture that for most integrable systems, the large eigenvalues have the statistics of a Poisson point process, i.e. rescaled gaps being asymptotically exponential random variables, like in Section 1. More precisely, by Weyl's law, we know that the number of such eigenvalues up to λ is

$$|\{i : \lambda_i \leq \lambda\}| \underset{\lambda\to\infty}{\sim} \frac{\text{area}(\mathcal{D})}{4\pi}\,\lambda. \tag{11}$$

To analyze the correlations between eigenvalues, consider the point process

$$\chi^{(n)} = \frac{1}{n}\sum_{i\leq n} \delta_{\frac{4\pi}{\text{area}(\mathcal{D})}(\lambda_{i+1}-\lambda_i)}.$$

Its expectation converges to 1 (as $n \to \infty$) from (11).

9. A billiard is a compact connected set with nonempty interior, with a generally piecewise regular boundary, so that the classical trajectories are straight lines reflecting with equal angles of incidence and reflection

10. Roughly speaking this means that there are many conserved quantities along the trajectory, and that explicit solutions can be given for the speed and position of the ball at any time

By the conjectured limiting Poissonian behavior, the spacing distribution converges to an exponential law : for any $I \subset \mathbb{R}^+$

$$\chi^{(n)}(I) \underset{n\to\infty}{\longrightarrow} \int_I e^{-x}\mathrm{d}x. \qquad (12)$$

In the chaotic case, the situation differs radically : the eigenvalues are supposed to repel each other, with gaps statistics conjecturally similar to those of a random matrix, from an ensemble depending on the symmetry properties of the system (e.g. time-reversibility for our quantum billiards correspond to the Gaussian Orthogonal Ensemble). This is known as the Bohigas-Giannoni-Schmidt Conjecture [5].

Numerical experiments were performed in [5] giving a correspondence between the eigenvalue spacings statistics for Sinai's billiard and those of the Gaussian Orthogonal Ensemble. The joint graphs, by A. Backer, present similar experiments for an integrable billiard (Figure 8) and a chaotic one (Figure 9). These statistics are perfectly coherent with both the Berry-Tabor and the Bohigas-Giannoni-Schmidt conjectures. This deepens the interest in these Random Matrix Theory distributions, which appear increasingly in many fields, including analytic number theory.

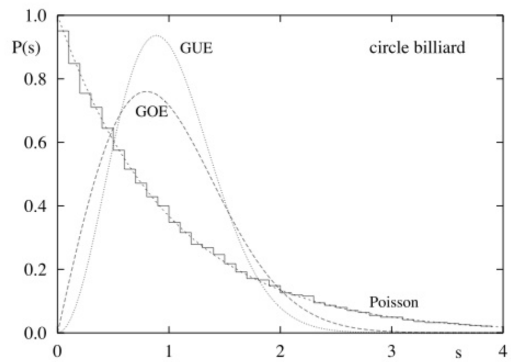

Figure 8 – Energy levels for the circular billiard compared to those of the Gaussian ensembles and Poissonian statistics (data and picture from [2]).

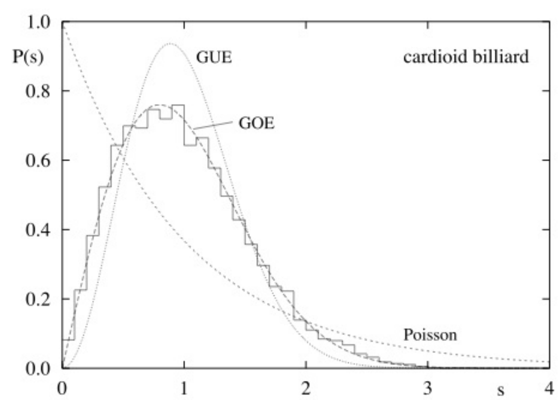

Figure 9 – Energy levels for the cardioid billiard compared to those of the Gaussian ensembles and Poissonian statistics (data and picture from [2]).

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