On random matrices and L-functions

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Abstract

Evidence for deep connections between number theory and random matrix theory has been noticed since the Montgomery-Dyson encounter in 1972 : the function fields case was studied by Katz and Sarnak, and the moments of the Riemann zeta function along its critical axis were conjectured by Keating and Snaith, in connection with similar calculations for random matrices on the unitary group. This thesis concentrates on the latter aspect : it aims first to give further evidence for this analogy in the number field case, second to develop probabilistic tools of interest for number theoretic questions.

The introduction is a survey about the origins and limits of analogies between random matrices in the compact groups and L-functions. We then state the main results of this thesis.

The first two chapters give a probabilistic flavor of results by Keating and Snaith, previously obtained by analytic methods. In particular, a common framework is set in which the notion of independence naturally appears from the Haar measure on a compact group. For instance, if g is a random matrix from a compact group endowed with its Haar measure, $\det(\mathrm{Id} - g)$ may be decomposed as a product of independent random variables.

Such independence results hold for the Hua-Pickrell measures, which generalize the Haar measure. Chapter 3 focuses on the point process induced on the spectrum by these laws on the unit circle : these processes are determinantal with an explicit kernel, called the hypergeometric kernel. The universality of this kernel is then shown : it appears for any measure with asymmetric singularities.

The characteristic polynomial of random matrices can be considered as an orthogonal polynomial associated to a spectral measure. This point of view combined with the widely developed theory of orthogonal polynomials on the unit circle yields results about the (asymptotic) independence of characteristic polynomials, a large deviations principle for the spectral measure and limit theorems for derivatives and traces. This is developed in Chapters 4 and 5.

Chapter 6 concentrates on a number theoretic issue : it contains a central limit theorem for $\log \zeta$ evaluated at distinct close points. This implies correlations when counting the zeros of ζ in distinct intervals at a mesoscopic level, confirming numerical experiments by Coram and Diaconis. A similar result holds for random matrices from the unitary group, giving further insight for the analogy at a local scale.

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Notations

Matrix groups

$\mathrm{S}(n)$	Symmetric group : set of permutations of $[\![1,n]\!]$
$\mathrm{U}(n)$	Unitary group : set of $n\times n$ complex matrices u satisfying $u\,{}^{\mathrm{t}}\overline{u}=\mathrm{Id}$
$\mathrm{O}(n)$	Orthogonal group : elements in $\mathrm{U}(n)$ with real entries
$\mathrm{SU}(n)$	Special unitary group : elements in $\mathrm{U}(n)$ with determinant 1
$\mathrm{SO}(n)$	Special orthogonal group : elements in $\mathrm{O}(n)$ with determinant 1
$\mathrm{USp}(2n)$	Unitary symplectic group : elements $u\in\mathrm{U}(2n)$ such that $uz\,{}^{\mathrm{t}}u=z$ with $z=\begin{pmatrix}0 & \mathrm{Id}_n\\ -\mathrm{Id}_n & 0\end{pmatrix}$
$\mathrm{U}(n,\mathrm{K})$	Unitary group over an arbitrary field : set of $n\times n$ complex matrices u with entries in K satisfying $u\,{}^{\mathrm{t}}\overline{u}=\mathrm{Id}$
$\mathrm{Z}_n(u,\mathrm{X})$	Characteristic polynomial : $\mathrm{Z}_n(u,\mathrm{X})=\det(\mathrm{Id}_n-u\overline{\mathrm{X}})$, u a $n\times n$ matrix, $\mathrm{X}\in\mathbb{C}^*$. When the dimension, the evaluation point or the matrix is implicit, $\mathrm{Z}_n(u,\mathrm{X})$ is written $\mathrm{Z}(u,\mathrm{X})$, $\mathrm{Z}(u,\varphi)$ ($\mathrm{X}=e^{\mathrm{i}\varphi}$) or Z_n ($\mathrm{X}=1$).

Number theory

$\mathcal{P}$	Set of prime numbers
$\pi(x)$	Prime-counting function : $\pi(x)=\lvert\mathcal{P}\cap[1,x]\rvert$
ζ	Riemann zeta function
$\mathrm{S}(t)$	Argument of ζ on the critical axis : $\mathrm{S}(t)=\frac{1}{\pi}\arg\zeta(1/2+\mathrm{i}t)$, defined continuously from 2 to $2+\mathrm{i}t$ to $1/2+\mathrm{i}t$ (if ζ vanishes on the horizontal line $\mathfrak{Im}z=t$, $\mathrm{S}(t)=\mathrm{S}(t^+)$)
$\mathrm{N}(t)$	Number of non-trivial zeros z of ζ with $0<\mathfrak{Im}z\leqslant t$: $\mathrm{N}(t)=\frac{t}{2\pi}\log\frac{t}{2\pi e}+\mathrm{S}(t)+\frac{7}{8}+\mathrm{R}(t),\mathrm{R}(t)\ll\frac{1}{t}$

Random variables

ω	Uniform random variable on (0,1)
$e^{\mathrm{i}\omega}$	Uniform random variable on the unit circle (here ω is uniform on $(0,2\pi)$)
$\mathrm{B}_{a,b}$	Beta random variable with parameter (a,b)
$\mathrm{Dir}_a^{(n)}$	Dirichlet distribution of order n with parameter a

Orthogonal polynomials on the unit circle

$\mathbb{D}$	Open unit disk
$\partial\mathbb{D}$	Unit circle
$\langle\cdot,\cdot\rangle$	Hermitian product on the unit circle associated to an implicit measure ν : $\langle f,g\rangle=\int_{\partial\mathbb{D}}\overline{f(x)}g(x)\mathrm{d}\nu(x)$
$(\Phi_k,k\geqslant 0)$	Monic orthogonal polynomials for $\langle\cdot,\cdot\rangle$ associated to a measure ν on $\partial\mathbb{D}$
$(\alpha_k,k\geqslant 0)$	Verblunsky coefficients associated to $(\Phi_k,k\geqslant 0)$
$(\gamma_k,k\geqslant 0)$	Modified Verblunsky coefficients

Special functions

Γ	Gamma function
$(x)_n$	Pochhammer symbol : $(x)_n = \Gamma(x+n)/\Gamma(x) = x(x+1)\dots(x+n-1)$ if $n \in \mathbb{N}^*$, $(x)_n = 1/(x+n)_{-n}$ if $n \in -\mathbb{N}$
${}_2\mathrm{F}_1(a,b,c;z)$	Gauss's hypergeometric function : if the series converges, ${}_2\mathrm{F}_1(a,b,c;z) = \sum_{k\geqslant 0} \frac{(a)_k(b)_k}{(c)_k k!} z^k$
$\Delta(x_1,\dots,x_n)$	Vandermonde determinant : $\Delta(x_1,\dots,x_n) = \prod_{1\leqslant i<j\leqslant n}(x_i - x_j)$

Measures on matrices

μ_{G}	Haar measure on a compact group G
$\mu_{\mathrm{G}}^{(\delta)}$	Hua-Pickrell measure on a compact group G : $\det_\delta$-sampling of μ_{G}
$\mathrm{J}_{a,b,\beta}^{(n)}$	Jacobi ensemble : the eigenvalues have density $c_{a,b,\beta}^{(n)} \lvert\Delta(x_1,\dots,x_n)\rvert^\beta \prod_{j=1}^n (2-x_j)^a(2+x_j)^b$ on $[-2,2]^n$
$\mathrm{CJ}_{\delta,\beta}^{(n)}$	Circular Jacobi ensemble : the eigenangles have density $c_{\delta,\beta}^{(n)} \lvert\Delta(e^{\mathrm{i}\theta_1},\dots,e^{\mathrm{i}\theta_n})\rvert^\beta \prod_{j=1}^n (1-e^{-\mathrm{i}\theta_j})^\delta (1-e^{\mathrm{i}\theta_j})^{\overline{\delta}}$ on $[-\pi,\pi]^n$

Various symbols

$[s,t)$	Set of real numbers x such that $s \leqslant x < t$
$a \wedge b$	$\min(a,b)$
$a \vee b$	$\max(a,b)$
$\mathrm{X} \sim \mu$	the random variable X is μ-distributed
$a_n \sim b_n$	$a_n/b_n \to 1$ as $n \to \infty$
o	$a_n = \mathrm{o}(b_n)$: $a_n/b_n \to 0$ as $n \to \infty$
O	$a_n = \mathrm{O}(b_n)$: there is $c > 0$ with $\lvert a_n\rvert < c\lvert b_n\rvert$ for all n
$\ll$	$a_n \ll b_n$, the Vinogradov symbol : $a_n = \mathrm{O}(b_n)$. Moreover, $a_n \gg b_n$ means $b_n \ll a_n$.

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Introduction on random matrices and number theory : historical analogies

I spent two years in Göttingen ending around the begin of 1914. I tried to learn analytic number theory from Landau. He asked me one day : "You know some physics. Do you know a physical reason that the Riemann hypothesis should be true ?" This would be the case, I answered, if the nontrivial zeros of the ξ-function were so connected with the physical problem that the Riemann hypothesis would be equivalent to the fact that all the eigenvalues of the physical problem are real.

George Pólya
Correspondence with Andrew Odlyzko, 1982

Hilbert and Pólya conjectured that the zeros of the Riemann zeta function may have a spectral origin : the values of t_n such that $\frac{1}{2} + \mathrm{i}t_n$ is a non trivial zero of ζ might be the eigenvalues of a Hermitian operator ; this would imply the Riemann hypothesis.

There was no real support for this spectral view of the Riemann zeros till the 50's : the resemblance between the Selberg trace formula, concerning the eigenvalues of the Laplacian of a Riemann surface, and the Weil explicit formula in number theory, provided evidence for the Hilbert-Pólya conjecture.

Montgomery's calculation of the pair correlation of the t_n's (1972) was a second deep sign : these zeros present the same repulsion as the eigenvalues of typical large unitary matrices, as noticed by Dyson. Montgomery conjectured more general analogies with these random matrices, which are fully confirmed by Odlyzko's numerical experiments in the 80's.

In the late 90's, two new viewpoints supported the Hilbert-Pólya conjecture. Firstly, the function field analogue (e.g. : zeta functions for curves over finite fields) was studied by Katz and Sarnak : they proved that the zeros of almost all such zeta functions satisfy the Montgomery-Odlyzko law. Secondly, the moments of L-functions along their critical axis were conjectured by Keating and Snaith. Their idea was to model these moments by similar expectations for the characteristic polynomial of random matrices. Their conjecture agrees with all the moments previously known or guessed.

This introduction reviews the evidence mentioned above and presents some of the results of this thesis along this historical account. The next chapters focus almost exclusively on the problem of moments and probabilistic digressions from it.

Many more precisions about the links between random matrices and number theory can be found in [98] (see also the survey by Emmanuel Royer [117], in French). Moreover, other analogies between number theory and probability exist : for surprising identities relating Brownian motion and L-functions, see Biane [12], Biane, Pitman and Yor [13] and Williams [141].

The Riemann zeta function : definition and some conjectures

The Riemann zeta function can be defined, for $\sigma = \mathfrak{Re}(s) > 1$, as a Dirichlet series or an Euler product :

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}=\prod_{p\in\mathcal{P}}\frac{1}{1-\frac{1}{p^s}},$$

where $\mathcal{P}$ is the set of all prime numbers. This function admits a meromorphic continuation to $\mathbb{C}$, and is regular except at the single pole $s=1$, with residue 1 (see [135] for a proof of all the results in this paragraph). Moreover, the classical functional equation holds :

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)=\pi^{-(1-s)/2}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s). \tag{0.1}$$

The zeta function admits trivial zeros at $s=-2,-4,-6,\dots$ corresponding to the poles of $\Gamma(s/2)$. From equation (0.1) one can check that all non-trivial zeros are confined to the critical strip $0\leqslant\sigma\leqslant 1$, and they are symmetrically positioned about the real axis and the critical line $\sigma=1/2$. It is supposed that all the non-trivial zeros of ζ lie on the critical line.

The Riemann hypothesis. If $\zeta(s)=0$ and $\mathfrak{Re}(s)>0$, then $\mathfrak{Re}(s)=1/2$.

This conjecture is known to be true for 2/5 of the zeta zeros (see Conrey [29]). The importance of the Riemann hypothesis in many mathematical fields justifies the intensive studies which continue to be made about the behavior of $\zeta(1/2+\mathrm{i}t)$ for $t\geqslant 0$. In particular, there is the following famous conjecture, which is implied by the Riemann hypothesis (see [135]).

The Lindelöf hypothesis. For every $\varepsilon>0$, as $\mathrm{T}\to\infty$,

$$\left|\zeta\left(\frac{1}{2}+\mathrm{iT}\right)\right|=\mathrm{O}(\mathrm{T}^{\varepsilon}).$$

This conjecture can be shown to be equivalent to the following one relative to the moments of ζ on the critical line.

The moments hypothesis. For every $\varepsilon>0$ and $k\in\mathbb{N}$, as $\mathrm{T}\to\infty$,

$$\mathrm{I}_k(\mathrm{T})=\frac{1}{\mathrm{T}}\int_0^{\mathrm{T}}\mathrm{d}s\left|\zeta\left(\frac{1}{2}+\mathrm{i}s\right)\right|^{2k}=\mathrm{O}(\mathrm{T}^{\varepsilon}).$$

The moments hypothesis may be easier to study than the Lindelöf hypothesis, because it deals with means of ζ and no values at specific points. Actually, more precise estimates on the moments are proved or conjectured[1] :

$$(\log\mathrm{T})^{k^2}\ll_k\mathrm{I}_k(\mathrm{T})\ll_{k,\varepsilon}(\log\mathrm{T})^{k^2+\varepsilon}.$$

for any $\varepsilon>0$. The lower bound was shown unconditionally by Ramachandra [115] for integers $2k$ and by Heath-Brown [66] for general integer k, and the upper bound was shown by Soundararajan [132] conditionally on the Riemann hypothesis.

By modeling the zeta function with the characteristic polynomial of a random unitary matrix, Keating and Snaith [80] even got a conjecture for the exact equivalent of $\mathrm{I}_k(\mathrm{T})$. This is presented in Section 2 of this introduction, and urges us to define precisely some families of random matrices in the classical compact groups.

Random matrices : classical compact groups and their Haar measures

To state the analogy between moments of ζ and the moments of the characteristic polynomial of random unitary matrices, the definition of a natural uniform measure

1. The subscripts $\ll_k$ and $\ll_{k,\varepsilon}$ mean that the involved constants only depend on the indicated parameters.

on $\mathrm{U}(n)$ is required. This *Haar measure* can be defined in great generality (see Halmos [63] for a proof of the following result, originally due in full generality to Weil [137]).

Existence and uniqueness of the Haar measure. *Let* G *be a locally compact group. There exists a unique (up to a multiplicative constant) measure* μ_{G} *on* G *such as :*

- $\mu_{\mathrm{G}}(\mathrm{A}) > 0$ *for all nonempty open sets* $\mathrm{A} \subset \mathrm{G}$ *;*
- $\mu_{\mathrm{G}}(g\mathrm{A}) = \mu_{\mathrm{G}}(\mathrm{A})$ *for all* $g \in \mathrm{G}$ *and nonempty open sets* $\mathrm{A} \subset \mathrm{G}$.

This measure μ_{G} *is called the Haar measure of* G.

Obviously, for finite groups this is the counting measure, and for $(\mathbb{R}, +)$ this is the Lebesgue measure. Hence the Haar measure corresponds to the idea of uniformity on a group.

The locally compact groups considered in this thesis are compact Lie groups, therefore in the sequel the Haar measure will always be normalized : in particular we will discuss probability Haar measure on the orthogonal group $\mathrm{O}(n)$ of $n \times n$ real matrices u such as $u\,{}^{\mathrm{t}}u = \mathrm{Id}_n$, the special orthogonal group, subgroup $\mathrm{SO}(n)$ of $\mathrm{O}(n)$ of matrices u with $\det(u) = 1$, the unitary group $\mathrm{U}(n)$ of $n \times n$ matrices u such as $u\,{}^{\mathrm{t}}\overline{u} = \mathrm{Id}_n$, the special unitary group $\mathrm{SU}(n)$, subgroup of $\mathrm{U}(n)$ of matrices u with $\det(u) = 1$, and finally the symplectic group $\mathrm{USp}(2n)$ of matrices $u \in \mathrm{U}(2n)$ such that $uz\,{}^{\mathrm{t}}u = z$ with

$$z = \begin{pmatrix} 0 & \mathrm{Id}_n \\ -\mathrm{Id}_n & 0 \end{pmatrix}.$$

For these groups, the characteristic polynomial $\mathrm{P(X)} = \det(\mathrm{XId} - u)$ verifies the functional equation

$$(-\mathrm{X})^n \mathrm{P}(1/\mathrm{X}) = \det(u)\overline{\mathrm{P}(\overline{\mathrm{X}})},$$

which can be compared to (0.1). Moreover, all the eigenvalues have modulus 1, which is an analogue of the Riemann Hypothesis : the unit circle corresponds to the critical axis.

There is no generic way to choose an element of a group endowed with the Haar measure, but for the special cases considered here, we mention the following ones.

First, take g be a random $n \times n$ matrix, with all g_{jk}'s independent standard normal variables. Then, the Gram-Schmidt orthonormalization of g is $\mu_{\mathrm{O}(n)}$-distributed (see [97] for a proof). The same method to generate an element of $\mathrm{U}(n)$ applies with, this time, $g_{jk} = a_{jk} + \mathrm{i}b_{jk}$, and all a_{jk}'s and b_{jk}'s standard independent normal variables (the distribution of g is often referred to as the Ginibre ensemble).

Another way to proceed, through a decomposition of any element of G as a product of independent reflections, is detailed in Chapters 1 and 2.

Let us now consider especially the unitary group $\mathrm{U}(n)$. The Haar measure induces a measure on the eigenvalues $(e^{\mathrm{i}\theta_1}, \dots, e^{\mathrm{i}\theta_n})$ of a unitary matrix. More precisely, the following formula holds (see [25] for a proof based on the theory of Lie groups and Chapter 1 for a probabilistic proof).

The Weyl integration formula. *With the previous notations, the joint distribution of* $(\theta_1, \dots, \theta_n)$ *is given by*

$$\mu_{\mathrm{U}(n)}(\mathrm{d}\theta_1, \dots, \mathrm{d}\theta_n) = \frac{1}{(2\pi)^n n!} \prod_{1 \leqslant j < k \leqslant n} \left| e^{\mathrm{i}\theta_j} - e^{\mathrm{i}\theta_k} \right|^2 \mathrm{d}\theta_1 \dots \mathrm{d}\theta_n.$$

Similar formulas exist for other compact groups (see e.g. Chapter 5 Section 1). The interest in densities of eigenvalues as in the above Theorem began in the 50's : Wigner's idea was to model the resonance lines of heavy nucleus by the spectrum of large random matrices, because explicit analytic calculations were impossible. His questions about the local (scaled) distribution of the eigenvalues for the above potential were answered by Gaudin [56] and Gaudin-Mehta [57]. Their results surprisingly also appear in the repulsion of the zeros of ζ, as discovered thanks to the following interactions between Montgomery and Dyson.

1. Correlations for the zeros : the Montgomery-Dyson encounter

We write $1/2 \pm \mathrm{i}t_n$ for all non-trivial roots of ζ, with $0 < \mathfrak{Re}(t_1) \leqslant \mathfrak{Re}(t_2) \dots$ and $w_n = \frac{\mathfrak{Re}(t_n)}{2\pi} \log \frac{\mathfrak{Re}(t_n)}{2\pi}$. Then

$$\frac{\mathcal{N}(x)}{x} \underset{x\to\infty}{\longrightarrow} 1,$$

where $\mathcal{N}(x) = |n \ : \ w_n < x|$: the mean spacing between consecutive t_n's tends to 0 and the mean spacing between w_n's tends to 1. This is a key step for an analytic proof of the prime number theorem, shown independently by Hadamard and de la Vallée Poussin in 1896 (see e.g. [135] for the complete proof). A more precise comprehension of the zeta zeros relies on the study of the pair correlation

$$\frac{1}{x}\left|(w_n, w_m) \in [0,x]^2 : \ \alpha \leqslant w_n - w_m \leqslant \beta\right|,$$

and more generally the operator

$$\mathrm{R}_2(f,x) = \frac{1}{x} \sum_{1\leqslant j,k\leqslant \mathcal{N}(x), j\neq k} f(w_j - w_k).$$

In the following, f is a localization function : this is a Schwartz function on $\mathbb{R}$ with $\hat{f}$ having compact support. The inverse Fourier transform of a $\mathscr{C}^\infty$ function with compact support is a localization function.

If the spaces between consecutive w_k's were uniformly distributed, $\mathrm{R}_2(f,x)$ would converge to $\int_{-\infty}^{\infty} \mathrm{d}y f(y)$ as $x \to \infty$. This is not the case, the following result showing repulsion between successive w_k's.

Theorem (Montgomery [100])**.** *Suppose f is a localization function with Fourier transform supported in $(-1,1)$. Then, assuming the Riemann Hypothesis,*

$$\mathrm{R}_2(f,x) \underset{x\to\infty}{\longrightarrow} \int_{-\infty}^{\infty} \mathrm{d}y f(y) \mathrm{R}_2(y) \tag{0.2}$$

with

$$\mathrm{R}_2(y) = 1 - \left(\frac{\sin \pi y}{\pi y}\right)^2.$$

This result is expected to hold with no restriction on the support of $\hat{f}$: this is Montgomery's conjecture, which perfectly agrees with Odlyzko's numerical experiments ([104], [105]). As shown by Goldston and Montgomery [60], this conjecture has a number theoretic equivalent, in terms of second moment for primes in short intervals.

What is the link with Random Matrix Theory ? In 1972, at the Princeton Institute for Advanced Studies, Montgomery and some colleagues stopped working for afternoon tea. There, he was introduced to the quantum physicist Freeman Dyson by Indian number theorist Daman Chowla. Montgomery, then a graduate student, described his model for the pair correlation of zeros of ζ, and Dyson recognized the pair correlation of eigenvalues of a unitary matrix[2]... The connection was born !

2. More precisely the link appeared with matrices from the GUE : its pair correlation asymptotically coincides with the one from Haar-distributed unitary matrices.

More precisely, let $(e^{\mathrm{i}\theta_1}, \dots, e^{\mathrm{i}\theta_n})$ be the eigenvalues of an element $u \in \mathrm{U}(n)$. We write

$$\varphi_k = \frac{n}{2\pi}\theta_k$$

for the normalized angles (θ_k is considered modulo 2π, but not φ_k). Then it can be shown that

$$\frac{1}{n}\int_{\mathrm{U}(n)} \mathrm{d}\mu_{\mathrm{U}(n)}(u)\,|(\ell, m) : \alpha < \varphi_\ell - \varphi_m < \beta, \ell \neq m| \underset{n\to\infty}{\longrightarrow} \int_\alpha^\beta \mathrm{d}y \left(1 - \left(\frac{\sin \pi y}{\pi y}\right)^2\right),$$

and, more generally, for a continuous function f with compact support, the asymptotic law of the pair correlation for eigenvalues in $\mathrm{U}(n)$ is given by

$$\frac{1}{n}\int_{\mathrm{U}(n)} \sum_{j\neq k} f(\varphi_j - \varphi_k)\mathrm{d}\mu_{\mathrm{U}(n)}(u) \underset{n\to\infty}{\longrightarrow} \int_{-\infty}^{\infty} \mathrm{d}y f(y)\mathrm{R}_2(y). \tag{0.3}$$

The similarity between formulas (0.2) and (0.3) suggests a strong connection between the zeros of the Riemann zeta function and the spectra of the random unitary matrices. Montgomery's result was extended in the following directions. Hejhal [67] proved that the triple correlations of the zeta zeros coincide with those of large Haar-distributed unitary matrices ; Rudnick and Sarnak [118] then showed that all correlations agree ; all those results are restricted to the condition that the Fourier transform of f is supported on some compact set. The proofs are based on the *explicit formula* that allows to express the correlations in terms of sums over prime numbers (see Proposition 2.1 in [118]). Moreover, Rudnick and Sarnak showed the asymptotic correlations for more general L-functions, $\mathrm{L}(s, f)$ where f is an automorphic cusp-form for $\mathrm{GL}_m/\mathbb{Q}$.

The depth of this analogy between eigenvalues and zeta zeros correlations needs to be slightly moderated : Bogomolny and Keating [14] showed that in the second order the two-point correlation function depends on the positions of the low Riemann zeros, something that clearly contrasts with random matrix theory. In the same vein, Berry and Keating [11] gave a clear explanation of a similar phenomenon in the number variance statistics first observed by Berry [10].

2. The problem of moments : the Keating-Snaith conjecture

2.1. The moments of ζ

The k^{th} moment of the zeta function, $\mathrm{I}_k(\mathrm{T})$, has been extensively studied, but its equivalent is well known in only two cases (see e.g. Titchmarsh [135] for proofs) : Hardy and Littlewood [65] showed in 1918 that

$$\mathrm{I}_1(\mathrm{T}) \underset{\mathrm{T}\to\infty}{\sim} \log \mathrm{T}$$

and Ingham [73] proved in 1926 that

$$\mathrm{I}_2(\mathrm{T}) \underset{\mathrm{T}\to\infty}{\sim} \frac{1}{2\pi^2}(\log \mathrm{T})^4.$$

For $k \geqslant 3$, no equivalent of $\mathrm{I}_k(\mathrm{T})$ has been proved. Keating and Snaith [80] have formulated the following conjecture about the asymptotics of $\mathrm{I}_k(\mathrm{T})$, which agrees with the 2^{nd} and 4^{th} moments above, and the conjectured 6^{th} and 8^{th} moments (see respectively Conrey and Ghosh [30], Conrey and Gonek [31])

The Keating-Snaith conjecture. *For every $k \in \mathbb{N}^*$*

$$\mathrm{I}_k(\mathrm{T}) \underset{\mathrm{T}\to\infty}{\sim} \mathrm{H}_{\mathrm{Mat}}(k)\mathrm{H}_{\mathcal{P}}(k)(\log \mathrm{T})^{k^2}$$

with the following notations :

- *the arithmetic factor*

$$\mathrm{H}_{\mathcal{P}}(k)=\prod_{p\in\mathcal{P}}\left(1-\frac{1}{p}\right)^{k^2}{}_2\mathrm{F}_1\left(k,k,1,\frac{1}{p}\right);$$

- *the matrix factor*

$$\mathrm{H}_{\mathrm{Mat}}(k)=\prod_{j=0}^{k-1}\frac{j!}{(j+k)!}. \tag{0.4}$$

Note that the above term makes sense for real k (it can be expressed by means of the Barnes function) so more generally the Keating-Snaith conjecture is originally stated for $\mathfrak{Re}(k)\geqslant -1/2$.

Here are some first steps to understand the origins of this conjecture. First suppose that $\sigma>1$. Then the absolute convergence of the Euler product and the linear independence of the $\log p$'s ($p\in\mathcal{P}$) over $\mathbb{Q}$ allow to show that

$$\frac{1}{\mathrm{T}}\int_0^{\mathrm{T}}\mathrm{d}s\,|\zeta(\sigma+\mathrm{i}s)|^{2k}\underset{\mathrm{T}\to\infty}{\sim}\prod_{p\in\mathcal{P}}\frac{1}{\mathrm{T}}\int_0^{\mathrm{T}}\frac{\mathrm{d}s}{\left|1-\frac{1}{p^s}\right|^{2k}}\underset{\mathrm{T}\to\infty}{\longrightarrow}\prod_{p\in\mathcal{P}}{}_2\mathrm{F}_1\left(k,k,1,\frac{1}{p^{2\sigma}}\right). \tag{0.5}$$

This asymptotic independence of the factors corresponding to distinct primes gives the intuition of a part of the arithmetic factor. Note that this equivalent of the k-th moment is guessed to hold also for $1/2<\sigma\leqslant 1$, which would imply the Lindelöf hypothesis (see Titchmarsh [135]).

Moreover, the factor $(1-1/p)^{k^2}$ can be interpreted as a compensator to allow the RHS in (0.5) to converge on $\sigma=1/2$. Note that $\mathrm{H}_{\mathcal{P}}(k)$ appears in a recipe to conjecture moments of L-functions, given in [32]. Moreover, the Riemann zeta function on the critical axis ($\mathfrak{Re}s=1/2,\mathfrak{Im}s>0$) is the (not absolutely) convergent limit of the partial sums

$$\zeta_n(s)=\sum_{k=1}^{n}\frac{1}{k^s}.$$

Conrey and Gamburd [29] showed that

$$\lim_{n\to\infty}\lim_{\mathrm{T}\to\infty}\frac{1}{\mathrm{T}(\log n)^{k^2}}\int_0^{\mathrm{T}}\left|\zeta_n\left(\frac{1}{2}+\mathrm{i}t\right)\right|^{2k}\mathrm{d}t=\mathrm{H}_{\mathrm{Sq}}(k)\mathrm{H}_{\mathcal{P}}(k),$$

where $\mathrm{H}_{\mathrm{Sq}}(k)$ is a factor distinct from $\mathrm{H}_{\mathrm{Mat}}(k)$ and appearing in counting magic squares. So the arithmetic factor appears when considering the moments of the partial sums.

This matrix factor, which is coherent with numerical experiments, comes from Keating and Snaith's idea [80] (enforced by Montgomery's theorem) that a good approximation for the zeta function is the determinant of a unitary matrix. Thanks to Selberg's integral[3] formula [3], they have calculated the generating function for the determinant of a $n\times n$ random unitary matrix ($\mathrm{Z}_n(u,\varphi)=\det(\mathrm{Id}-e^{-\mathrm{i}\varphi}u)$) with respect to the Haar measure :

$$\mathrm{P}_n(s,t)=\mathbb{E}_{\mu_{\mathrm{U}(n)}}\left(|\mathrm{Z}_n(u,\varphi)|^t e^{\mathrm{i}s\arg\mathrm{Z}_n(u,\varphi)}\right)=\prod_{j=1}^{n}\frac{\Gamma(j)\Gamma(t+j)}{\Gamma(j+\frac{t+s}{2})\Gamma(j+\frac{t-s}{2})}. \tag{0.6}$$

Note that a similar result was given by Brézin and Hikami [24] for the GUE. This closed form implies in particular

$$\mathbb{E}_{\mu_{\mathrm{U}(n)}}\left(|\mathrm{Z}_n(u,\varphi)|^{2k}\right)\underset{n\to\infty}{\sim}\mathrm{H}_{\mathrm{Mat}}(k)n^{k^2}.$$

3. More about the Selberg integrals and its numerous applications can be found in [50]

This led Keating and Snaith to introduce this matrix factor in the conjectured asymptotics of $\mathrm{I}_k(\mathrm{T})$. This matrix factor is supposed to be universal : it should for example appear in the asymptotic moments of Dirichlet L-functions.

However, these explanations are not sufficient to understand clearly how these arithmetic and matrix factors must be combined to get the Keating-Snaith conjecture. A clarification of this point is the purpose of the two following paragraphs.

Remark. As observed in Chapter 1, the scission on the Mellin-Fourier transform in formula (0.6) has the following probabilistic meaning : $\det(\mathrm{Id}-e^{-\mathrm{i}\varphi}u)$ has the same law as a product of n independent random variables. This will be proven by probabilistic means in Chapter 1 and extended to other compact groups in Chapter 2. An analogue of this scission in independent random variables for ζ remains mysterious.

2.2. Understanding the conjecture : the hybrid model

Gonek, Hughes and Keating [61] gave a partial justification for the Keating-Snaith conjecture based on a particular factorization of the zeta function.

More precisely, let $s=\sigma+\mathrm{i}t$ with $\sigma\geqslant 0$ and $|t|\geqslant 2$, let $\mathrm{X}\geqslant 2$ be a real parameter, and let K be any fixed positive integer. Let $u(x)$ be a nonnegative $\mathscr{C}^\infty$ function of mass 1, supported on $[e^{1-1/\mathrm{X}},e]$, and set $\mathrm{U}(z)=\int_0^\infty u(x)\mathrm{E}_1(z\log x)\mathrm{d}x$, where $\mathrm{E}_1(z)$ is the exponential integral $\int_z^\infty (e^{-w}/w)\ \mathrm{d}w$. Let also

$$\mathrm{P}_\mathrm{X}(s)=\exp\left(\sum_{n\leqslant \mathrm{X}}\frac{\Lambda(n)}{n^s\log n}\right)$$

where Λ is Van Mangoldt's function ($\Lambda(n)=\log p$ if n is an integral power of a prime p, 0 otherwise), and

$$\mathrm{Z}_\mathrm{X}(s)=\exp\left(-\sum_{\rho_n}\mathrm{U}\left((s-\rho_n)\log \mathrm{X}\right)\right)$$

where $(\rho_n,n\geqslant 0)$ are the imaginary parts of the zeros of ζ. Then the following unconditional result was proved in [61].

Theorem. *With the previous notations,*

$$\zeta(s)=\mathrm{P}_\mathrm{X}(s)\mathrm{Z}_\mathrm{X}(s)\left(1+\mathrm{O}\left(\frac{\mathrm{X}^{\mathrm{K}+2}}{(|s|\log \mathrm{X})^{\mathrm{K}}}\right)+\mathrm{O}\left(\mathrm{X}^{-\sigma}\log \mathrm{X}\right)\right),$$

where the constants in front of the O *only depend on the function* u *and* K.

The P_X term corresponds to the *arithmetic* factor of the Keating-Snaith conjecture, while the Z_X term corresponds to the *matrix* factor. More precisely, this decomposition suggests a proof for the Keating-Snaith conjecture along the following steps.

First, for a value of the parameter X chosen such that $\mathrm{X}=\mathrm{O}(\log(\mathrm{T})^{2-\varepsilon})$ here and in the sequel, the following conjecture A suggests that the moments of zeta are well approximated by the product of the moments of P_X and Z_X (they are sufficiently *independent*).

Conjecture A : the splitting hypothesis. With the previous notations

$$\frac{1}{\mathrm{T}}\int_0^{\mathrm{T}}\mathrm{d}s\left|\zeta\left(\frac{1}{2}+\mathrm{i}s\right)\right|^{2k} \underset{\mathrm{T}\to\infty}{\sim}\left(\frac{1}{\mathrm{T}}\int_0^{\mathrm{T}}\mathrm{d}s\left|\mathrm{P}_\mathrm{X}\left(\frac{1}{2}+is\right)\right|^{2k}\right)\left(\frac{1}{\mathrm{T}}\int_0^{\mathrm{T}}\mathrm{d}s\left|\mathrm{Z}_\mathrm{X}\left(\frac{1}{2}+is\right)\right|^{2k}\right)$$

Assuming that conjecture A is true, we then need to approximate the moments of $\mathrm{P_X}$ and $\mathrm{Z_X}$. The moments of $\mathrm{P_X}$ were evaluated in [61], where the following result is proven.

Theorem. *With the previous notations,*

$$\frac{1}{\mathrm{T}}\int_0^{\mathrm{T}} \mathrm{d}s \left|\mathrm{P_X}\left(\frac{1}{2}+\mathrm{i}s\right)\right|^{2k} = \mathrm{H}_{\mathcal{P}}(k)\ (e^{\gamma}\log \mathrm{X})^{k^2}\left(1+\mathrm{O}\left(\frac{1}{\log \mathrm{X}}\right)\right).$$

Finally, an additional conjecture about the moments of $\mathrm{Z_X}$, if proven, would be the last step to prove the Keating-Snaith conjecture.

Conjecture B : the moments of $\mathrm{Z_X}$. With the previous notations,

$$\frac{1}{\mathrm{T}}\int_0^{\mathrm{T}} \mathrm{d}s \left|\mathrm{Z_X}\left(\frac{1}{2}+\mathrm{i}s\right)\right|^{2k} \underset{t\to\infty}{\sim} \mathrm{H_{Mat}}(k)\ \left(\frac{\log \mathrm{T}}{e^{\gamma}\log \mathrm{X}}\right)^{k^2}.$$

The reasoning which leads to conjecture B is the following. First of all, the function $\mathrm{Z_X}$ is not as complicated as it seems, because as X tends to ∞, the function u tends to the Dirac measure at point e, so

$$\mathrm{Z_X}\left(\frac{1}{2}+\mathrm{i}t\right) \approx \prod_{\rho_n}\left(\mathrm{i}(t-\rho_n)e^{\gamma}\log \mathrm{X}\right).$$

The ordinates ρ_n (where ζ vanishes) are supposed to have many statistical properties identical to those of the eigenangles of a random element of $\mathrm{U}(n)$. In order to make an adequate choice for n, we recall that the γ_n are spaced $2\pi/\log \mathrm{T}$ on average, whereas the eigenangles have average spacing $2\pi/n$: thus n should be chosen to be the greatest integer less than or equal to $\log \mathrm{T}$. Then the calculation for the moments of this model leads to conjecture B.

2.3. Understanding the conjecture : the multiple Dirichlet series

Diaconu, Goldfeld and Hoffstein [43] proposed an explanation of the Keating-Snaith conjecture relying only on a supposed meromorphy property of the multiple Dirichlet series

$$\int_1^{\infty} \zeta(s_1+\varepsilon_1\mathrm{i}t)\dots\zeta(s_{2m}+\varepsilon_{2m}\mathrm{i}t)\left(\frac{2\pi e}{t}\right)^{k\mathrm{i}t} t^{-w}\mathrm{d}t,$$

with $w, s_k \in \mathbb{C}$, $\varepsilon_k = \pm 1$ $(1 \leqslant k \leqslant 2m)$. They make no use of any analogy with random matrices to predict the moments of ζ, and recover the Keating-Snaith conjecture. Important tools in their method are a group of approximate functional equations for such multiple Dirichlet series and a Tauberian theorem to connect the asymptotics as $w \to 1^+$ and the moments $\int_1^{\mathrm{T}}$.

An intriguing question is whether their method applies or not to predict the joint moments of ζ,

$$\frac{1}{\mathrm{T}}\int_1^{\mathrm{T}} \mathrm{d}t \prod_{j=1}^{\ell}\left|\zeta\left(\frac{1}{2}+\mathrm{i}(t+s_j)\right)\right|^{2k_j}$$

with $k_j \in \mathbb{N}^*$, $(1 \leqslant j \leqslant \ell)$, the s_j's being distinct elements in $\mathbb{R}$. If such a conjecture could be stated, independently of any considerations about random matrices, this would be an accurate test for the correspondence between random matrices and L-functions. One could compare if the conjecture perfectly agrees with the analogue result on the unitary group, which is a special case of the Fisher-Hartwig asymptotics of Toeplitz determinants and first proven by Widom [139] :

$$
\begin{aligned}
&\mathbb{E}_{\mu_{\mathrm{U}(n)}}\left(\prod_{j=1}^{\ell}\left|\det(\mathrm{Id}-e^{\mathrm{i}\varphi_j}u)\right|^{2k_j}\right)\\
\underset{n\to\infty}{\sim}&\prod_{1\leqslant i<j\leqslant \ell}|e^{\mathrm{i}\varphi_i}-e^{\mathrm{i}\varphi_j}|^{-2k_ik_j}\prod_{j=1}^{\ell}\mathbb{E}_{\mu_{\mathrm{U}(n)}}\left(\left|\det(\mathrm{Id}-e^{\mathrm{i}\varphi_j}u)\right|^{2k_j}\right)\\
\underset{n\to\infty}{\sim}&\prod_{1\leqslant i<j\leqslant \ell}|e^{\mathrm{i}\varphi_i}-e^{\mathrm{i}\varphi_j}|^{-2k_ik_j}\prod_{j=1}^{\ell}\mathrm{H}_{\mathrm{Mat}}(k_j)n^{k_j^2},
\end{aligned}
$$

the φ_j's being distinct elements modulo 2π.

Note that the joint moments of ζ have already been conjectured by Conrey, Farmer, Keating, Rubinstein, and Snaith [32]. Their formula is remarkable in the sense that it gives not only the leading order but also a full expansion of the joint moments. Their conjecture was found thanks to an analogue in random matrix theory, and is expressed in terms of multiple integrals ; the link with the Fisher-Hartwig asymptotics is not obvious in this form.

3. Gaussian fluctuations

3.1. Central limit theorems.

The law of a unitary matrix determinant gives another example of similarity between Random Matrix Theory and the zeta function. First, we recall the following result due to Selberg in the 40's and popularized in the 80's.

Theorem (Selberg [120]). *Let ω be a uniform random variable on $(0,1)$. Then*

$$
\frac{\log\zeta\left(\frac{1}{2}+\mathrm{i}\omega\mathrm{T}\right)}{\sqrt{\frac{1}{2}\log\log\mathrm{T}}}\xrightarrow{\text{law}}\mathcal{N}_1+\mathrm{i}\mathcal{N}_2
$$

as $\mathrm{T}\to\infty$, with $\mathcal{N}_1$ and $\mathcal{N}_2$ independent standard real normal variables (here, and in the following, $\log\zeta$ is a continuous determination of the complex logarithm of ζ, precisely defined at the beginning of Chapter 6).

This theorem admits a cousin concerning the characteristic polynomial of a generic unitary matrix $u_n\in\mathrm{U}(n)$, and originally proven after its number theoretic counterpart. Precisely, we define

$$
\log\mathrm{Z}(u_n,\varphi)=\log\det\left(\mathrm{Id}_n-e^{-\mathrm{i}\varphi}u\right)=\sum_{k=1}^{n}\log\left(1-\mathrm{e}^{\mathrm{i}(\theta_k-\varphi)}\right),\tag{0.7}
$$

where $e^{\mathrm{i}\theta_1},\dots,e^{\mathrm{i}\theta_n}$ are the eigenvalues of u_n, and the logarithms on the RHS are defined as the almost surely convergent series $\log(1-x)=-\sum_{j\geqslant1}\frac{x^j}{j}$, $|x|\leqslant1, x\neq1$.

Theorem (Keating-Snaith [80]). *If $u_n\sim\mu_{\mathrm{U}(n)}$, then*

$$
\frac{\log\mathrm{Z}(u_n,\varphi)}{\sqrt{\frac{1}{2}\log n}}\xrightarrow{\text{law}}\mathcal{N}_1+\mathrm{i}\mathcal{N}_2
$$

as $n\to\infty$, with $\mathcal{N}_1$ and $\mathcal{N}_2$ independent standard real normal variables.

Note that the limit does not depend on φ : indeed, by definition, the Haar measure is invariant under the action of the unitary group (especially under the action of $e^{\mathrm{i}\varphi}\mathrm{Id}_n$), so the law of $\mathrm{Z}(u_n,\varphi)$ is the same as the law of $\mathrm{Z}(u_n,0)$.

The Keating-Snaith central limit theorem admits an easy proof from the decomposition as a product of independent random variables shown in Chapter 1. Such a decomposition holds for a wider class of potentials for the eigenvalues on the unit circle, for which similar central limit theorems, with speed of convergence, easily follow (see Chapters 1, 2, 4 and 5 for more details).

In the hope to transform the *static* limit in law of Selberg's central limit theorem into a *dynamic* one (i.e. : showing the convergence in law of a sequence of random processes towards another random process), Hughes, Nikeghbali, Yor [72] defined, for any fixed $\lambda > 0$,

$$\mathcal{L}_\lambda(\omega, n) = \frac{\log \zeta\left(\frac{1}{2} + \mathrm{i}\omega e^{n^\lambda}\right)}{\sqrt{\frac{1}{2}\log n}}.$$

Then, if ω is uniformly distributed on $[0,1]$, $\mathcal{L}_\lambda(u,n) \overset{\text{law}}{\longrightarrow} \mathcal{N}_\lambda$, from Selberg's theorem, where $\mathcal{N}_\lambda$ is a complex Gaussian variable with density

$$\mathbb{P}(\mathcal{N}_\lambda \in \mathrm{d}x\mathrm{d}y) = \frac{1}{2\pi\lambda} e^{-\frac{x^2+y^2}{2\lambda}} \mathrm{d}x\mathrm{d}y. \tag{0.8}$$

As λ varies, there is the following multidimensional extension of Selberg's central limit theorem.

Theorem (Hughes, Nikeghbali, Yor [72]). *Let ω be uniform on $[0,1]$. Then, for $0 < \lambda_1 < \lambda_2 < \dots < \lambda_k$,*

$$(\mathcal{L}_{\lambda_1}(\omega,n), \dots, \mathcal{L}_{\lambda_k}(\omega,n)) \overset{\text{law}}{\longrightarrow} (\mathcal{N}_{\lambda_1}, \dots, \mathcal{N}_{\lambda_k})$$

as $n \to \infty$, which means the convergence of the finite dimensional distributions of $\{\mathcal{L}_\lambda(\omega,n), \lambda > 0\}$ towards those of $\{\mathcal{N}_\lambda, \lambda > 0\}$, which denotes the totally disordered Gaussian process : its components are all independent, $\mathcal{N}_\lambda$ being distributed as indicated in (0.8).

Note that the process $(\mathcal{N}_\lambda, \lambda > 0)$ does not admit a measurable version. Assuming the contrary we would obtain by Fubini, for every $a < b$, $\mathbb{E}\left((\int_a^b \mathrm{d}\lambda\, \mathcal{N}_\lambda)^2\right) = 0$. Thus $\mathcal{N}_\lambda = 0$, $\mathrm{d}\lambda$ a. s., which is absurd.

Moreover, a totally disordered Gaussian process has also been encountered by Hugues, Keating and O'Connell in [71]. They have proven that

$$\frac{\log \mathrm{Z}(u_n, \varphi)}{\sqrt{\frac{1}{2}\log n}}$$

converges weakly towards a totally disordered process $\mathrm{X}(\varphi) + \mathrm{i}\mathrm{Y}(\varphi)$ with covariance

$$\mathbb{E}\left(\mathrm{X}(\varphi_1)\mathrm{X}(\varphi_2)\right) = \mathbb{E}\left(\mathrm{Y}(\varphi_1)\mathrm{Y}(\varphi_2)\right) = \delta(\varphi_1 - \varphi_2).$$

3.2. Counting the zeros.

The above central limit theorems have consequences for the counting of zeros of ζ in the critical strip, or of eigenvalues of u_n on arcs of the circle. First, write $\mathrm{N}(t)$ for the number of non-trivial zeros z of ζ with $0 < \mathfrak{Im} z \leqslant t$, counted with multiplicity. Then (see e.g. [135])

$$\mathrm{N}(t) = \frac{t}{2\pi}\log\frac{t}{2\pi e} + \frac{1}{\pi}\mathfrak{Im}\log\zeta\left(1/2 + \mathrm{i}t\right) + \frac{7}{8} + \mathrm{O}\left(\frac{1}{t}\right).$$

with $\mathfrak{Im}\log\zeta\left(1/2+\mathrm{i}t\right) = \mathrm{O}(\log t)$. Hence Selberg's central limit theorem implies that,

$$\frac{\mathrm{N}(\omega t) - \frac{\omega t}{2\pi}\log\frac{\omega t}{2\pi e}}{\frac{1}{\pi}\sqrt{\frac{1}{2}\log\log t}}$$

converges in law to a standard Gaussian random variable as $t \to \infty$, ω being uniform on $(0,1)$. This result was extended by Fujii [55] to count the number of zeros in a shorter interval : writing

$$\Delta(t_1,t_2) = (\mathrm{N}(t_2) - \mathrm{N}(t_1)) - \left(\frac{t_2}{2\pi}\log\frac{t_2}{2\pi e} - \frac{t_1}{2\pi}\log\frac{t_1}{2\pi e}\right),$$

which represents the fluctuations of the number of zeros z ($t_1 < \mathfrak{Im} z \leqslant t_2$) minus its expectation, he showed among other results that for any constant $c > 0$

$$\frac{\Delta(\omega t, \omega t + c)}{\frac{1}{\pi}\sqrt{\log\log t}} \xrightarrow{\text{law}} \mathcal{N}$$

as $t \to \infty$. Note the very small normalization $\sqrt{\log\log t}$: this indicates the repulsion of the zeros. Central limit theorems in counting the number of eigenvalues of random matrices in domains were shown by Soshnikov [130] in great generality and by Wieand, in particular through her following important result, then extended by Diaconis and Evans [40]. We write $\mathrm{N}_n(\alpha,\beta)$ for the number of eigenvalues $e^{\mathrm{i}\theta}$ of $u_n \sim \mu_{\mathrm{U}(n)}$ with $0 \leqslant \alpha < \theta < \beta < 2\pi$.

Theorem (Wieand [140])**.** *As $n \to \infty$, the finite-dimensional distributions of the process*

$$\frac{\mathrm{N}_n(\alpha,\beta) - \mathbb{E}\left(\mathrm{N}_n(\alpha,\beta)\right)}{\frac{1}{\pi}\sqrt{\log n}},\ 0 \leqslant \alpha < \beta < 2\pi,$$

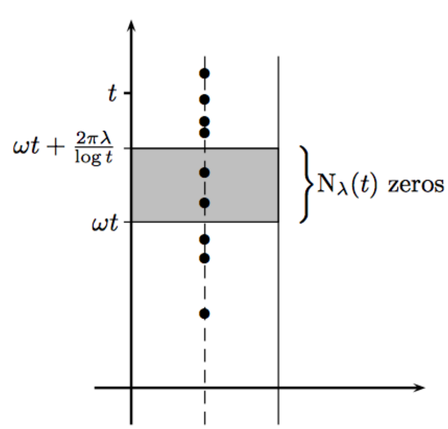
converge to those of a Gaussian process $\{\delta(\alpha,\beta), 0 \leqslant \alpha < \beta < 2\pi\}$ with covariance function

$$\mathbb{E}\left(\delta(\alpha,\beta)\delta(\alpha',\beta')\right) = \begin{cases} 1 & \text{if } \alpha = \alpha' \text{ and } \beta = \beta' \\ 1/2 & \text{if } \alpha = \alpha' \text{ and } \beta \neq \beta' \\ 1/2 & \text{if } \alpha \neq \alpha' \text{ and } \beta = \beta' \\ -1/2 & \text{if } \beta = \alpha' \\ 0 & \text{elsewhere} \end{cases}.$$

The above correlation structure is surprising : it states for example that the deviations from the mean are uncorrelated if the arc (α,β) is strictly included in (α',β'), and positively correlated if this inclusion is not strict. Coram and Diaconis [33] made numerical experiments to give evidence that the same covariance appears when counting the zeros of ζ in distinct intervals. This is proved in Chapter 6.

Concerning counting of zeta zeros in much shorter intervals, limits in law seem out of reach with today's techniques. The mean spacing between successive zeta zeros z ($0 \leqslant \mathfrak{Im}(z) \leqslant t$) is asymptotically $2\pi/\log t$. For fixed $\lambda > 0$ and ω uniform on (0,1), it is therefore realistic to expect that the integer-valued random variables

$$\mathrm{N}_\lambda(t) = \mathrm{N}\left(\omega t + \frac{2\pi\lambda}{\log t}\right) - \mathrm{N}(\omega t)$$



converge weakly as $t \to \infty$. If the zeta zeros were independent and uniform on the critical axis, then $\mathrm{N}_\lambda(t)$ would converge in law to a Poisson variable ; this is not the case, as shown by Fujii in [54] : this indicates the existence of correlations between the zeros. Furthermore, he conjectured that $\mathrm{N}_\lambda(t)$ converges weakly to the Gram distribution, described below (see [54] for an historical overview of this problem and the origins of the Gram distribution). Consider the eigenvalues λ_j's of the operator

$$\{f(y), -1 \leqslant y \leqslant 1\} \mapsto \left\{ \int_{-1}^{1} \frac{\sin((y-x)\pi\lambda)}{(y-x)\pi\lambda} f(x)\mathrm{d}x, -1 \leqslant y \leqslant 1 \right\}$$

and note, for any $k \in \mathbb{N}$,

$$p_\lambda(k) = \left(\prod_{j \geqslant 0} (1 - \lambda_j) \right) \left(\sum_{j_1 < \dots < j_k} \prod_{\ell=1}^{k} \frac{\lambda_{j_\ell}}{1 - \lambda_{j_\ell}} \right).$$

Then it can be shown that $\sum k p_\lambda(k) = \lambda$ (see Bohigas and Giannoni [15]) : the above probabilities define a positive random variable with expectation λ. Fujii's conjecture is

$$\mathbb{P}(\mathrm{N}_\lambda(t) = k) \underset{t \to \infty}{\longrightarrow} p_\lambda(k) \tag{0.9}$$

for any $k \geqslant 0$. This means that $\mathrm{N}_\lambda(t)$ weakly converges to the so-called Gram distribution, which appears when counting the eigenvalues of the GUE or $\mathrm{U}(n)$ matrices in short intervals, as shown by Mehta and des Cloizeaux [95] : this is the origin of Fujii's conjecture. Note that equation (0.9) exactly means that $\mathrm{N}_\lambda(t)$ converges in law to a random variable

$$\mathrm{N}_\lambda(\infty) \overset{\text{law}}{=} \sum_{j=0}^{\infty} \mathrm{X}_j$$

where the X_j's are independent, and X_j is a Bernoulli random variable with parameter λ_j. For general results about the decomposition of the counting function as a sum of independent random variables, see [9].

The mesoscopic fluctuations (Chapter 6) and microscopic fluctuations (Fujii's results) of the zeta zeros are of different nature : the first ones are governed by Selberg's results and the second ones require precise conjectures of the same nature as Montgomery's. However, the repulsion of the zeros appears in both cases : Selberg's central limit theorem requires a remarkably small normalization and the Gram law has a small variance compared to Poisson variables.

4. Over function fields : results by Katz-Sarnak

The following discussion presents some of the numerous results by Katz and Sarnak [78] on consecutive spacings of zeros of Artin L-functions. Much more can be found in [79], which is the main source of the following lines.

Among other types of L-functions, Katz and Sarnak considered the zeta function of a curve over the finite field with q elements, $\mathbb{F}_q$. More precisely, let $\mathrm{F}(\mathrm{X}, \mathrm{Y}, \mathrm{Z})$ be a homogeneous polynomial of degree d and nonsingular (i.e. its first partial derivatives have no common zero in $\overline{\mathbb{F}}_q$). Then the projective plane of equation

$$\mathrm{F} = 0 \tag{0.10}$$

in P^2 is a curve, noted $\mathrm{C}/\mathbb{F}_q$, with genus $g = (d-1)(d-2)/2$ (see e.g. [107] for a precise definition of the genus and a proof of this formula). For $n \geqslant 1$, let N_n be the number of projective solutions of (0.10) in $\mathbb{F}_{q^n}$. Then the zeta function of $\mathrm{C}/\mathbb{F}_q$ is the formal series in T

$$\zeta(\mathrm{T}, \mathrm{C}/\mathbb{F}_q) = \exp\left(\sum_{n=1}^{\infty} \frac{\mathrm{N}_n \mathrm{T}^n}{n} \right).$$

Thanks to this geometrical definition, it can be shown that

$$\zeta(\mathrm{T}, \mathrm{C}/\mathbb{F}_q) = \frac{\mathrm{P}(\mathrm{T})}{(1-\mathrm{T})(1-q\mathrm{T})}$$

with $\mathrm{P} \in \mathbb{Z}[\mathrm{T}]$ a polynomial with degree $2g$. Moreover, the Riemann-Roch theorem on the curve $\mathrm{C}/\mathbb{F}_q$ plays the role of Poisson summation formula and yields the functional equation $\mathrm{P}(\mathrm{T}) = q^g \mathrm{T}^{2g}\, \mathrm{P}(1/q\mathrm{T})$. There is a Riemann hypothesis for $\zeta(\mathrm{T}, \mathrm{C}/\mathbb{F}_q)$, which asserts that all the (complex) zeros lie on the circle $|\mathrm{T}| = 1/\sqrt{q}$. This was proven by Weil [138] and extended to a larger class of L-functions by Deligne [38].

To study the distribution of the zeta zeros, we write them as

$$\frac{e^{\mathrm{i}\theta_j}}{\sqrt{q}},\ 0 \leqslant \theta_1 \leqslant \cdots \leqslant \theta_{2g} < 2\pi.$$

Katz and Sarnak consider the k-th consecutive spacing, the measure $\mu_k^{(\mathrm{C}/\mathbb{F}_q)}$ on $[0, \infty)$ defined by

$$\mu_k^{(\mathrm{C}/\mathbb{F}_q)}[a, b] = \frac{\#\left\{1 \leqslant j \leqslant 2d : \frac{d}{\pi}(\theta_{j+k} - \theta_j) \in [a, b]\right\}}{2d}$$

for any $0 \leqslant a < b \leqslant 2\pi$, where the indexes are considered modulo $2g$ and the differences $\theta_{j+k} - \theta_j$ modulo 2π. The factor d/π normalizes μ_k to have mean k. Take a matrix g from the unitary, special orthogonal or symplectic group $\mathrm{G}(n)$, with n eigenvalues $e^{\mathrm{i}\theta_k}$'s ordered as previously. The analogous measure is

$$\mu_k^{(g)}[a, b] = \frac{\#\left\{1 \leqslant j \leqslant 2d : \frac{n}{2\pi}(\theta_{j+k} - \theta_j) \in [a, b]\right\}}{n},$$

giving a weight $1/n$ to each of the normalized spacings. The Kolmogorov-Smirnov distance between two measures is defined by

$$\mathrm{D}(\nu_1, \nu_2) = \sup\{|\nu_1(\mathrm{I}) - \nu_2(\mathrm{I})| : \mathrm{I} \subset \mathbb{R} \text{ an interval}\}.$$

Katz and Sarnak show that for any $k \geqslant 1$ there exists a measure $\mu_k^{(\mathrm{GUE})}$ such that

$$\int \mathrm{D}\left(\mu_k^{(g)}, \mu_k^{(\mathrm{GUE})}\right) \mathrm{d}\mu_{\mathrm{G}(n)}(g) \underset{n\to\infty}{\longrightarrow} 0,$$

the group $\mathrm{G}(n)$ being any of the three ones previously mentioned, $\mu_{\mathrm{G}(n)}$ its Haar probability measure. Hence the consecutive spacings, for the classical compact groups, are asymptotically universal. Note that the measures $\mu_k^{(\mathrm{GUE})}$ are not those of Poisson random variables, corresponding to independent and uniform eigenvalues.

In the following asymptotics of the measures $\mu_k^{(\mathrm{C}/\mathbb{F}_q)}$, the average is taken with respect to the counting measure on $\mathcal{M}_g(\mathbb{F}_q)$, the finite set of $\mathbb{F}_q$-isomorphism classes of curves over $\mathbb{F}_q$ with genus g. More precisely, a morphism $\varphi : \mathrm{C} \to \mathrm{C}'$ between two curves over $\mathbb{F}_q$ has the property that, at each point $\mathrm{P} \in \mathrm{C}$, φ is represented in an open neighborhood of P by homogenous polynomials of the same degree ; an isomorphism is a bijective morphism whose inverse is a morphism.

Theorem (Katz, Sarnak [78])**.** *For any $k \geqslant 1$,*

$$\lim_{g\to\infty} \lim_{q\to\infty} \frac{1}{\#\mathcal{M}_g(\mathbb{F}_q)} \sum_{\mathrm{C}\in\mathcal{M}_g(\mathbb{F}_q)} \mathrm{D}(\mu_k^{(\mathrm{C}/\mathbb{F}_q)}, \mu_k^{(\mathrm{GUE})}) = 0.$$

As all consecutive spacings universally converge to those of the GUE, this theorem states that the zeta functions of curves over finite fields follow, on average over g and q, the Montgomery-Odlyzko law. In the case of fixed q, we do not know if random matrix statistics appear in the limit $g \to \infty$.

The proof of this theorem requires deep algebraic arguments, a little idea of them being given in the following lines. First, an important ingredient is the spectral interpretation of the zeros of $\zeta(\mathrm{T}, \mathrm{C}/\mathbb{F}_q)$ in terms of the Frobenius : N_n is the number of fixed points when raising the coordinates of C in $\overline{\mathbb{F}}_q$ to the power q, iterated n times.

Based on this point of view, Katz and Sarnak show that the geometric monodromy group of this family of curves is the symplectic one (see Theorem 10.1.18.3 in [78] for a precise statement). This allows to use Deligne's equidistribution theorem [38] : it implies that for fixed genus g, the consecutive spacings of the zeros converge on average, as $q \to \infty$, to those of a Haar-distributed element of $\mathrm{USp}(2g)$. As the genus g goes to infinity, the asymptotics of the consecutive spacings are universally the $\mu_k^{(\mathrm{GUE})}$'s, included for the symplectic group, leading to the result.

Katz and Sarnak also give families of curves with other possible monodromy groups, such as $\mathrm{SO}(2n)$. The statistics of the eigenangles closest to 1 are not universal, they depend on the corresponding group. This corresponds to distinct statistical properties of the low-lying zeros of zeta functions of curves. Moreover, they show that for families of Kloosterman sums, the zeros of the corresponding L-functions have the statistical properties of conjugation classes in $\mathrm{USp}(2n)$, $\mathrm{SO}(2n)$ or $\mathrm{SU}(n)$, depending on the considered family (see Theorem 11.10.5 in [78]).

Main results

This PhD thesis represents my work, during the years 2007-2009, at the Université Pierre et Marie Curie under supervision of M. Yor and at the Institut Telecom under supervision of A.S. Üstünel. This led to the following accepted publications.

Mesoscopic fluctuations of the zeta zeros, Probability Theory and Related Fields, Volume 148, Numbers 3-4, 479-500.
Ewens measures on compact groups and hypergeometric kernels, with A. Nikeghbali, A. Rouault, Séminaire de Probabilités XLIII, Lecture Notes in Mathematics, 2011, Volume 2006/2011, 351-377, Springer.
Circular Jacobi ensembles and deformed Verblunsky coefficients, with A. Nikeghbali, A. Rouault, International Mathematics Research Notices, 4357-4394 (2009).
Conditional Haar measures on classical compact groups, Annals of Probability vol 37, Number 4, 1566-1586, (2009).
The characteristic polynomial of a random unitary matrix : a probabilistic approach, with C.P. Hughes, A. Nikeghbali, M. Yor, Duke Mathematical Journal Volume 145, Number 1, 45-69 (2008).
The characteristic polynomial on compact groups with Haar measure : some equalities in law, with A. Nikeghbali, A. Rouault, Comptes Rendus de l'Académie des Sciences, Série I 345, 4, 229-232 (2007).
Euler's formula for $\zeta(2n)$ and products of Cauchy variables, with T. Fujita, M. Yor, Electronic Communications in Probability 12, 73-80 (2007).

This last work is too disjoint from the main topic of this thesis (analogies between random matrices and number theory) to be presented in this document. The following articles also constitute parts of this thesis and were submitted for publication.

The chapters do not follow the chronological order of the original findings, and do not exactly correspond to the distinct publications. Some results, characteristic of this thesis, are summarized below, in three sets of themes, in which we also mention some questions related to this thesis.

First the Haar measure on a compact group can be obtained as a product of independent transformations, which implies a scission in law for the characteristic polynomial and many limit theorems for these random matrix analogues of zeta functions.

This led us then to consider problems about random spectral measures on the unit circle : they are characterized by a set of independent coordinates, and a large deviations principle for spectral measures follows, when the dimension grows. On account of the behavior of such measures *at the edge*, we show that the kernel associated to asymmetric singularities is universally, asymptotically, the so-called hypergeometric kernel.

Finally we give a joint central limit theorem for $\log\zeta$, which implies in particular that a Gaussian process appears at the limit when counting the zeta zeros in distinct intervals.

1. Haar measure and independence

To prove their central limit theorem on $\log \mathrm{Z}_n = \log\det(\mathrm{Id} - u_n), u_n \sim \mu_{\mathrm{U}(n)}$ Keating and Snaith rely on the explicit Mellin-Fourier transform

$$\mathbb{E}_{\mu_{\mathrm{U}(n)}}\left(|\mathrm{Z}_n|^t e^{\mathrm{i}s\arg \mathrm{Z}_n}\right)=\prod_{j=1}^{n}\frac{\Gamma(j)\Gamma(t+j)}{\Gamma(j+\frac{t+s}{2})\Gamma(j+\frac{t-s}{2})}.$$

This shows that Z_n is equal in law to a product of n independent random variables, but hides the geometric interpretation of these random variables. Actually, thanks to a recursive construction of the Haar measure, we directly get the following scission in law, with no need of either the Weyl integration formula or the Selberg integrals.

Theorem. *Let $u_n \sim \mu_{\mathrm{U}(n)}$. Then $\det(\mathrm{Id}-u_n)\overset{\text{law}}{=}\prod_{k=1}^{n}(1-\gamma_k)$ with independent random variables γ_k's, and $\gamma_k \overset{\text{law}}{=} e^{\mathrm{i}\omega_k}\sqrt{\mathrm{B}_{1,k}}$: ω_k is uniform on $(-\pi,\pi)$ and $\mathrm{B}_{1,k}$ is a beta variable with the indicated parameters.*

These coefficients γ_k's have a clear geometric meaning : they are entries of independent reflections whose composition generates the Haar measure $\mu_{\mathrm{U}(n)}$. The above result implies easily a central limit theorem with rate of convergence and a law of the iterated logarithm for $\log \mathrm{Z}_n$, both of them being difficult to obtain without the above interpretation in terms of independent random variables. The recursive construction of the Haar measure has other consequences, such as a probabilistic proof of the Weyl integration formula (Chapter 1, section 3), and a scission in law of $\det(\mathrm{Id}-u)$ for u Haar-distributed in other compact groups, for example $\mathrm{SO}(2n)$ or $\mathrm{USp}(2n)$ (Corollaries 2.8 and 2.11). Such a scission in law also holds for the group of permutations : this led us to prove an analogue of the Ewens sampling formula for general compact groups (Theorem 2.15).

A natural question is whether the above scission in law holds in a more general setting, when the eigenvalues of u_n have a density given by the Jacobi circular ensemble :

$$c_{n,\beta,\delta}\prod_{1\leqslant k<l\leqslant n}|e^{\mathrm{i}\theta_k}-e^{\mathrm{i}\theta_l}|^{\beta}\prod_{j=1}^{n}(1-e^{-\mathrm{i}\theta_j})^{\delta}(1-e^{\mathrm{i}\theta_j})^{\overline{\delta}}.$$

Relying on the theory of orthogonal polynomials on the unit circle (OPUC), Killip and Nenciu [84] have given a very simple matrix model for the above density of eigenvalues in the case $\delta=0$. The theory of OPUC and our products of independent reflections jointly yield a matrix model for the above density for general δ (Theorem 4.16). This implies in particular

Theorem. *Let u_n have its eigenvalues with the above density. Then $\det(\mathrm{Id}-u_n)\overset{\text{law}}{=}\prod_{k=0}^{n-1}(1-\gamma_k)$ with independent random variables γ_k's.*

The explicit distribution of the γ_k's is given in Chapter 4. Moreover, these independent random variables are well understood : called *deformed Verblunsky coefficients*, they are natural coordinates coding the spectral measure associated to (u_n,e_1). The definition of the γ_k's associated to the spectral measure actually coincides with the definition of the γ_k's in terms of matrix entries of reflections (Theorem 4.12).

Concerning zeta functions, what could be the analogue of the above decompositions in law ? For the Riemann zeta function, the Keating-Snaith conjecture involves a matrix factor

$$\mathrm{H}_{\mathrm{Mat}}(k)=\lim_{n\to\infty}\frac{1}{n^{k^2}}\,\mathbb{E}_{\mu_{\mathrm{U}(n)}}(|\det(\mathrm{Id}-u_n)|^{2k}).$$

The above results suggest that $\zeta(s)/\prod_{\mathcal{P}}(1-1/p^s)$, for $s\in[1/2,1/2+\mathrm{i}t]$ might be decomposed as a product of $\log t$ asymptotically independent factors, provided that the Euler product converges. We have no intuition of what the γ_k's might be.

For function-field zeta functions, the scission in law is a little better understood. We keep the example of the Introduction, Section 4, about a family of curves over a finite field $\mathbb{F}_q$ with q elements. Let $\theta(\mathrm{C}/\mathbb{F}_q)=\{e^{\mathrm{i}\theta_1},\dots,e^{\mathrm{i}\theta_{2g}}\}$ where the $e^{\mathrm{i}\theta_k}/\sqrt{q}$'s are the zeros of the zeta function $\zeta(\mathrm{T},\mathrm{C}/\mathbb{F}_q)$, and g is the fixed genus of the curve. We note $\mu_{q,g}$ the counting probability measure on $\mathcal{M}_g(\mathbb{F}_q)$. Katz and Sarnak have shown ([78], [79]) that the Deligne equidistribution theorem yields

$$\lim_{q\to\infty}\int f(\theta(\mathrm{C}/\mathbb{F}_q))\mathrm{d}\mu_{q,g}(\mathrm{C})=\int f(u)\mathrm{d}\mu_{\mathrm{USp}(2g)}(u),$$

for any continuous class function f. Consider the special case $f(u)=\det(\mathrm{Id}-u)$, the above result can be read, for $\mathrm{C}\sim\mu_{q,g}$ and $u\sim\mu_{\mathrm{USp}(2g)}$,

$$(1-\sqrt{q})\,\zeta\left(\frac{1}{\sqrt{q}},\mathrm{C}/\mathbb{F}_q\right)\overset{\text{law}}{\longrightarrow}\det(\mathrm{Id}-u)$$

as $q\to\infty$. For the symplectic group, we know that

$$\det(\mathrm{Id}-u)\overset{\text{law}}{=}\prod_{j=1}^{2g}(1-\gamma_j),$$

with independent γ_j's (see Theorem 5.3). Hence the two above results imply for example a central limit theorem, large deviations, or iterated logarithm laws for $\log\zeta\left(1/\sqrt{q},\mathrm{C}/\mathbb{F}_q\right)$, and the asymptotics of its moments for $\mathrm{C}\sim\mu_{q,g}$, as $q,g\to\infty$.

An interesting problem is about the geometric meaning of the Verblunsky coefficients γ_k's in this function field context. In other words :

Given a curve $\mathrm{C}/\mathbb{F}_q$*, what are the associated Verblunsky coefficients, and why are they asymptotically independent as* $q\to\infty$ *?*

Note that the Verblunsky coefficients depend on the spectral measure, that is to say not only the conjugacy class (the $e^{\mathrm{i}\theta_k}$'s) but also the weight on each zero of $\zeta(\mathrm{T},\mathrm{C}/\mathbb{F}_q)$. Therefore, the asymptotic independence of the Verblunsky coefficients associated to a curve C would require an equidistribution theorem about the whole monodromy group, not only its conjugacy classes.

2. Limiting spectral measures

The asymptotic properties ($n\to\infty$) of the Jacobi circular ensemble

$$c_{n,\beta,\delta}\prod_{1\leqslant k<l\leqslant n}|e^{\mathrm{i}\theta_k}-e^{\mathrm{i}\theta_l}|^{\beta}\prod_{j=1}^{n}(1-e^{-\mathrm{i}\theta_j})^{\delta}(1-e^{\mathrm{i}\theta_j})^{\overline{\delta}}$$

are of interest in statistical physics for example. In this thesis, we are interested in two distinct regimes.

First in Chapter 3, for the special case $\beta=2$, the distribution of these n particles on the unit circle is a determinantal point process, characterized by a kernel, whose asymptotics are given (Theorem 3.1). This limiting *hypergeometric kernel* $\tilde{\mathrm{K}}_\infty^{(\delta)}(\alpha,\beta)$ is actually universal, in the sense that it appears for all potentials having an asymmetric singularity. In the theorem below, $\lambda^{(\delta)}(\theta)$ stands for the density $(2-2\cos\theta)^{\mathfrak{Re}(\delta)}e^{-\mathfrak{Im}(\delta)(\pi\,\mathrm{sgn}\,\theta-\theta)}$ on $(-\pi,\pi)$, and $\tilde{\mathrm{K}}_n^{(\mu)}$ is the kernel associated to n particles placed on the unit circle with the potential μ (see Chapter 3 for more precisions).

Theorem. *Let μ be a measure on $\partial\mathbb{D}$, such that the set of points with $\mu'=0$ has Lebesgue measure 0. Suppose that μ is absolutely continuous in a neighborhood of 1 and $\mu'(\theta)=h(\theta)\lambda^{(\delta)}(\theta)$ in this neighborhood, with h continuous at* 0 *and* $h(0)>0$*. Then for all α,β in $\mathbb{R}$,*

$$\frac{1}{n}\tilde{\mathrm{K}}_n^{(\mu)}(e^{\mathrm{i}\frac{\alpha}{n}},e^{\mathrm{i}\frac{\beta}{n}})\underset{n\to\infty}{\longrightarrow}\tilde{\mathrm{K}}_\infty^{(\delta)}(\alpha,\beta).$$

Furthermore, the empirical spectral measure ($\mu_{\mathsf{esd}}^{(n)} = 1/n\ \sum_{k=1}^n \delta_{e^{\mathrm{i}\theta_k}}$) associated to the circular Jacobi ensemble has an equilibrium measure as $n \to \infty$ in the regime $\delta = \delta(n) = (\beta/2)n\mathrm{d}$ for a constant d ($\mathfrak{Re}(d) \geqslant 0$). This explicit limiting measure is supported by an arc of the circle (see Theorem 4.19), and a large deviations principle is given concerning the convergence to this equilibrium measure.

More precisely we work with the set $\mathcal{M}_1(\partial\mathbb{D})$ of probability measures on the torus and write

$$\Sigma(\mu) = \iint \log|e^{\mathrm{i}\theta} - e^{\mathrm{i}\theta'}|\mathrm{d}\mu(\theta)\mathrm{d}\mu(\theta')\ , \mathrm{B}(\mathrm{d}) = \int_0^1 x \log \frac{x(x+\mathfrak{Re}\,\mathrm{d})}{|x+\mathrm{d}|^2}\,\mathrm{d}x$$

$$\mathrm{Q}(\theta) = -(\mathfrak{Re}\,\mathrm{d})\log\left(2\sin\frac{\theta}{2}\right) - (\mathfrak{Im}\,\mathrm{d})\frac{\theta-\pi}{2}\ ,\quad (\theta \in (0,2\pi))\,.$$

Theorem. *In the regime $\delta(n) = (\beta/2)n\mathrm{d}$, the sequence of empirical measures*

$$\mu_{\mathsf{esd}}^{(n)} = \frac{\delta_{\theta_1} + \cdots + \delta_{\theta_n}}{n}$$

satisfies the large deviations principle at scale $(\beta/2)n^2$ with good rate function defined for $\mu \in \mathcal{M}_1(\partial\mathbb{D})$ by

$$\mathrm{I}(\mu) = -\Sigma(\mu) + 2\int \mathrm{Q}(\theta)\mathrm{d}\mu(\theta) + \mathrm{B}(\mathrm{d})\,.$$

3. Gaussian fluctuations of ζ

The result below could be enounced for any L-function in the Selberg class [120], for simplicity we give it only for ζ. It is a special case of Theorem 6.1 in Chapter 6, which actually holds for shrinking shifts.

Theorem. *Let ω be uniform on $(0,1)$ and constants $0 \leqslant f^{(1)} < \cdots < f^{(\ell)}$. Then the vector*

$$\frac{1}{\sqrt{\log\log t}}\left(\log\zeta\left(\frac{1}{2} + \mathrm{i}f^{(1)} + \mathrm{i}\omega t\right), \ldots, \log\zeta\left(\frac{1}{2} + \mathrm{i}f^{(\ell)} + \mathrm{i}\omega t\right)\right)$$

converges in law to a standard complex Gaussian vector $(\mathrm{Y}_1, \ldots, \mathrm{Y}_\ell)$ of independent random variables (see Corollary 6.2).

If the shifts $f^{(k)}$ decrease with t, correlations may appear in the limiting Gaussian vector : this allows us to obtain convergence of $\log\zeta$ values to Gaussian processes.

Moreover, a consequence of this joint central limit theorem is an analogue for ζ of a result by Wieand [140] : she showed that correlations appear when counting the eigenvalues of unitary matrices which fall in distinct intervals. More precisely, let $\mathrm{N}(t)$ be the number of non-trivial zeros z of ζ with $0 < \mathfrak{Im} z \leqslant t$, counted with their multiplicity, and for any $0 < t_1 < t_2$

$$\Delta(t_1, t_2) = (\mathrm{N}(t_2) - \mathrm{N}(t_1)) - \left(\frac{t_2}{2\pi}\log\frac{t_2}{2\pi e} - \frac{t_1}{2\pi}\log\frac{t_1}{2\pi e}\right),$$

which represents the fluctuations of the number of zeros z ($t_1 < \mathfrak{Im} z \leqslant t_2$) minus its *expectation.*

Corollary. *The finite dimensional distributions of the process*

$$\frac{\Delta\left(\omega t + \alpha, \omega t + \beta\right)}{\frac{1}{\pi}\sqrt{\log\log t}},\ 0 \leqslant \alpha < \beta < \infty$$

converge to those of a centered Gaussian process $(\tilde{\Delta}(\alpha,\beta), 0 \leqslant \alpha < \beta < \infty)$ *with the covariance structure*

$$\mathbb{E}\left(\tilde{\Delta}(\alpha,\beta)\tilde{\Delta}(\alpha',\beta')\right) = \begin{cases} 1 & \textit{if } \alpha=\alpha' \textit{ and } \beta=\beta' \\ 1/2 & \textit{if } \alpha=\alpha' \textit{ and } \beta\neq\beta' \\ 1/2 & \textit{if } \alpha\neq\alpha' \textit{ and } \beta=\beta' \\ -1/2 & \textit{if } \beta=\alpha' \\ 0 & \textit{elsewhere} \end{cases}.$$

This correlation structure is surprising : for example $\tilde{\Delta}(\alpha,\beta)$ and $\tilde{\Delta}(\alpha',\beta')$ are independent if the segment $[\alpha,\beta]$ is strictly included in $[\alpha',\beta']$, and positively correlated if this inclusion is not strict.

Finally, as noted by Laurincikas [88], the asymptotics of the moments of the Riemann zeta functions, $\frac{1}{\mathrm{T}}\int_0^{\mathrm{T}} \mathrm{d}s\,|\zeta\,(1/2+\mathrm{i}s)|^{2k}$, are known only for $k=1,2$ or $u/\sqrt{\log\log \mathrm{T}}$, for fixed $u>0$. Theorem 6.1 also implies joint moments of the form $(0\leqslant\delta\leqslant 1)$

$$\frac{1}{\mathrm{T}}\int_0^{\mathrm{T}} \mathrm{d}s \left|\zeta\left(\frac{1}{2}+\mathrm{i}s\right)\zeta\left(\frac{1}{2}+\mathrm{i}s+\frac{\mathrm{i}}{(\log t)^{\delta}}\right)\right|^{\frac{u}{\sqrt{\log\log \mathrm{T}}}} \underset{\mathrm{T}\to\infty}{\longrightarrow} e^{\frac{u^2}{2}(1+\delta)}.$$

For future research on such topics, we distinguish two directions. First the convergence in law, as $n\to\infty$ of integrals

$$\int_0^1 \log\zeta\left(\frac{1}{2}+\mathrm{i}\omega t+\delta\right)\mathrm{d}\delta,\ \int_0^1 \log\zeta\left(\frac{1}{2}+\mathrm{i}\omega t+\frac{\mathrm{i}}{(\log t)^{\delta}}\right)\mathrm{d}\delta,$$

for which the convergence to normal variables certainly requires distinct normalizations.

Moreover, to predict the extreme $\log\zeta$ values up to height t, it is not clear whether ζ should be modeled by a product of independent random variables (like our decomposition for $\log\det(\mathrm{Id}-u)$) and use an iterated logarithm law : if there were a fast convergence in the multidimensional central limit theorem previously stated, the prediction would rather be that these extreme values are of order $\sqrt{\log t\log_2 t}$, as predicted also in [47]. This point needs to be clarified.

Chapter 1

Haar measure and independence : the unitary group

The first two sections of this chapter are a synthesis of *The characteristic polynomial of a random unitary matrix : a probabilistic approach* [19], a joint work with C.P. Hughes, A. Nikeghbali, M. Yor, Duke Mathematical Journal, Vol 145, 1 (2008), 45-69. The last section is extracted from *Conditional Haar measures on classical compact groups* [17], Annals of Probability vol 37, Number 4, 1566-1586, (2009).

Let u denote a generic $n \times n$ random matrix drawn from the unitary group $\mathrm{U}(n)$ with the Haar measure $\mu_{\mathrm{U}(n)}$. The characteristic polynomial of u is

$$\mathrm{Z}(u, \varphi) = \det(\mathrm{Id} - e^{-\mathrm{i}\varphi} u) = \prod_{j=1}^{n} \left(1 - e^{\mathrm{i}(\theta_n - \varphi)}\right)$$

where $e^{\mathrm{i}\theta_1}, \dots, e^{\mathrm{i}\theta_n}$ are the eigenvalues of u. Note that by the rotational invariance of Haar measure, if φ is real then $\mathrm{Z}(u, \varphi) \overset{\text{law}}{=} \mathrm{Z}(u, 0)$. Therefore here and in the following we may simply write Z_n for $\mathrm{Z}(u, \varphi)$. In [80] and in this thesis, $\arg \mathrm{Z}_n$ is defined as the imaginary part of

$$\log \mathrm{Z}_n = \sum_{k=1}^{n} \log(1 - e^{\mathrm{i}\theta_k})$$

with $\mathfrak{Im} \log(1 - e^{\mathrm{i}\theta}) = (\theta - \pi)/2$ if $\theta \in [0, \pi)$, $(\theta + \pi)/2$ if $\theta \in (-\pi, 0)$. An equivalent definition for $\log \mathrm{Z}_n$ is the value at $x = 1$ of the unique continuous function $\log \det(\mathrm{Id} - xu)$ (on $[0, 1]$) which is 0 at $x = 0$.

In [80], Keating and Snaith give evidence to model the Riemann zeta function on the critical line by the characteristic polynomial of a random unitary matrix considered on the unit circle. In their development of the model they showed that the logarithm of the characteristic polynomial weakly converges to a normal distribution :

$$\frac{\log \mathrm{Z}_n}{\sqrt{\frac{1}{2} \log n}} \xrightarrow{\text{law}} \mathcal{N}_1 + \mathrm{i}\mathcal{N}_2, \tag{1.1}$$

where $\mathcal{N}_1$ and $\mathcal{N}_2$ are two independent standard Gaussian random variables. This is the analogue to Selberg's result on the normal distribution of values of the logarithm of the Riemann zeta function [120].

To prove this central limit theorem, Keating and Snaith evaluated the Mellin-Fourier transform of Z_n. Integrating against the Haar measure on $\mathrm{U}(n)$, they obtained, for all t and s with $\mathfrak{Re}(t \pm s) > -1$,

$$\mathbb{E}\left(|\mathrm{Z}_n|^t e^{\mathrm{i}s \arg \mathrm{Z}_n}\right) = \prod_{k=1}^{n} \frac{\Gamma(k)\, \Gamma(k+t)}{\Gamma\left(k + \frac{t+s}{2}\right) \Gamma\left(k + \frac{t-s}{2}\right)}. \tag{1.2}$$

By calculating the asymptotics of the cumulants of (1.2), they were able to show that for any fixed s, t,

$$\mathbb{E}\left(|\mathrm{Z}_n|^{t/\sqrt{(\log n)/2}} e^{\mathrm{i}s \arg \mathrm{Z}_n/\sqrt{(\log n)/2}}\right) \underset{n\to\infty}{\longrightarrow} e^{\frac{t^2-s^2}{2}}$$

as $n \to \infty$, and from this deduce the central limit theorem (1.1). Therefore their proof relies on two ingredients :

1. the Weyl integration formula to write the LHS of (1.2) as a n-dimensional integral ;
2. Selberg's integral formula [121] to perform the calculation.

One purpose of this chapter is to prove (1.2) from different tools, including a recursive way to generate the Haar measure on the unitary group. As a consequence (1.2) may be simply interpreted as an identity in law involving a certain product of independent random variables. In particular, $\mathfrak{Re} \log \mathrm{Z}_n$ and $\mathfrak{Im} \log \mathrm{Z}_n$ can be written in law as sums of independent random variables. Sums of independent random variables are well known and well studied objects in probability theory, and we can thus have a better understanding of the distribution of the characteristic polynomial with such a representation.

The classical limit theorems are then applied to such sums to obtain asymptotic properties of Z_n when $n \to \infty$. In particular, the Keating-Snaith limit theorem for $\log \mathrm{Z}_n$ is a consequence of the classical central limit theorem. The rate of convergence in (1.1) and an iterated logarithm law also follow from the decomposition of the characteristic polynomial as a product of independent random variables.

Finally, the recursive way we obtain here to generate the Haar measure yields a new proof of the Weyl integration formula, giving the density of the eigenvalues for the Haar measure. This proof does not require the theory of Lie groups, but only elementary probability theory.

1. Decomposition of the Haar measure

For r a $n \times n$ complex matrix, the subscript r_{ij} stands for $\langle e_i, r(e_j)\rangle$, where $\langle x, y\rangle = \sum_{k=1}^n \overline{x}_k y_k$.

1.1. Reflections.

Many distinct definitions of reflections on the unitary group exist, the most well-known may be the Householder reflections. The transformations we need in this work are the following.

Definition 1.1. *An element r in $\mathrm{U}(n)$ will be referred to as a reflection if $r - \mathrm{Id}$ has rank 0 or 1.*

The reflections can also be described in the following way. Let $\mathcal{M}(n)$ be the set of $n \times n$ complex matrices m that can be written

$$m = \left(m_1, e_2 - k\frac{\overline{m}_{12}}{1-\overline{m}_{11}}, \dots, e_n - k\frac{\overline{m}_{1n}}{1-\overline{m}_{11}}\right),$$

with the vector $m_1 = {}^{\mathrm{t}}(m_{11}, \dots, m_{1,n}) \neq e_1$ on the n-dimensional unit complex sphere and $k = m_1 - e_1$. Then the reflections are exactly the elements

$$r = \begin{pmatrix} \mathrm{Id}_{k-1} & 0 \\ 0 & m \end{pmatrix}$$

with $m \in \mathcal{M}(n-k+1)$ for some $1 \leqslant k \leqslant n$. For fixed k, the set of these elements is noted $\mathcal{R}^{(k)}$. If the first column of m, m_1, is uniformly distributed on the unit complex sphere of dimension $n-k+1$, it induces a measure on $\mathcal{R}^{(k)}$, noted $\nu^{(k)}$.

The non-trivial eigenvalue $e^{\mathrm{i}\theta}$ of a reflection $r \in \mathcal{R}^{(k)}$ is

$$e^{\mathrm{i}\theta} = -\frac{1-r_{kk}}{1-\overline{r_{kk}}}. \tag{1.3}$$

A short proof of it comes from $e^{\mathrm{i}\theta} = \mathrm{Tr}(r) - (n-1)$. We see from (1.3) that for $r \sim \nu^{(k)}$ this eigenvalue is not uniformly distributed on the unit circle, and converges in law to -1 as $n \to \infty$.

1.2. Haar measure as the law of a product of independent reflections.

The following two results are the starting point of this work : Theorems 1.2 and 1.3 below will allow us to properly define the Haar measure on $\mathrm{U}(n)$ conditioned to have eigenvalues equal to 1.

In the following we make use of this notation : if $u_1 \sim \mu^{(1)}$ and $u_2 \sim \mu^{(2)}$ are independent elements in $\mathrm{U}(n)$, then $\mu_1 \times \mu_2$ stands for the law of $u_1 u_2$.

The following theorem gives a way to generate the Haar measure recursively. Relations between Haar measures on a group and a subgroup can be found in Diaconis and Shahshahani [41], and Mezzadri [97] gives a decomposition based on Householder reflections and proved through the Ginibre ensemble.

Theorem 1.2. *Let $\mu_{\mathrm{U}(n)}$ be the Haar measure on $\mathrm{U}(n)$. Then*

$$\mu_{\mathrm{U}(n)} = \nu^{(1)} \times \cdots \times \nu^{(n)}.$$

Proof. Take independently $r \sim \nu^{(1)}$ and $u \sim \mu_{\mathrm{U}(n-1)}$. Let $v = \begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix}$. If we can prove that

$$rv \sim \mu_{\mathrm{U}(n)}, \tag{1.4}$$

then the result will follow by an immediate induction. The Haar probability measure is unique, so (1.4) is equivalent to

$$grv \overset{\text{law}}{=} rv$$

for all fixed $g \in \mathrm{U}(n)$. Since $r(e_1)$ is uniform on the unit complex sphere, so is $gr(e_1)$. Therefore in an orthonormal basis with first element e_1, the matrix gr can be written $(p(e_1), \tilde{p})$ with $p(e_1) \overset{\text{law}}{=} r(e_1)$. Consequently, by conditioning on the value $p(e_1) = r(e_1) = w$, it is sufficient to show that

$$(w, p')v \overset{\text{law}}{=} (w, r')v,$$

for some distributions on p' and r', still assumed to be independent of v. Take $u_w \in \mathrm{U}(n)$ so that $u_w(w) = e_1$. By multiplication of the above equality by u_w, we only need to show that

$$p''v \overset{\text{law}}{=} r''v$$

for some elements p'' and r'' once again independent of v, satisfying $p''(e_1) = r''(e_1) = e_1$. By conditioning on p'' (resp r''), $p''v \overset{\text{law}}{=} v$ (resp $r''v \overset{\text{law}}{=} v$) by definition of the Haar measure $\mu_{\mathrm{U}(n-1)}$. This gives the desired result. □

1.3. First scission of the characteristic polynomial.

Theorem 1.3. *Take $r^{(k)} \in \mathcal{R}^{(k)}$ $(1 \leqslant k \leqslant n)$. Then*

$$\det\left(\mathrm{Id} - r^{(1)} \dots r^{(n)}\right) = \prod_{k=1}^{n} \left(1 - r_{kk}^{(k)}\right).$$

Proof. Take $r \in \mathcal{R}^{(1)}$, $u \in \mathrm{U}(n-1)$ and $v = \begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix}$. If we can show that

$$\det(\mathrm{Id}_n - rv) = (1 - r_{11})\det(\mathrm{Id}_{n-1} - u),$$

then the result will follow by an immediate induction. First note that

$$\det(\mathrm{Id}_n - rv) = \det\left({}^{\mathrm{t}}\overline{v} - r\right)\det u.$$

As r is a reflection, it can be written $r = (e_1 + k, e_2 + \lambda_2 k, \dots, e_n + \lambda_n k)$ for some complex numbers $\lambda_2, \dots, \lambda_n$ and $k = r_1 - e_1$. Write ${}^{\mathrm{t}}\overline{v} = (e_1, v_2, \dots, v_n)$. Then the multilinearity of the determinant gives

$$\begin{aligned}
\det({}^{\mathrm{t}}\overline{v} - r) &= \det(-k, v_2 - e_2 - k\lambda_2, \dots, v_n - e_n - k\lambda_n) \\
&= \det(-k, v_2 - e_2, \dots, v_n - e_n) \\
\det({}^{\mathrm{t}}\overline{v} - r) &= (1 - r_{11})\det\left({}^{\mathrm{t}}\overline{u} - \mathrm{Id}_{n-1}\right),
\end{aligned}$$

completing the proof. □

This decomposition gives another proof for equation (1.2). In reality, the following Corollary 1.4 gives much more, as we get a representation of Z_n as a product of n simple independent random variables.

Corollary 1.4. *Let $u \in \mathrm{U}(n)$ be distributed with the Haar measure $\mu_{\mathrm{U}(n)}$. Then*

$$\det(\mathrm{Id} - u) \stackrel{\text{law}}{=} \prod_{k=1}^{n} \left(1 - e^{\mathrm{i}\omega_k}\sqrt{\mathrm{B}_{1,k-1}}\right),$$

with $\omega_1, \dots, \omega_n, \mathrm{B}_{1,0}, \dots, \mathrm{B}_{1,n-1}$ independent random variables, the ω_k's uniformly distributed on $(0, 2\pi)$ and the $\mathrm{B}_{1,j}$'s $(0 \leqslant j \leqslant n-1)$ being beta distributed with parameters 1 and j (by convention, $\mathrm{B}_{1,0} \sim \delta_1$, the Dirac distribution on 1).

Proof. Theorems 1.2 and 1.3 together yield

$$\det(\mathrm{Id} - u) \stackrel{\text{law}}{=} \prod_{k=1}^{n} \left(1 - r_{kk}^{(k)}\right),$$

where $r^{(1)}, \dots, r^{(n)}$ are independent reflections with distributions $\nu^{(1)}, \dots, \nu^{(n)}$ respectively. The proof will therefore be complete if $r_{kk}^{(k)} \stackrel{\text{law}}{=} e^{\mathrm{i}\omega_{n-k}}\sqrt{\mathrm{B}_{1,n-k}}$. This is straightforward because, since $r^{(k)}(e_k)$ (restricted to the last $n-k+1$ coordinates) is a random vector chosen uniformly on $\mathscr{S}_{\mathbb{C}}^{n-k+1}$,

$$r_{kk}^{(k)} \stackrel{\text{law}}{=} \frac{x_1 + \mathrm{i}y_1}{\sqrt{(x_1^2 + y_1^2) + \dots + (x_{n-k+1}^2 + y_{n-k+1}^2)}} \stackrel{\text{law}}{=} e^{\mathrm{i}\omega_{n-k+1}}\sqrt{\mathrm{B}_{1,n-k}},$$

with the x_i's and y_i's all independent standard normal variables, ω_{n-k+1} and $\mathrm{B}_{1,n-k}$ as desired. □

To end the proof of (1.2), thanks to the independence property, we now only need to show the following result : if $\mathrm{X} = 1 - e^{\mathrm{i}\omega}\sqrt{\mathrm{B}}$, where ω has uniform distribution on $(-\pi, \pi)$ and, independently B has a beta law with parameters 1 and $n-1$, then, for all t and s with $\mathfrak{Re}(t \pm s) > -1$

$$\mathbb{E}\left(|\mathrm{X}|^t e^{\mathrm{i}s \arg \mathrm{X}}\right) = \frac{\Gamma(n)\Gamma(n+t)}{\Gamma\left(n + \frac{t+s}{2}\right)\Gamma\left(n + \frac{t-s}{2}\right)}.$$

This is the consequence of an elementary calculation, given in the Appendix as Lemma 7.2. Consequently, the Mellin-Fourier transform of Z_n has been found by a probabilistic method.

Corollary 1.5. *For all t and s with $\mathfrak{Re}(t \pm s) > -1$,*

$$\mathbb{E}\left(|\mathrm{Z}_n|^t e^{\mathrm{i}s \arg \mathrm{Z}_n}\right) = \prod_{k=1}^{n} \frac{\Gamma(k)\,\Gamma(k+t)}{\Gamma\left(k+\frac{t+s}{2}\right)\Gamma\left(k+\frac{t-s}{2}\right)}.$$

Corollary 1.4 can be extended to the law of the characteristic polynomial of a random unitary matrix off the unit circle where we replace $e^{\mathrm{i}\varphi}$ by a fixed x. Once more, due to the rotational invariance of the unitary group, we may take x to be real.

Corollary 1.6. *Take $x \in \mathbb{R}$, u_{n-1} distributed with the Haar measure $\mu_{\mathrm{U}(n-1)}$ and, independently, $r \sim \nu^{(n)}$. Let $\tilde{r}$ denote the vector with coordinates $r_{12}, \dots, r_{1n}$.*

Then, if u_n is distributed with the Haar measure $\mu_{\mathrm{U}(n)}$,

$$\begin{aligned}\det(\mathrm{Id}_n - x\,u_n) &\overset{\text{law}}{=} (1 - x\,r_{11})\,\det(\mathrm{Id}_{n-1} - x\,u_{n-1})\\ &\quad + \frac{x(1-x)}{1-\overline{r_{11}}}\,{}^{\mathrm{t}}\overline{\tilde{r}}({}^{\mathrm{t}}\overline{u_{n-1}} - x\,\mathrm{Id}_{n-1})^{-1}\tilde{r}\,\det(\mathrm{Id}_{n-1} - x\,u_{n-1}).\end{aligned} \tag{1.5}$$

Proof. The method is the same as the one used to prove Corollary 1.4. If $k = r(e_1) - e_1$ we can write more precisely

$$v = \left(r_{11}, e_2 + \frac{-\overline{r}_{12}}{1-\overline{r}_{11}}k, \dots, e_n + \frac{-\overline{r}_{1n}}{1-\overline{r}_{11}}k\right).$$

Thus, using multi-linearity of the determinant and using one step of the recursion in the proof of Theorem 1.2, we get after some straightforward calculation

$$\begin{aligned}\det(\mathrm{Id} - x\ u_n) &\overset{\text{law}}{=} \det\left(\mathrm{Id} - x\,r\begin{pmatrix} 1 & 0\\ 0 & u_{n-1}\end{pmatrix}\right)\\ &= b\,\det\begin{pmatrix} a & {}^{\mathrm{t}}\overline{\tilde{r}}\\ \tilde{r} & {}^{\mathrm{t}}\overline{u}_{n-1} - x\mathrm{Id}\end{pmatrix}\det\begin{pmatrix} 1 & 0\\ 0 & u_{n-1}\end{pmatrix}\end{aligned}$$

with $b = \frac{-x(1-x)}{1-\overline{r}_{11}}$ and $a = \frac{(1-xr_{11})(1-\overline{r}_{11})}{-x(1-x)}$. As we want to express these terms with respect to $\det(\mathrm{Id} - x\,u_{n-1})$, writing $v = {}^{\mathrm{t}}\overline{u}_{n-1} - x\,\mathrm{Id}$ leads to

$$\begin{aligned}\det(\mathrm{Id} - x\,u_n) &\overset{\text{law}}{=} b\,\det\begin{pmatrix} a & {}^{\mathrm{t}}\overline{\tilde{r}}\\ \tilde{r} & v\end{pmatrix}\det\begin{pmatrix} 1 & 0\\ -v^{-1}\tilde{r} & u_{n-1}\end{pmatrix}\\ &= b\,\det\begin{pmatrix} a - {}^{\mathrm{t}}\overline{\tilde{r}}v^{-1}\tilde{r} & \cdots\\ 0 & v\,u_{n-1}\end{pmatrix}\\ &= b\,(a - {}^{\mathrm{t}}\overline{\tilde{r}}v^{-1}\tilde{r})\,\det(\mathrm{Id} - x\,u_{n-1}).\end{aligned}$$

This is the desired result. □

Remark. Remember that $\log \mathrm{Z}_n$ is defined by continuity of $\log\det(\mathrm{Id} - x\,u)$ along $(0,1)$. If, for $|\varepsilon| < 1$, $\log(1-\varepsilon)$ is defined through the Taylor expansion $-\sum_{j\geqslant 1}\varepsilon^j/j$, is the equation

$$\log \mathrm{Z}_n \overset{\text{law}}{=} \sum_{k=1}^{n} \log(1 - e^{\mathrm{i}\omega_k}\sqrt{\mathrm{B}_{1,k-1}}) \tag{1.6}$$

still true ? The result is an obvious consequence of Corollary 1.4 for the real parts of both terms, but it is not so obvious for the imaginary parts which could have a $2k\pi$ difference. The main tool to prove (1.6) is Corollary 1.6. Indeed, its proof shows that its result remains true for the whole trajectory $x \in (0,1)$:

$$(\det(\mathrm{Id} - x\,u_n), 0 \leqslant x \leqslant 1) \overset{\text{law}}{=} ((1 - f(x, u_{n-1}, r(e_1)))\det(\mathrm{Id} - x\,u_{n-1}), 0 \leqslant x \leqslant 1),$$

with the suitable f from (1.5). Let the logarithm be defined as in the Introduction (i.e. by continuity from $x=0$). The previous equation then implies, as f is continuous in x,

$$\log\det(\mathrm{Id}-x\,u_n)\overset{\text{law}}{=}\log(1-f(x,u_{n-1},r(e_1)))+\log\det(\mathrm{Id}-x\,u_{n-1}).$$

One can easily check that $|f(x,u_{n-1},r(e_1))|<1$ for all $x\in(0,1)$ a.s., so

$$\log(1-f(x,u_{n-1},r(e_1)))=-\sum_{j\geqslant 0}\frac{f(x,u_{n-1},r(e_1))^j}{j}$$

for all $x\in(0,1)$ almost surely. In particular, for $x=1$, we get

$$\log\det(\mathrm{Id}-u_n)\overset{\text{law}}{=}\sum_{j\geqslant 1}\frac{r_{11}^j}{j}+\log\det(\mathrm{Id}-u_{n-1}),$$

which gives the desired result (1.6) by an immediate induction.

1.4. Second scission of the characteristic polynomial

Corollary 1.7. *Let* $(\mathrm{B}_{k,k-1})_{1\leqslant k\leqslant n}$ *be independent beta variables of parameters* k *and* $k-1$ *respectively (with the convention that* $\mathrm{B}_{1,0}\equiv 1$*). Define* $\mathrm{W}_1,\dots,\mathrm{W}_n$ *as independent random variables which are independent of the* $(\mathrm{B}_{k,k-1})_{1\leqslant k\leqslant n}$*, with* W_k *having the density*

$$\sigma_k\,(\mathrm{d}v)=c_k\,\cos^{2(k-1)}(v)\,\mathbb{1}_{\left(\frac{-\pi}{2},\frac{\pi}{2}\right)}\mathrm{d}v,$$

c_k *being the normalization constant. Then*

$$(\arg \mathrm{Z}_n,|\mathrm{Z}_n|)\overset{\text{law}}{=}\left(\sum_{k=1}^n \mathrm{W}_k,\prod_{k=1}^n 2\mathrm{B}_{k,k-1}\cos\mathrm{W}_k\right).$$

Remark. The normalization constant is $c_k=\dfrac{2^{2(k-1)}\left((k-1)!\right)^2}{\pi\,(2k-2)!}$, but we will not need it in the following.

Proof. From the remark after Corollary 1.6, we know that

$$\log \mathrm{Z}_n\overset{\text{law}}{=}\sum_{k=1}^n\log(1-e^{\mathrm{i}\omega_k}\sqrt{\mathrm{B}_{1,k-1}}).$$

Moreover, by identifying the Mellin-Fourier transforms, the case $\delta=0$ in Lemma 7.4 in the Appendix shows that

$$1-e^{\mathrm{i}\omega_k}\sqrt{\mathrm{B}_{1,k-1}}\overset{\text{law}}{=}2\mathrm{B}_{k,k-1}\cos\mathrm{W}_k e^{\mathrm{i}\mathrm{W}_k},$$

which concludes the proof. ☐

2. On the central limit theorem by Keating and Snaith

2.1. The central limit theorem.

In this section, we give an alternative proof of the following central limit theorem by Keating and Snaith [80]. This relies on the first decomposition in Corollary 1.4. The original proof by Keating and Snaith relies on an expansion of formula (1.2) with cumulants.

Theorem 1.8. *Let u_n be distributed with the Haar measure on the unitary group* $\mathrm{U}(n)$*. Then,*

$$\frac{\log\det(\mathrm{Id}-u_n)}{\sqrt{\frac{1}{2}\log n}}\xrightarrow{\text{law}}\mathcal{N}_1+\mathrm{i}\mathcal{N}_2,$$

as $n\to\infty$, with $\mathcal{N}_1$ and $\mathcal{N}_2$ independent standard normal variables.

Proof. The idea is basically that $\mathrm{B}_{1,k-1}$ tends in law to the Dirac distribution at 0 as k tends to ∞. So $\log(1-e^{\mathrm{i}\omega_k}\sqrt{\mathrm{B}_{1,k-1}})$ is well approximated by $e^{\mathrm{i}\omega_k}\sqrt{\mathrm{B}_{1,k-1}}$, whose distribution is invariant by rotation. Hence the central limit theorem will be easily proven from classical results in dimension 1.

More precisely, $\log\det(\mathrm{Id}-u_n)$ can be decomposed, thanks to (1.6), as

$$\log\det(\mathrm{Id}-u_n)\overset{\text{law}}{=}-\underbrace{\sum_{k=1}^{n}e^{\mathrm{i}\omega_k}\sqrt{\mathrm{B}_{1,k-1}}}_{\mathrm{X}_1(n)}-\underbrace{\sum_{k=1}^{n}\frac{e^{2\mathrm{i}\omega_k}}{2}\mathrm{B}_{1,k-1}}_{\mathrm{X}_2(n)}-\underbrace{\sum_{j\geqslant 3}\sum_{k=1}^{n}\frac{1}{j}\left(e^{\mathrm{i}\omega_k}\sqrt{\mathrm{B}_{1,k-1}}\right)^j}_{\mathrm{X}_3(n)}$$

where all the terms are absolutely convergent. We study these three terms separately.

Clearly, the distribution of $\mathrm{X}_1(n)$ is invariant by rotation, so to prove that $\frac{\mathrm{X}_1(n)}{\sqrt{\frac{1}{2}\log n}}\xrightarrow{\text{law}}\mathcal{N}_1+\mathrm{i}\mathcal{N}_2$, we only need to prove the following result for the real part :

$$\frac{\sum_{k=1}^{n}\cos\omega_k\sqrt{\mathrm{B}_{1,k-1}}}{\sqrt{\frac{1}{2}\log n}}\xrightarrow{\text{law}}\mathcal{N},$$

where $\mathcal{N}$ is a standard normal variable. As $\mathbb{E}(\cos^2(\omega_k)\mathrm{B}_{1,k-1})=\frac{1}{2k}$, this is a direct consequence of the central limit theorem (our random variables satisfy the Lyapunov condition).

To deal with $\mathrm{X}_2(n)$, as $\sum_{k\geqslant 1}1/k^2<\infty$, there exists a constant $c>0$ such as $\mathbb{E}(|\mathrm{X}_2(n)|^2)<c$ for all $n\in\mathbb{N}$. Thus $(\mathrm{X}_2(n),n\geqslant 1)$ is a L^2-bounded martingale, so it converges almost surely. Hence

$$\mathrm{X}_2(\mathrm{N})/\sqrt{\tfrac{1}{2}\log\mathrm{N}}\to 0\quad a.s.$$

Finally, for $\mathrm{X}_3(n)$, let $\mathrm{Y}=\sum_{j=3}^{\infty}\sum_{k=1}^{\infty}\frac{1}{j}(\mathrm{B}_{1,k-1})^{j/2}$. One can easily check that $\mathbb{E}(\mathrm{Y})<\infty$, so $\mathrm{Y}<\infty$ a.s., hence as $n\to\infty$

$$|\mathrm{X}_3(n)|/\sqrt{\tfrac{1}{2}\log n}<\mathrm{Y}/\sqrt{\tfrac{1}{2}\log n}\to 0\quad a.s.$$

Gathering all these convergences, we get the desired result. □

Remark. Still another proof can be found in [19], relying on the second decomposition in Corollary 1.7 and on a multidimensional central limit theorem by Petrov [108].

2.2. The rate of convergence.

With the representation in Corollary 1.7 it is possible to obtain estimates on the rate of convergence in the central limit theorem, using the following Berry-Esseen inequalities (see [108] for example).

We use the traditional notations :

- $(\mathrm{X}_k,k\geqslant 1)$ are independent real random variables such that $\mathbb{E}(\mathrm{X}_k)=0$;
- $\sigma_n=\sum_{k=1}^{n}\mathbb{E}\left(\mathrm{X}_k^2\right)$;
- $\mathrm{L}_n=\sum_{k=1}^{n}\mathbb{E}\left(|\mathrm{X}_k|^3\right)/\sigma_n^{3/2}$

- the distribution functions of the partial sums are defined as

$$\mathrm{F}_n(x) = \mathbb{P}\left(\frac{1}{\sqrt{\sigma_n}}\sum_{j=1}^{n}\mathrm{X}_j \leqslant x\right);$$

- the distribution function of the standard Gaussian variable is

$$\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-t^2/2}\mathrm{d}t.$$

Theorem 1.9 (Petrov [108])**.** *With the above notations, if moreover $\mathbb{E}\left(|\mathrm{X}_k|^3\right) < \infty$ for all $k \geqslant 1$, there exists a constant c not depending on n such that for any $x \in \mathbb{R}$*

$$|\mathrm{F}_n(x) - \Phi(x)| \leqslant \frac{c\,\mathrm{L}_n}{(1+|x|)^3}.$$

Now, easy calculations allow us to apply the above theorem to the variables $(\mathrm{W}_k, k \geqslant 0)$ and $(2\mathrm{B}_{k,k-1}\cos\mathrm{W}_k, k \geqslant 0)$. Hence the rate of convergence in the Keating Snaith central limit theorem is of order $\mathrm{O}(1/(\log n)^{3/2})$.

Proposition 1.10. *There exists a universal constant $c > 0$ such that for any $n \geqslant 1$ and $x \in \mathbb{R}$*

$$\left|\mathbb{P}\left(\log|\mathrm{Z}_n|/\sqrt{\frac{1}{2}\log n} \leqslant x\right) - \Phi(x)\right| \leqslant \frac{c}{(\log n)^{3/2}(1+|x|)^3},$$

$$\left|\mathbb{P}\left(\arg \mathrm{Z}_n/\sqrt{\frac{1}{2}\log n} \leqslant x\right) - \Phi(x)\right| \leqslant \frac{c}{(\log n)^{3/2}(1+|x|)^3}.$$

Remark. If $u \sim \mu_{\mathrm{U}(n)}$, then for any $k \in \mathbb{N}^*$, $\left(\mathrm{Tr}\,u, \frac{\mathrm{Tr}(u^2)}{\sqrt{2}}, \dots, \frac{\mathrm{Tr}(u^k)}{\sqrt{k}}\right)$ converges in law (as $n \to \infty$) to a Gaussian vector with independent standard normal complex entries (see [42]). Moreover, this speed of convergence is extremely fast (extra-exponential), as shown by Johansson [75]. This contrasts with the much slower rate of convergence found in the above proposition for the infinite sum

$$\log \mathrm{Z}_n = -\sum_{\ell \geqslant 1}\frac{\mathrm{Tr}(u^\ell)}{\ell}.$$

2.3. An iterated logarithm law

We first state a general version of the iterated logarithm law by Petrov (see [109, 110]). Our interest here is in the asymptotic behavior of the maximum of the partial sums

$$\mathrm{S}_n = \sum_{k=1}^{n}\mathrm{X}_k.$$

Theorem 1.11. *Take $(\mathrm{X}_k, k \geqslant 1)$ and $(\sigma_n, n \geqslant 1)$ as previously defined, and suppose that $\mathbb{E}\left(\mathrm{X}_k^2\right) < \infty$ for any $k \geqslant 1$. Suppose that $\sigma_n \underset{n\to\infty}{\longrightarrow} \infty$, $\frac{\sigma_{n+1}}{\sigma_n} \underset{n\to\infty}{\longrightarrow} 1$ and $\sup_{x\in\mathbb{R}}|\mathrm{F}_n(x) - \Phi(x)| = \mathrm{O}\left((\log\sigma_n)^{-1-\delta}\right)$, for some $\delta > 0$. Then*

$$\limsup_{n\to\infty}\frac{\mathrm{S}_n}{\sqrt{2\sigma_n\log\log\sigma_n}} = 1 \quad a.s..$$

Remark. If the conditions of the theorem are satisfied, then we also have

$$\liminf \frac{\mathrm{S}_n}{\sqrt{2\sigma_n\log\log\sigma_n}} = -1 \quad \text{a.s.}$$

Before using Theorem 1.11 for the real and imaginary parts of $\log \mathrm{Z}_n$, we need to give the explicit meaning of the almost sure convergence for matrices with different sizes.

Consider the set $\mathrm{O} = \mathscr{S}_{\mathbb{C}}^1 \times \mathscr{S}_{\mathbb{C}}^2 \times \mathscr{S}_{\mathbb{C}}^3 \dots$ endowed with the measure $\lambda = \lambda_1 \times \lambda_2 \times \lambda_3 \dots$, where λ_k is the uniform measure on the unit sphere $\mathscr{S}_{\mathbb{C}}^k$ (this can be a probability measure by defining the measure of a set as the limiting measure of the finite-dimensional cylinders). Consider the application f which transforms $\omega \in \mathrm{O}$ into an element of $\mathrm{U}(1) \times \mathrm{U}(2) \times \mathrm{U}(3) \dots$ with iterations of compositions of reflections, as in Theorem 1.2. Then $\Omega = \mathfrak{Im}(f)$ is naturally endowed with a probability measure $f(\lambda)$, and the marginal distribution of $f(\lambda)$ on the k^{th} coordinate is the Haar measure on $\mathrm{U}(k)$.

Let g be a function of a unitary matrix u, no matter the size of u (for example $g = \det(\mathrm{Id} - u)$). The introduction of the set Ω with measure μ_{U} allows us to define the almost sure convergence of $(g(u_k), k \geqslant 0)$, where $(u_k)_{k \geqslant 0} \in \Omega$. This is, for instance, the sense of the "a.s" in the following iterated logarithm law.

Proposition 1.12. *The following almost sure convergence (defined previously) holds :*

$$\begin{aligned} \limsup_{n\to\infty} \frac{\log |\mathrm{Z}_n|}{\sqrt{\log n \log \log \log n}} &= 1, \\ \limsup_{n\to\infty} \frac{\arg \mathrm{Z}_n}{\sqrt{\log n \log \log \log n}} &= 1. \end{aligned}$$

Remark. The representations in law as sums of independent random variables we have obtained could as well be used to obtain all sorts of refined large and moderate deviations estimates for the characteristic polynomial.

3. A proof of the Weyl integration formula

The Weyl integration formula states that, for any continuous class function f (i.e. : functions invariant on conjugation classes),

$$\mathbb{E}_{\mu_{\mathrm{U}(n)}}(f(u)) = \frac{1}{n!} \int \cdots \int |\Delta(e^{\mathrm{i}\theta_1}, \dots, e^{\mathrm{i}\theta_n})|^2 f(e^{\mathrm{i}\theta_1}, \dots, e^{\mathrm{i}\theta_n}) \frac{\mathrm{d}\theta_1}{2\pi} \cdots \frac{\mathrm{d}\theta_n}{2\pi},$$

where Δ denotes the Vandermonde determinant. Classical proofs of this density of the eigenvalues make use of the theory of Lie groups (see e.g. [25]), raising the question of a more probabilistic proof of it.

3.1. Conditional Haar measure as the law of a product of independent reflections.

What might be the conditional expectation of $u \sim \mu_{\mathrm{U}(n)}$, conditioned to have one eigenvalue at 1 ? As this conditioning is with respect to an event of measure 0, such a choice of conditional expectation is not trivial.

As previously, suppose we generate the Haar measure as the law of a product of independent reflections : $u = r^{(1)} \dots r^{(n)}$. Since $\mathrm{Id} - r^{(k)}$ has rank 1 a.s., our conditional distribution will naturally be constructed as a product of $n-1$ of these reflections : the unitary matrix u has one eigenvalue $e^{\mathrm{i}\theta} = 1$ if and only if $r^{(k)} = \mathrm{Id}$ for some $1 \leqslant k \leqslant n$, which yields $r_{kk}^{(k)} = 1$. As $r_{kk}^{(k)} \overset{\text{law}}{=} e^{\mathrm{i}\omega} \sqrt{\mathrm{B}_{1,n-k}}$, with the previous notations, $r_{nn}^{(n)}$ is more likely to be equal to 1 than any other $r_{kk}^{(k)}$ $(1 \leqslant k \leqslant n-1)$.

Consequently, a good definition for the conditional distribution of $u \sim \mu_{\mathrm{U}(n)}$, conditionally to have one eigenvalue at 1, is $r^{(1)} \dots r^{(n-1)}$. This idea is formalized in the following way.

Proposition 1.13. *Let* $\mathrm{Z}(\mathrm{X}) = \det(\mathrm{X}\mathrm{Id} - u)$ *and* $\mathrm{d}x$ *be the measure of* $|\mathrm{Z}(1)|$ *under Haar measure on* $\mathrm{U}(n)$. *There exists a continuous family of probability measures* $\mathrm{P}^{(x)}$

$(0 \leqslant x \leqslant 2^n)$ such that for any Borel subset Γ of $\mathrm{U}(n)$

$$\mu_{\mathrm{U}(n)}(\Gamma) = \int_0^{2^n} \mathrm{P}^{(x)}(\Gamma)\mathrm{d}x. \tag{1.7}$$

Moreover $\mathrm{P}^{(0)} = \nu^{(1)} \times \cdots \times \nu^{(n-1)}$ necessarily.

Remark. The continuity of the probability measures is in the sense of weak topology : the map

$$x \mapsto \int_{\mathrm{U}(n)} f(\omega)\mathrm{d}\mathrm{P}^{(x)}(\omega) \tag{1.8}$$

is continuous for any continuous function f on $\mathrm{U}(n)$.

Proof. We give an explicit expression of this conditional expectation, thanks to Theorem 1.3. Take $x > 0$. If $\prod_{k=1}^{n-1} |1 - r_{kk}^{(k)}| > x/2$, then there are two $r_{nn}^{(n)}$'s on the unit circle such that $\prod_{k=1}^{n} |1 - r_{kk}^{(k)}| = x$:

$$r_{nn}^{(n)} = \exp\left(\pm 2\mathrm{i} \arcsin \frac{x}{2\prod_{k=1}^{n-1} |1 - r_{kk}^{(k)}|}\right). \tag{1.9}$$

These two numbers will be denoted r_+ et r_-. We write $\nu_\pm$ for the distribution of $r_\pm$, the random matrix in $\mathcal{R}^{(n)}$ equal to $\mathrm{Id}_{n-1} \oplus r_+$ with probability $1/2$, $\mathrm{Id}_{n-1} \oplus r_-$ with probability $1/2$. We define the conditional expectation, for any bounded continuous function f, by

$$\mathbb{E}_{\mu_{\mathrm{U}(n)}}\left(f(u) \mid |\mathrm{Z}(1)| = x\right) = \frac{\mathbb{E}\left(f(r^{(1)} \dots r^{(n-1)} r_\pm) \mathbb{1}_{\prod_{k=1}^{n-1} |1 - r_{kk}^{(k)}| > x/2}\right)}{\mathbb{E}\left(\mathbb{1}_{\prod_{k=1}^{n-1} |1 - r_{kk}^{(k)}| > x/2}\right)}, \tag{1.10}$$

the expectations on the RHS being with respect to $\nu^{(1)} \times \cdots \times \nu^{(n-1)} \times \nu_\pm$. For such a choice of the measures $\mathrm{P}^{(x)}$ $(x > 0)$, (1.7) holds thanks to Theorems 1.2 and 1.3. Moreover, these measures are continuous in x, and from (1.9) and (1.10) they converge to $\nu^{(1)} \times \cdots \times \nu^{(n-1)}$ as $x \to 0$. The continuity condition and formula (1.7) impose uniqueness for $(\mathrm{P}^{(x)}, 0 \leqslant x \leqslant 2^n)$. Consequently, $\mathrm{P}^{(0)}$ necessarily coincides with $\nu^{(1)} \times \cdots \times \nu^{(n-1)}$. □

For any $1 \leqslant k \leqslant n-1$, we can state some analogue of Proposition 1.13, conditioning now with respect to

$$(|\mathrm{Z}(1)|, |\mathrm{Z}'(1)|, \dots, |\mathrm{Z}^{(k-1)}(1)|). \tag{1.11}$$

This leads to the following definition of the conditional expectation, which is the unique suitable choice preserving the continuity of measures with respect to (1.11).

Definition 1.14. *For any $1 \leqslant p \leqslant n-1$, $\nu^{(1)} \times \cdots \times \nu^{(n-p)}$ is called the Haar measure on $\mathrm{U}(n)$ conditioned to have p eigenvalues equal to 1.*

Remark. The above discussion can be held for the orthogonal group : the Haar measure on $\mathrm{O}(n)$ conditioned to the existence of a stable subspace of dimension p $(0 \leqslant p \leqslant n-1)$ is

$$\nu_{\mathbb{R}}^{(1)} \times \cdots \times \nu_{\mathbb{R}}^{(n-p)},$$

where $\nu_{\mathbb{R}}^{(k)}$ is defined as the real analogue of $\nu^{(k)}$: a reflection r is $\nu_{\mathbb{R}}^{(k)}$-distributed if $r(e_k)$ has its first $k-1$ coordinates equal to 0 and the others are uniformly distributed on the real unit sphere.

More generally, we can define this conditional Haar measure for any compact group generated by reflections, more precisely any compact group checking condition (R) in the sense of the next chapter.

Take $p = n-1$ in Definition 1.14 : the distribution of the unique eigenangle distinct from 1 coincides with the distribution of the non-trivial eigenangle of a reflection $r \sim \nu^{(1)}$, that is to say from (1.3)

$$e^{\mathrm{i}\varphi} = -\frac{1-r_{11}}{1-\overline{r_{11}}} \stackrel{\text{law}}{=} -\frac{1-e^{\mathrm{i}\omega}\sqrt{\mathrm{B}_{1,n-1}}}{1-e^{-\mathrm{i}\omega}\sqrt{\mathrm{B}_{1,n-1}}}.$$

In particular, this eigenvalue is not uniformly distributed on the unit circle : it converges in law to -1 as $n \to \infty$. This agrees with the idea of repulsion of the eigenvalues : we make it more explicit with the following probabilistic proof of the Weyl integration formula.

3.2. The conditioning and slipping lemmas.

The following two lemmas play a key role in our proof of the Weyl integration formula : the first shows that the spectral measure on $\mathrm{U}(n)$ can be generated by $n-1$ reflections (instead of n) and the second one gives a transformation from this product of $n-1$ reflections in $\mathrm{U}(n)$ to a product of $n-1$ reflections in $\mathrm{U}(n-1)$, preserving the spectrum.

In the following, $u \stackrel{\text{sp}}{=} v$ means that the spectra of the matrices u and v are equally distributed.

Recall that the measures $\nu^{(k)}$ $(1 \leqslant k \leqslant n)$ are supported on the set of reflections : the following lemma would not be true by substituting our reflections with Householder transformations, for example.

Lemma 1.15. *Take $r^{(k)} \sim \nu^{(k)}$ $(1 \leqslant k \leqslant n)$, ω uniform on $(-\pi,\pi)$ and $u \sim \mu_{\mathrm{U}(n)}$, all being independent. Then*

$$u \stackrel{\text{sp}}{=} e^{\mathrm{i}\omega} r^{(1)} \dots r^{(n-1)}.$$

Proof. From Proposition 1.13, the spectrum of $r^{(1)} \dots r^{(n-1)}$ is equal in law to the spectrum of u conditioned to have one eigenvalue equal to 1.

Moreover, the Haar measure on $\mathrm{U}(n)$ is invariant by translation, in particular by multiplication by $e^{\mathrm{i}\varphi}\mathrm{Id}$, for any fixed φ : the distribution of the spectrum is invariant by rotation.

Consequently, the spectral distribution of $u \sim \mu_{\mathrm{U}(n)}$ can be realized by successively conditioning to have one eigenvalue at 1 and then shifting by an independent uniform eigenangle, that is to say $u \stackrel{\text{sp}}{=} e^{\mathrm{i}\omega} r^{(1)} \dots r^{(n-1)}$, giving the desired result. $\square$

Take $1 \leqslant k \leqslant n$ and δ a complex number. We first define a modification $\nu_\delta^{(k)}$ of the measure $\nu^{(k)}$ on the set of reflections $\mathcal{R}^{(k)}$. Let

$$\exp_\delta^{(k)} : \left\{ \begin{array}{ccc} \mathcal{R}^{(k)} & \to & \mathbb{R}^+ \\ r & \mapsto & (1-r_{kk})^{\overline{\delta}}(1-\overline{r_{kk}})^{\delta} \end{array} \right. .$$

Then $\nu_\delta^{(k)}$ is defined as the $\exp_\delta^{(k)}$-sampling of a measure $\nu^{(k)}$ on $\mathcal{R}^{(k)}$, in the sense of the following definition.

Definition 1.16. *Let $(\mathrm{X},\mathcal{F},\mu)$ be a probability space, and $h : \mathrm{X} \mapsto \mathbb{R}^+$ a measurable function with $\mathbb{E}_\mu(h) > 0$. Then a measure μ' is said to be the h-sampling of μ if for all bounded measurable functions f*

$$\mathbb{E}_{\mu'}(f) = \frac{\mathbb{E}_\mu(f\,h)}{\mathbb{E}_\mu(h)}.$$

For $\mathfrak{Re}(\delta) > -1/2$, $0 < \mathbb{E}_{\nu^{(k)}}(\exp_\delta^{(k)}(r)) < \infty$, so $\nu_\delta^{(k)}$ is properly defined.

Lemma 1.17. *Let $r^{(k)} \sim \nu^{(k)}$ $(1 \leqslant k \leqslant n-1)$ and $r_1^{(k)} \sim \nu_1^{(k)}$ $(2 \leqslant k \leqslant n)$ be $n \times n$ independent reflections. Then*

$$r^{(1)} \dots r^{(n-1)} \stackrel{\text{sp}}{=} r_1^{(2)} \dots r_1^{(n)}.$$

Proof. We proceed by induction on n. For $n = 2$, take $r \sim \nu^{(1)}$. Consider the unitary change of variables

$$\Phi : \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} \mapsto \frac{1}{|1-r_{11}|^2 + |r_{12}|^2} \begin{pmatrix} \overline{r_{12}} & -(1-\overline{r_{11}}) \\ 1-r_{11} & r_{12} \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}. \tag{1.12}$$

In this new basis, r is diagonal with eigenvalues 1 and $r_{11} - |r_{12}|^2/(1-\overline{r_{11}})$, so we only need to check that this last random variable is equal in law to the $|1-\mathrm{X}|^2$-sampling of a random variable X which is uniform on the unit circle. This is a particular case of the identity in law given in Theorem 7.1.

We now reproduce the above argument for general $n > 2$. Suppose the result is true at rank $n-1$. Take $u \sim \mu_{\mathrm{U}(n)}$, independent of all the other random variables. Obviously,

$$r^{(1)} \dots r^{(n-1)} \stackrel{\text{sp}}{=} (u^{-1} r^{(1)} u)(u^{-1} r^{(2)} \dots r^{(n-1)} u).$$

As the uniform measure on the sphere is invariant by a unitary change of basis, then by conditioning on $(u, r^{(2)}, \dots, r^{(n-1)})$ we get

$$r^{(1)} \dots r^{(n-1)} \stackrel{\text{sp}}{=} r^{(1)} (u^{-1} r^{(2)} \dots r^{(n-1)} u)$$

whose spectrum is equal in law (by induction) to the one of

$$r^{(1)} (u^{-1} r_1^{(3)} \dots r_1^{(n)} u) \stackrel{\text{sp}}{=} (u\, r^{(1)} u^{-1})\; r_1^{(3)} \dots r_1^{(n)} \stackrel{\text{sp}}{=} r^{(1)}\; r_1^{(3)} \dots r_1^{(n)}.$$

Consider now the change of basis Φ in (1.12), extended to keep $(e_3, \dots, e_n)$ invariant. As this transition matrix commutes with $r_1^{(3)} \dots r_1^{(n)}$, to conclude we only need to show that $\Phi(r^{(1)}) \stackrel{\text{law}}{=} r_1^{(2)}$. Both transformations are reflections, so a sufficient condition is $\Phi(r^{(1)})(e_2) \stackrel{\text{law}}{=} r_1^{(2)}(e_2)$. A simple calculation gives

$$\Phi(r^{(1)})(e_2) = {}^{\mathrm{t}}\Big(0, r_{11} - \frac{|r_{12}|^2}{1-\overline{r_{11}}}, \mathrm{c}\; r_{13}, \dots, \mathrm{c}\; r_{1n}\Big)$$

where the constant c depends uniquely on r_{11} and r_{12}. Hence the desired result is a direct consequence of the identity in law from Theorem 7.1. □

Remark. The above method and the identity in law stated in Theorem 7.1 can be used to prove the following more general version of the slipping lemma. Let $1 \leqslant m \leqslant n-1$, $\delta_1, \dots, \delta_m$ be complex numbers with real part greater than $-1/2$. Let $r_{\delta_k}^{(k)} \sim \nu_{\delta_k}^{(k)}$ $(1 \leqslant k \leqslant m)$ and $r_{\delta_{k-1}+1}^{(k)} \sim \nu_{\delta_{k-1}+1}^{(k)}$ $(2 \leqslant k \leqslant m+1)$ be independent $n \times n$ reflections. Then

$$r_{\delta_1}^{(1)} \dots r_{\delta_m}^{(m)} \stackrel{\text{sp}}{=} r_{\delta_1+1}^{(2)} \dots r_{\delta_m+1}^{(m+1)}.$$

In particular, iterating the above result,

$$r^{(1)} \dots r^{(n-p)} \stackrel{\text{sp}}{=} r_p^{(p+1)} \dots r_p^{(n)}.$$

3.3. The proof by induction.

The two previous lemmas give a recursive proof of the following well-known result.

Theorem 1.18. *Let f be a class function on $\mathrm{U}(n)$: $f(u) = f(\theta_1, \dots, \theta_n)$, where the θ's are the eigenangles of u and f is symmetric. Then*

$$\mathbb{E}_{\mu_{\mathrm{U}(n)}}(f(u)) = \frac{1}{n!} \int_{(-\pi,\pi)^n} f(\theta_1, \dots, \theta_n) \prod_{1 \leqslant k < l \leqslant n} |e^{\mathrm{i}\theta_k} - e^{\mathrm{i}\theta_l}|^2 \frac{\mathrm{d}\theta_1}{2\pi} \dots \frac{\mathrm{d}\theta_n}{2\pi}.$$

Proof. We proceed by induction on n. The case $n = 1$ is obvious. Suppose the result holds at rank $n-1$. Successively using the Conditioning Lemma and the Slipping Lemma, if $u \sim \mu_{\mathrm{U}(n)}$, we get

$$u \stackrel{\text{sp}}{=} e^{\mathrm{i}\omega} r^{(1)} \dots r^{(n-1)} \stackrel{\text{sp}}{=} e^{\mathrm{i}\omega} r_1^{(2)} \dots r_1^{(n)}.$$

Hence, using the recurrence hypothesis, for any class function f

$$\begin{aligned}\mathbb{E}_{\mu_{\mathrm{U}(n)}}(f(u)) &= \frac{1}{\mathrm{cst}} \int_{-\pi}^{\pi} \frac{\mathrm{d}\omega}{2\pi} \frac{1}{(n-1)!} \int_{(-\pi,\pi)^{n-1}} f(\omega, \theta_2 + \omega, \dots, \theta_n + \omega) \\ &\qquad \prod_{2 \leqslant k < l \leqslant n} |e^{\mathrm{i}\theta_k} - e^{\mathrm{i}\theta_l}|^2 \prod_{j=2}^{n} |1 - e^{\mathrm{i}\theta_j}|^2 \frac{\mathrm{d}\theta_2}{2\pi} \dots \frac{\mathrm{d}\theta_n}{2\pi} \\ &= \frac{1}{\mathrm{cst}} \frac{1}{(n-1)!} \int_{(-\pi,\pi)^n} f(\theta_1, \dots, \theta_n) \prod_{1 \leqslant k < l \leqslant n} |e^{\mathrm{i}\theta_k} - e^{\mathrm{i}\theta_l}|^2 \frac{\mathrm{d}\theta_1}{2\pi} \dots \frac{\mathrm{d}\theta_n}{2\pi}.\end{aligned}$$

Here cst comes from the sampling : $\mathrm{cst} = \mathbb{E}_{\mu_{\mathrm{U}(n-1)}}(|\det(\mathrm{Id} - u)|^2)$. This is equal to n from Corollary 1.5, which is probabilistically proven. $\square$

Chapter 2

Hua-Pickrell measures on compact groups

The first two sections of this chapter are extracted from *Ewens measures on compact groups and hypergeometric kernels* [21], with A. Nikeghbali, A. Rouault, to appear in Séminaire de Probabilités.

In this chapter, we note $\mathrm{U}(n, \mathrm{K})$ the unitary group over any field K endowed with a norm (in practice, $\mathrm{K} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$). These groups are defined in the following way : their elements preserve the inner product on K^n,

$$\langle a, b\rangle = \sum_{i=1}^{n} \overline{a_i} b_i.$$

In other words, for $u \in \mathrm{U}(n, \mathrm{K})$, ${}^{\mathrm{t}}\overline{u}u = \mathrm{Id}$, which is equivalent to $u\,{}^{\mathrm{t}}\overline{u} = \mathrm{Id}$ (if ${}^{\mathrm{t}}\overline{u}u = \mathrm{Id}$ then $u\,{}^{\mathrm{t}}\overline{u} = u\,{}^{\mathrm{t}}\overline{u}uu^{-1} = uu^{-1} = \mathrm{Id}$).

Let g be distributed with the Haar measure on the n-dimensional complex unitary group. As we have seen in the introduction, the random variable $\det(\mathrm{Id}_n - g)$ has played a crucial role in recent years in the study of the connections between random matrix theory and analytic number theory, according to the Keating-Snaith paradigm.

In Chapter 1, we treated $\det(\mathrm{Id}_n - g)$ as a random variable : we showed that it can be decomposed as a product of n independent random variables :

$$\det(\mathrm{Id}_n - g) \overset{\text{law}}{=} \prod_{k=1}^{n} \left(1 - e^{i\omega_k} \sqrt{\mathrm{B}_{1,k-1}}\right), \tag{2.1}$$

where $\omega_1, \dots, \omega_n, \mathrm{B}_{1,0}, \dots, \mathrm{B}_{1,n-1}$ are independent random variables, the $\omega_k' s$ being uniformly distributed on $(-\pi, \pi)$ and the $\mathrm{B}_{1,j}$'s ($0 \leqslant j \leqslant n-1$) being beta distributed with parameters 1 and j (with the convention that $\mathrm{B}_{1,0} = 1$). From this decomposition, the Mellin-Fourier transform, as well as the central limit theorem, follow at once (one can actually deduce easily from this decomposition some information about the rate of convergence in the central limit theorem, see Chapter 1).

Such a decomposition could not be easily obtained for the symplectic group, which also plays an important role in the connections between random matrix theory and the study of families of L functions; see [78], [79].

The present chapter first aims at extending (2.1) to other compact groups, including the case of the symplectic group which was left unsolved in Chapter 1. We shall prove that if a subgroup $\mathcal{G}$ of $\mathrm{U}(n, k)$ contains *enough reflections*, in a sense to be made precise in Section 1, then an element of $\mathcal{G}$ drawn according to the Haar measure can be written as a product of n elementary independent reflections (the fact that we allow K to be the field of Quaternions is important in solving the problem for the symplectic group).

In particular, our method applies to the discrete subgroup of (matrices of) permutations of dimension n, $\mathcal{S}_n$, or more precisely to the symmetrized group $\tilde{\mathcal{S}}_n = \{(e^{\mathrm{i}\theta_j}\delta^j_{\sigma(i)})_{1\leqslant i,j\leqslant n} \mid \sigma \in \mathcal{S}_n, (\theta_1, \dots, \theta_n) \in (-\pi, \pi)^n\}$: the corresponding decomposition writes

$$\det(\mathrm{Id}_n - g) \overset{\text{law}}{=} \prod_{k=1}^{n} \left(1 - e^{\mathrm{i}\omega_k} \mathrm{X}_k\right), \tag{2.2}$$

where $\omega_1, \dots, \omega_n, \mathrm{X}_1, \dots, \mathrm{X}_n$ are independent random variables, the ω_k's being uniformly distributed on $(-\pi, \pi)$ and the X_k's being Bernoulli variables : $\mathbb{P}(\mathrm{X}_k = 1) = 1/k$, $\mathbb{P}(\mathrm{X}_k = 0) = 1 - 1/k$.

The method we use to decompose the Haar measure also applies to a two parameters deformation (or sampling) of the Haar measure to yield a generalization of the Ewens sampling formula on $\mathcal{S}_n$, which plays an important role in statistics and applications to mathematical biology (see [4] for more details and references about the Ewens sampling formula). Let us recall this formula for the symmetric group. Define :

a. $\sigma_1 = \tau_1 \circ \dots \circ \tau_n$ where the τ_k's are independent transpositions in $\mathcal{S}_n$, $\tau_k = [k, j]$, $(k \leqslant j \leqslant n)$, with

$$\mathbb{P}(\tau_k(k) = j) = \begin{cases} \frac{\theta}{\theta+n-k} & \text{if} \quad j = k \\ \frac{1}{\theta+n-k} & \text{if} \quad j > k \end{cases} ;$$

b. σ_2 with law $\mu^{(\theta)}$, the sampling of the Haar measure μ on $\mathcal{S}_n$ by a factor θ^{k_σ} (k_σ : the number of cycles of a permutation σ) :

$$\mathbb{E}_{\mu^{(\theta)}}(f(\sigma_2)) = \frac{\mathbb{E}_\mu(f(\sigma_2)\theta^{k_{\sigma_2}})}{\mathbb{E}_\mu(\theta^{k_{\sigma_2}})}$$

for any bounded measurable function f.

Then the Ewens sampling formula can be expressed as the simple equality

$$\sigma_1 \overset{\text{law}}{=} \sigma_2. \tag{2.3}$$

We generalize (2.3) to unitary groups and a particular class of their subgroups. The analogues of transpositions in decomposition (a) are reflections and the sampling (b) is considered relatively to the factor $\det(\mathrm{Id}-g)^{\overline{\delta}}\det(\mathrm{Id}-\overline{g})^{\delta}$, $\delta \in \mathbb{C}$, $\mathfrak{Re}(\delta) > -1/2$: the measure $\mu_{\mathrm{U}(n)}^{(\delta)}$ on $\mathrm{U}(n)$, which is defined by

$$\mathbb{E}_{\mu_{\mathrm{U}(n)}^{(\delta)}}(f(u)) = \frac{\mathbb{E}_{\mu_{\mathrm{U}(n)}}\left(f(u)\det(\mathrm{Id}-u)^{\overline{\delta}}\det(\mathrm{Id}-\overline{u})^{\delta}\right)}{\mathbb{E}_{\mu_{\mathrm{U}(n)}}\left(\det(\mathrm{Id}-u)^{\overline{\delta}}\det(\mathrm{Id}-\overline{u})^{\delta}\right)}$$

for any continuous function f, is the analogue of the Ewens measure and generalizes the Haar measure $\mu_{\mathrm{U}(n)}$.

Such samplings with $\delta \in \mathbb{R}$ have already been studied on the finite-dimensional unitary group by Hua [70], and results about the infinite dimensional case (on complex Grassmannians) were given by Pickrell ([111] and [112]). More recently, Neretin [103] also studied these sampled measures, introducing the possibility $\delta \in \mathbb{C}$. Borodin and Olshanski [16] have used the analogue of this sampled Haar measure on the infinite dimensional unitary group and studied resulting ergodic properties. Following their work about the unitary group, we will refer to these sampled Haar measures as the Hua-Pickrell probability measures. Forrester and Witte [53] also studied these measures (or their projection on the spectrum), referring to them as the circular Jacobi ensemble.

The organization of the chapter is as follows : Section 1 extends (2.1) to other compact groups, such as the unitary group over other fields and the symplectic group. In Section 2 we use the decomposition of Section 1 to derive central limit theorems for the characteristic polynomial. Section 3 generalizes (2.3) to unitary groups and a particular class of their subgroups.

1. Splitting of the characteristic polynomial

In this section, we give conditions under which an element of a subgroup of $\mathrm{U}(n, \mathrm{K})$ (under the Haar measure) can be generated as a product of independent elementary transformations. This will lead to some remarkable identities for the characteristic polynomial.

1.1. Reflections

In the same manner as in Chapter 1, we first define the reflections of a unitary group. This requires some precisions because the field is not necessarily commutative anymore.

In the following definition, the rank of a matrix u is defined as the dimension of the K$-$vector space generated by the columns by multiplication *on the right* :

$$\operatorname{rank}(u) = \dim \left\{ \sum_1^n u_i \lambda_i \mid \lambda_1, \dots, \lambda_n \in \mathrm{K} \right\}.$$

Definition 2.1. *An element r in $\mathrm{U}(n, \mathrm{K})$ will be referred to as a reflection if $r - \mathrm{Id}$ has rank 0 or 1.*

The reflections can also be described in the following way, exactly as in Chapter 1. Let $\mathcal{M}(n)$ be the set of $n \times n$ quaternionic matrices m that can be written

$$m = \left(m_1, e_2 - k \frac{\overline{m}_{12}}{1 - \overline{m}_{11}}, \dots, e_n - k \frac{\overline{m}_{1n}}{1 - \overline{m}_{11}} \right),$$

with the vector $m_1 = {}^{\mathrm{t}}(m_{11}, \dots, m_{1,n}) \neq e_1$ on the n-dimensional unit quaternionic sphere and $k = m_1 - e_1$. Then the reflections are exactly the elements

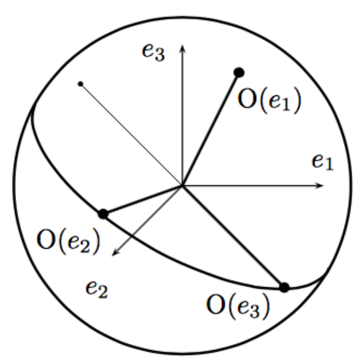
$$r = \begin{pmatrix} \mathrm{Id}_{k-1} & 0 \\ 0 & m \end{pmatrix}$$

with $m \in \mathcal{M}(n-k+1)$ for some $1 \leqslant k \leqslant n$ (note that the above explicit construction yields ${}^{\mathrm{t}}\overline{r} r = \mathrm{Id}$, as expected). For fixed k, the set of these elements is noted $\mathcal{R}^{(k)}$.

1.2. The general equality in law.

There exist distinct ways to generate the Haar measure recursively : for example, Diaconis and Shahshahani [41] give relations between Haar measures on a group and a subgroup, Mezzadri [97] presents a method based on Householder reflections and proved through the Ginibre ensemble. We present here a general simple framework that includes permutation groups or Lie groups.

For example, to generate a Haar-distributed element of $\mathrm{O}(3)$, it seems natural to proceed as follows :



- $\mathrm{O}(e_1)$ is uniform on the unit sphere ;
- $\mathrm{O}(e_2)$ is uniform on the unit circle orthogonal to $\mathrm{O}(e_1)$;
- $\mathrm{O}(e_3)$ is uniform on $\{\mathrm{O}(e_1) \wedge \mathrm{O}(e_2), -\mathrm{O}(e_1) \wedge \mathrm{O}(e_2)\}$.

The lines below are a formalization of this simple idea, for general groups.

Let $\mathcal{G}$ be a subgroup of $\mathrm{U}(n, \mathrm{K})$, the group of unitary matrices of size n over K. Let $(e_1, \dots, e_n)$ be an orthonormal basis of K^n and $\mathcal{H} = \{h \in \mathcal{G} \mid h(e_1) = e_1\}$, the subgroup of $\mathcal{G}$ which stabilizes e_1. For a generic compact group $\mathcal{A}$, we write $\mu_{\mathcal{A}}$ for the unique Haar probability measure on $\mathcal{A}$. The following result holds :

Proposition 2.2. *Let g and h be independent random matrices, $g \in \mathcal{G}$ and $h \in \mathcal{H}$ with distribution $\mu_{\mathcal{H}}$. Then $gh \sim \mu_{\mathcal{G}}$ if and only if $g(e_1) \sim p_1(\mu_{\mathcal{G}})$, where p_1 is the map $u \mapsto u(e_1)$.*

Proof. If $gh \sim \mu_{\mathcal{G}}$, then $g(e_1) = gh(e_1) \sim p_1(\mu_{\mathcal{G}})$.

Suppose now that $g(e_1) \sim p_1(\mu_{\mathcal{G}})$. Thanks to the uniqueness of the Haar probability measure, to prove $gh \sim \mu_{\mathcal{G}}$, it suffices to show

$$agh \overset{\text{law}}{=} gh$$

for any fixed $a \in \mathcal{G}$. Since $g(e_1) \sim p_1(\mu_{\mathcal{G}})$, $ag(e_1) \sim p_1(\mu_{\mathcal{G}})$. Therefore in an orthonormal basis with first element e_1, the matrix ag can be written $(p(e_1), \tilde{p})$ with $p(e_1) \overset{\text{law}}{=} g(e_1)$. Consequently, by conditioning on the value $g(e_1) = p(e_1) = v$, it is sufficient to show that

$$(v, p')h \overset{\text{law}}{=} (v, q')h,$$

for some distributions on p' and q', still assumed to be independent of h. As $v = g(e_1) \sim p_1(\mu_{\mathcal{G}})$, there exists almost surely an element $a_v \in \mathcal{G}$ with $a_v(e_1) = v$. By multiplication of the above equality by a_v^{-1}, we only need to show that

$$p''h \overset{\text{law}}{=} q''h$$

for some elements p'' and q'' in $\mathcal{H}$, again assumed to be independent of h. By conditioning on p'' (resp q''), we know that $p''h \overset{\text{law}}{=} h$ (resp $q''h \overset{\text{law}}{=} h$) by definition of the Haar measure $\mu_{\mathcal{H}}$. This gives the desired result. □

This proposition will enable us to give a simple way to generate the Haar measure on the group $\mathcal{G}$. Before stating the corresponding theorem, we need to introduce the following definition.

Definition 2.3. *Let $\mathcal{G}$ be a subgroup of $\mathrm{U}(n, \mathrm{K})$ and, for all $1 \leqslant k \leqslant n-1$, let μ_k be the Haar measure on this subgroup of $\mathcal{G}$: $\mathcal{H}_k = \{g \in \mathcal{G} \mid g(e_j) = e_j, 1 \leqslant j \leqslant k\}$. We also set $\mathcal{H}_0 = \mathcal{G}$, $\mu_0 = \mu_{\mathcal{G}}$. Moreover, for all $1 \leqslant k \leqslant n$ we define p_k as the map $p_k : u \mapsto u(e_k)$.*

A sequence $(\nu_0, \dots, \nu_{n-1})$ of probability measures on $\mathcal{G}$ is said to be coherent with $\mu_{\mathcal{G}}$ if for all $0 \leqslant k \leqslant n-1$, $\nu_k(\mathcal{H}_k) = 1$ and the probability measures $p_{k+1}(\nu_k)$ and $p_{k+1}(\mu_k)$ are the same.

In the following, $\nu_0 \times \nu_1 \times \cdots \times \nu_{n-1}$ stands for the law of a random variable $h_0 h_1 \dots h_{n-1}$ where all h_i's are independent and $h_i \sim \nu_i$. Now we can provide a general method to generate an element of $\mathcal{G}$ endowed with its Haar measure.

Theorem 2.4. *Let $\mathcal{G}$ be a subgroup of $\mathrm{U}(n, \mathrm{K})$. Let $(\nu_0, \dots, \nu_{n-1})$ be a sequence of coherent measures with $\mu_{\mathcal{G}}$. Then $\mu_{\mathcal{G}}$ and $\nu_0 \times \nu_1 \times \cdots \times \nu_{n-1}$ are the same :*

$$\mu_{\mathcal{G}} = \nu_0 \times \nu_1 \times \cdots \times \nu_{n-1}.$$

Proof. It is sufficient to prove by induction on $1 \leqslant k \leqslant n$ that

$$\nu_{n-k} \times \nu_{n-k+1} \times \cdots \times \nu_{n-1} = \mu_{\mathcal{H}_{n-k}},$$

which gives the desired result for $k = n$. If $k = 1$ this is obvious. If the result is true at rank k, it remains true at rank $k+1$ by a direct application of Proposition 2.2 to the groups $\mathcal{H}_{n-k-1}$ and its subgroup $\mathcal{H}_{n-k}$. □

As an example, take the orthogonal group $\mathrm{O}(n)$. Let $\mathscr{S}_{\mathbb{R}}^{(k)} = \{x \in \mathbb{R}^k \mid |x| = 1\}$ and, for $x_k \in \mathscr{S}_{\mathbb{R}}^{(k)}$, $r_k(x_k)$ the matrix representing the reflection which transforms x_k in the first element of the basis. If the x_k's are uniformly distributed on the $\mathscr{S}_{\mathbb{R}}^{(k)}$'s and independent, then Theorem 2.4 implies that

$$r_n(x_n) \begin{pmatrix} \mathrm{Id}_1 & 0 \\ 0 & r_{n-1}(x_{n-1}) \end{pmatrix} \dots \begin{pmatrix} \mathrm{Id}_{n-2} & 0 \\ 0 & r_2(x_2) \end{pmatrix} \begin{pmatrix} \mathrm{Id}_{n-1} & 0 \\ 0 & r_1(x_1) \end{pmatrix} \sim \mu_{\mathrm{O}(n)}.$$

1.3. Decomposition of determinants as products of independent random variables.

From now on, in the remaining of this chapter, the determinants are supposed to be over commutative fields : the study of determinants over the quaternionic field is not our purpose here.

Let $\mathcal{G}$ and $\mathcal{H}$ be as in the previous subsection and let $\mathcal{R}$ be the set of elements of $\mathcal{G}$ which are reflections : the rank of $\mathrm{Id}-u$, $u\in\mathcal{R}$, is 0 or 1. Define also

$$\mathrm{pr}:\left\{\begin{array}{ccc}\mathcal{H} & \to & \mathrm{U}(n-1,\mathrm{K})\\ h & \mapsto & h_{\mathrm{span}(e_2,\dots,e_n)}\end{array}\right.,$$

where $h_{\mathrm{span}(e_2,\dots,e_n)}$ is the restriction of h to $\mathrm{span}(e_2,\dots,e_n)$. Now suppose that

$$\{g(e_1)\mid g\in\mathcal{G}\}=\{r(e_1)\mid r\in\mathcal{R}\}. \tag{2.4}$$

Under this additional condition the following proposition allows to represent the characteristic polynomial of $\mathcal{G}$ as a product of two independent variables.

Proposition 2.5. *Let g $(\sim\mu_{\mathcal{G}})$, g' $(\sim\mu_{\mathcal{G}})$ and h $(\sim\mu_{\mathcal{H}})$ be independent. Suppose that condition (2.4) holds. Then*

$$\det(\mathrm{Id}_n-g)\overset{\text{law}}{=}(1-\langle e_1,g'(e_1)\rangle)\det(\mathrm{Id}_{n-1}-\mathrm{pr}(h)).$$

Proof. Note that in Proposition 2.2, we can choose any matrix in $\mathrm{U}(n,\mathrm{K})$ with its first column having distribution $p_1(\mu_{\mathcal{G}})$. Let us choose the simplest suitable transformation : namely r, the reflection mapping e_1 onto $r(e_1)$ if $r(e_1)\neq e_1$ (Id if $r(e_1)=e_1$) with $r(e_1)\overset{\text{law}}{=}g(e_1)$ independent of h. Thanks to condition (2.4), $r\in\mathcal{G}$. Define the vector k as $k=r(e_1)-e_1$. There exists $(\lambda_2,\dots,\lambda_n)\in\mathrm{K}^{n-1}$ such that

$$r=(e_1+k,e_2+k\lambda_2,\dots,e_n+k\lambda_n)\,.$$

Hence from Proposition 2.2, one can write

$$\det(\mathrm{Id}-g)\overset{\text{law}}{=}\det(\mathrm{Id}-rh)=\det\left({}^{\mathrm{t}}\overline{h}-r\right)\det h.$$

If we call $(u_1,\dots,u_{n-1})={}^{\mathrm{t}}\overline{\mathrm{pr}(h)}$ then using the multi-linearity of the determinant, we get

$$\begin{aligned}
\det({}^{\mathrm{t}}\overline{h}-r) &= \det\left(-k,\begin{pmatrix}0\\u_1\end{pmatrix}-e_2-k\lambda_2,\dots,\begin{pmatrix}0\\u_{n-1}\end{pmatrix}-e_n-k\lambda_n\right)\\
&= \det\left(-k,\begin{pmatrix}0\\u_1\end{pmatrix}-e_2,\dots,\begin{pmatrix}0\\u_{n-1}\end{pmatrix}-e_n\right)\\
&= \det\left(\begin{array}{c|c}-k_1 & 0\\ \dots & {}^{\mathrm{t}}\overline{\mathrm{pr}(h)}-\mathrm{Id}_{n-1}\end{array}\right)\\
\det({}^{\mathrm{t}}\overline{h}-r) &= -k_1\det\left({}^{\mathrm{t}}\overline{\mathrm{pr}(h)}-\mathrm{Id}_{n-1}\right).
\end{aligned}$$

Finally, $\det(\mathrm{Id}-g)\overset{\text{law}}{=}-k_1\det(\mathrm{Id}-\mathrm{pr}(h))$, with $-k_1=1-\langle e_1,r(e_1)\rangle\overset{\text{law}}{=}1-\langle e_1,g'(e_1)\rangle$ and h independent. $\square$

This decomposition can be iterated to write the determinant as a product of independent random variables. We first need the equivalent of condition (2.4) for every dimension.

Definition 2.6. *Note $\mathcal{R}_k$ the set of elements in $\mathcal{H}_k$ which are reflections. If for all $0\leqslant k\leqslant n-1$*

$$\{r(e_{k+1})\mid r\in\mathcal{R}_k\}=\{h(e_{k+1})\mid h\in\mathcal{H}_k\},$$

the group $\mathcal{G}$ will be said to satisfy condition (R) (R standing for reflection).

Remark. We do not know an easy classification of groups satisfying condition (R). For example, the symmetric group belongs to this class but not the alternate group.

The following result now follows immediately from Proposition 2.5, combined with an induction on n :

Theorem 2.7. *Let $\mathcal{G}$ be a subgroup of $\mathrm{U}(n,\mathrm{K})$ satisfying condition (R), and let $(\nu_0,\dots,\nu_{n-1})$ be coherent with $\mu_{\mathcal{G}}$. Take $h_k \sim \nu_k$, $0 \leqslant k \leqslant n-1$, and $g \sim \mu_{\mathcal{G}}$, all being assumed independent. Then*

$$\det(\mathrm{Id}-g) \overset{\text{law}}{=} \prod_{k=0}^{n-1} \left(1 - \langle h_k(e_{k+1}), e_{k+1}\rangle\right).$$

1.4. Unitary groups.

Take $\mathcal{G} = \mathrm{U}(n,\mathbb{C})$. Then $\mu_{\mathcal{H}_k} = f_k(\mu_{\mathrm{U}(n-k,\mathbb{C})})$ where $f_k : \mathrm{A} \in \mathrm{U}(n-k,\mathbb{C}) \mapsto \mathrm{Id}_k \oplus \mathrm{A}$. As all reflections with respect to a hyperplane of $\mathbb{C}^{n-k}$ are elements of $\mathrm{U}(n-k,\mathbb{C})$, one can apply Theorem 2.7. The Hermitian products $\langle e_k, h_k(e_k)\rangle$ are distributed as the first coordinate of the first vector of an element of $\mathrm{U}(n-k,\mathrm{K})$, that is to say the first coordinate of the $(n-k)$-dimensional unit complex sphere with uniform measure : $\langle h_k(e_{k+1}), e_{k+1}\rangle \overset{\text{law}}{=} e^{\mathrm{i}\omega_n}\sqrt{\mathrm{B}_{1,n-k-1}}$ with ω_n uniform on $(-\pi,\pi)$ and independent of $\mathrm{B}_{1,n-k-1}$, a beta variable with parameters 1 and $n-k-1$.

Therefore, as a consequence of Theorem 2.7, we obtain the following decomposition formula derived in Chapter 1 : let $g \in \mathrm{U}(n,\mathbb{C})$ be $\mu_{\mathrm{U}(n,\mathbb{C})}$ distributed. Then

$$\det(\mathrm{Id}-g) \overset{\text{law}}{=} \prod_{k=1}^{n} \left(1 - e^{\mathrm{i}\omega_k}\sqrt{\mathrm{B}_{1,k-1}}\right),$$

with $\omega_1,\dots,\omega_n,\mathrm{B}_{1,0},\dots,\mathrm{B}_{1,n-1}$ independent random variables, the ω_k's uniformly distributed on $(-\pi,\pi)$ and the $\mathrm{B}_{1,j}$'s $(0 \leqslant j \leqslant n-1)$ being beta distributed with parameters 1 and j (by convention, $\mathrm{B}_{1,0}=1$).

The generality of Theorem 2.7 allows us to apply it to other groups such as $\mathrm{SO}(2n) = \{g \in \mathrm{O}(2n) \mid \det(g)=1\}$. A similar reasoning (with the complex unit spheres replaced by the real ones) yields :

Corollary 2.8. Special orthogonal group. *Let $g \in \mathrm{SO}(2n)$ be $\mu_{\mathrm{SO}(2n)}$ distributed. Then*

$$\det(\mathrm{Id}-g) \overset{\text{law}}{=} 2\prod_{k=2}^{2n} \left(1 - \varepsilon_k\sqrt{\mathrm{B}_{\frac{1}{2},\frac{k-1}{2}}}\right),$$

with $\varepsilon_1,\dots,\varepsilon_{2n},\mathrm{B}_{1/2,1/2},\dots,\mathrm{B}_{1/2,(2n-1)/2}$ independent random variables, $\mathbb{P}(\varepsilon_k = 1) = \mathbb{P}(\varepsilon_k = -1) = 1/2$, and the B's being beta distributed with the indicated parameters.

1.5. The symplectic group.

Following the Katz-Sarnak philosophy ([78] and [79]), the study of moments of families of L-functions gives great importance to $\det(\mathrm{Id}-g)$ for $g \in \mathrm{U}(n,\mathbb{C})$, $\mathrm{SO}(2n)$ but also for g in $\mathrm{USp}(2n,\mathbb{C}) = \{u \in \mathrm{U}(2n,\mathbb{C}) \mid uz\,{}^{\mathrm{t}}u = z\}$, with

$$z = \begin{pmatrix} 0 & \mathrm{Id}_n \\ -\mathrm{Id}_n & 0 \end{pmatrix}. \tag{2.5}$$

Getting a decomposition as a product of independent random variables for $\mathcal{G} = \mathrm{USp}(2n,\mathbb{C})$ requires some additional work, since condition (R) does not apply in that case. We shall overcome this obstacle by using condition (R) after application of a suitable ring morphism.

Let K be a subfield of $\mathbb{H}$ and let K' be a subfield of $\mathbb{C}$. We write $\mathrm{M}(m,\mathrm{K}')$ for the ring of linear transformations on K'^m. Let $\varphi:\mathrm{K}\to\mathrm{M}(m,\mathrm{K}')$ be a continuous injective ring morphism such that $\varphi(\overline{x})={}^{\mathrm{t}}\overline{\varphi(x)}$. This morphism trivially induces the ring morphism (abusively noted the same way)

$$\varphi:\left\{\begin{array}{ccc}\mathrm{M}(n,\mathrm{K}) & \to & \mathrm{M}(nm,\mathrm{K}')\\ (a_{ij})_{1\leqslant i,j\leqslant n} & \mapsto & (\varphi(a_{ij}))_{1\leqslant i,j\leqslant n}\end{array}\right. .$$

More generally, for any matrix A of size $s\times t$ with entries in K, we write $\varphi(\mathrm{A})$ the $sm\times tm$ matrix with entries the $\varphi(a_{ij})$'s.

Let $\mathcal{G}$ be a subgroup of $\mathrm{U}(n,\mathrm{K})$; then $\varphi(\mathcal{G})$ is a subgroup of $\mathrm{U}(nm,\mathrm{K}')$. The action of φ can be applied to Theorem 2.4 and implies, with the notation of this Theorem,

$$\varphi(\mu_{\mathcal{G}})=\varphi(\nu_0)\times\varphi(\nu_1)\times\cdots\times\varphi(\nu_{n-1}),$$

because $\varphi(uv)=\varphi(u)\varphi(v)$. The measure $\varphi(\mu_{\mathcal{G}})$ is invariant by left translation by any element of $\varphi(\mathcal{G})$. Indeed, for $x\sim\mu_{\mathcal{G}}$ and $a\in\mathcal{G}$, $ax\overset{\text{law}}{=}x$, which yields

$$\varphi(a)\varphi(x)=\varphi(ax)\overset{\text{law}}{=}\varphi(x).$$

Hence, by uniqueness of the Haar measure, $\varphi(\mu_{\mathcal{G}})=\mu_{\varphi(\mathcal{G})}$, so that

$$\mu_{\varphi(\mathcal{G})}=\varphi(\nu_0)\times\varphi(\nu_1)\times\cdots\times\varphi(\nu_{n-1}).$$

This constitutes an analogue of Theorem 2.4 about the decomposition of the Haar measure. What would be the counterpart of Theorem 2.7 about the decomposition of the determinant ? We have the following extension.

Theorem 2.9. *Let $\mathcal{G}$ be a subgroup of $\mathrm{U}(n,\mathrm{K})$ checking (R), and $(\nu_0,\dots,\nu_{n-1})$ coherent with $\mu_{\mathcal{G}}$. Take h_k $(\sim\nu_k)$, $0\leqslant k\leqslant n-1$, and $g(\sim\mu_{\varphi(\mathcal{G})})$, all being assumed independent. Then*

$$\det(\mathrm{Id}_{nm}-g)\overset{\text{law}}{=}\prod_{k=0}^{n-1}\det\left(\mathrm{Id}_m-\varphi(\langle e_{k+1},h_k(e_{k+1})\rangle)\right).$$

To prove this theorem, we only need the following analogue of Proposition 2.5.

Proposition 2.10. *Suppose $\mathcal{G}$ contains a reflection r. Then, for $h\in\mathcal{H}$,*

$$\det\left(\mathrm{Id}_{nm}-\varphi(rh)\right)=\det(\mathrm{Id}_m-\varphi(\langle e_1,r(e_1)\rangle))\det(\mathrm{Id}_{m(n-1)}-\varphi(\mathrm{pr}(h))).$$

Proof. Define the vector k as $k=r(e_1)-e_1$. There exists $(\lambda_2,\dots,\lambda_n)\in\mathrm{K}^{n-1}$ such that

$$r=\left(e_1+k,e_2+k\lambda_2,\dots,e_n+k\lambda_n\right).$$

One can write

$$\det\left(\mathrm{Id}-\varphi(rh)\right)=\det\left(\varphi({}^{\mathrm{t}}\overline{h})\varphi(h)-\varphi(r)\varphi(h)\right)\det\left(\varphi({}^{\mathrm{t}}\overline{h})-\varphi(r)\right)\det(\varphi(h)).$$

If we call $(u_1,\dots,u_{n-1})={}^{\mathrm{t}}\overline{\mathrm{pr}(h)}$ then using the multi-linearity of the determinant, we get

$$\begin{aligned}\det(\varphi({}^{\mathrm{t}}\overline{h})-\varphi(r))&=\det\Bigg(\varphi(-k),\varphi\left(\begin{pmatrix}0\\u_1\end{pmatrix}\right)-\varphi(e_2)-\varphi(k)\varphi(\lambda_2),\dots,\\&\qquad\varphi\left(\begin{pmatrix}0\\u_{n-1}\end{pmatrix}\right)-\varphi(e_n)-\varphi(k)\varphi(\lambda_n)\Bigg)\\&=\det\left(\varphi(-k),\varphi\left(\begin{pmatrix}0\\u_1\end{pmatrix}\right)-\varphi(e_2),\dots,\varphi\left(\begin{pmatrix}0\\u_{n-1}\end{pmatrix}\right)-\varphi(e_n)\right)\\&=\det\left(\begin{array}{c|c}\varphi(-k_1) & 0\\ \dots & \varphi({}^{\mathrm{t}}\overline{\mathrm{pr}(h)}-\mathrm{Id}_{n-1})\end{array}\right)\\&=\varphi(-k_1)\det\left(\varphi({}^{\mathrm{t}}\overline{\mathrm{pr}(h)})-\mathrm{Id}_{m(n-1)}\right).\end{aligned}$$

We can use the multi-linearity of the determinant from line 2 to 3 for the following reason : if $k = (l_1, \dots, l_m)$, every column of the $m \times mn$ matrix $\varphi(k)\varphi(\lambda)$ is a linear combination of the l_j's, hence it can be removed by adding the appropriate linear combinations of columns of $\varphi(-k)$. This concludes the proof. $\square$

Remark. In the previous proof, we see a good reason for our choice of reflections : if they were defined relatively to multiplication *on the left*, the matrices $\varphi(\lambda)\varphi(k)$ would not make sense.

Consequently, $\det(\mathrm{Id} - g)$, for $g \sim \mu_{\mathrm{USp}(2n,\mathbb{C})}$, can be split in a product of n independent random variables. This is an easy application of Theorem 2.9 with

$$\varphi : \left\{ \begin{array}{ccc} \mathbb{H} & \to & \mathrm{M}(2,\mathbb{C}) \\ a + \mathrm{i}b + \mathrm{j}c + \mathrm{k}d & \mapsto & \begin{pmatrix} a+\mathrm{i}b & c+\mathrm{i}d \\ -c+\mathrm{i}d & a-\mathrm{i}b \end{pmatrix} \end{array} \right. ,$$

the usual representation of quaternions. Indeed, for such a choice of φ, $\Phi(\mathrm{U}(n,\mathbb{C}))$ is precisely the set of elements in $g \in \mathrm{U}(2n,\mathbb{C})$ satisfying $g\tilde{z}\,{}^{\mathrm{t}}g = \tilde{z}$. Here, $\mathrm{J}_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\tilde{z} = \mathrm{J}_2 \oplus \dots \oplus \mathrm{J}_2$ is conjugate to z, defined by (2.5). The set $\Phi(\mathrm{U}(n,\mathbb{C}))$ is therefore conjugate to $\mathrm{USp}(2n,\mathbb{C})$, so the law of $\det(\mathrm{Id} - g)$ is the same in both sets endowed with their respective Haar measure. As

$$\det\left(\mathrm{Id} - \begin{pmatrix} a+\mathrm{i}b & c+\mathrm{i}d \\ -c+\mathrm{i}d & a-\mathrm{i}b \end{pmatrix}\right) = (a-1)^2 + b^2 + c^2 + d^2,$$

the desired decomposition follows from Theorem 2.9.

Corollary 2.11. Symplectic group. *Let $g \in \mathrm{USp}(2n,\mathbb{C})$ be $\mu_{\mathrm{USp}(2n,\mathbb{C})}$ distributed. Then*

$$\det(\mathrm{Id} - g) \overset{\text{law}}{=} \prod_{k=1}^{n} \left((a_k - 1)^2 + b_k^2 + c_k^2 + d_k^2\right),$$

with the vectors (a_k, b_k, c_k, d_k), $1 \leqslant k \leqslant n$, being independent and (a_k, b_k, c_k, d_k) 4 coordinates of the $4k$-dimensional real unit sphere endowed with the uniform measure ; hence $(a_k, b_k, c_k, d_k) \overset{\text{law}}{=} \frac{1}{\sqrt{\mathcal{N}_1^2 + \dots + \mathcal{N}_{4k}^2}}(\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3, \mathcal{N}_4)$, with the $\mathcal{N}_i'$s independent standard normal variables.

Remark. If $\mathcal{N}_1, \mathcal{N}_2, \dots, \mathcal{N}_k, \dots, \mathcal{N}_n$ are independent standard normal variables, then

$$\frac{\mathcal{N}_1^2 + \dots + \mathcal{N}_k^2}{\mathcal{N}_1^2 + \dots + \mathcal{N}_n^2} \overset{\text{law}}{=} \mathrm{B}_{\frac{k}{2}, \frac{n-k}{2}}.$$

Consequently, with the notation of Corollary 2.11,

$$\left(a_k^2, b_k^2 + c_k^2 + d_k^2\right) \overset{\text{law}}{=} \left(\mathrm{B}_{\frac{1}{2}, 2k-\frac{1}{2}}, \left(1 - \mathrm{B}_{\frac{1}{2}, 2k-\frac{1}{2}}\right) \mathrm{B}'_{\frac{3}{2}, 2k-2}\right),$$

with B and B′ independent beta variables with the specified parameters. This gives the somehow more tractable identity in law

$$\det(\mathrm{Id} - g) \overset{\text{law}}{=} \prod_{k=1}^{n} \left(\left(1 + \varepsilon_k \sqrt{\mathrm{B}_{\frac{1}{2}, 2k-\frac{1}{2}}}\right)^2 + \left(1 - \mathrm{B}_{\frac{1}{2}, 2k-\frac{1}{2}}\right) \mathrm{B}'_{\frac{3}{2}, 2k-2} \right),$$

with all variables independent, $\mathbb{P}(\varepsilon_k = 1) = \mathbb{P}(\varepsilon_k = -1) = 1/2$.

Moreover, note that our method can be applied to other interesting groups such as $\mathrm{USp}(2n,\mathbb{R}) = \{u \in \mathrm{U}(2n,\mathbb{R}) \mid uz\,{}^{\mathrm{t}}u = z\}$ thanks to the morphism

$$\varphi : \left\{ \begin{array}{ccc} \mathbb{C} & \to & \mathrm{M}(2,\mathbb{R}) \\ a + \mathrm{i}b & \mapsto & \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \end{array} \right. .$$

The traditional representation of the quaternions in $\mathrm{M}(4,\mathbb{R})$

$$\varphi : \left\{ \begin{array}{ccc} \mathbb{H} & \to & \mathrm{M}(4,\mathbb{R}) \\ a+\mathrm{i}b+\mathrm{j}c+\mathrm{k}d & \mapsto & \begin{pmatrix} a & -b & -c & -d \\ b & a & -d & -c \\ c & d & a & -b \\ d & -c & b & a \end{pmatrix} \end{array} \right.$$

gives another identity in law for a compact exotic subgroup of $\mathrm{U}(4n,\mathbb{R})$.

1.6. The symmetric group.

Consider now $\mathcal{S}_n$ the group of permutations of size n. An element $\sigma \in \mathcal{S}_n$ can be identified with the matrix $(\delta^j_{\sigma(i)})_{1\leqslant i,j\leqslant \mathrm{N}}$ (δ is Kronecker's symbol).

Let H be a subgroup of $\left\{x \in \mathbb{H} \mid |x|^2 = 1\right\}$, endowed with the Haar probability measure μ_{H}, and let H_n be the group of diagonal matrices of size n with diagonal elements in H. Then the semidirect product $\mathcal{G} = \mathrm{H}^n \cdot \mathcal{S}_n$ gives another example of determinant-splitting. More explicitly,

$$\mathcal{G} = \{(h_j\delta^j_{\sigma(i)})_{1\leqslant i,j\leqslant n} \mid \sigma \in \mathcal{S}_n, (h_1,\dots,h_n) \in \mathrm{H}^n\}.$$

As the reflections correspond now to the transpositions, condition (R) holds. Moreover, with the notation of Theorem 2.7, $h_k(e_{k+1})$ is uniformly distributed on the unit sphere $\{xe_{k+1} \mid x \in \mathrm{H}\} \cup \cdots \cup \{xe_n \mid x \in \mathrm{H}\}$. Therefore, following Theorem 2.7, we can state the following decomposition result :

Corollary 2.12. *Let $g \in \mathcal{G}(= \mathrm{H}^n \cdot \mathcal{S}_n)$ be $\mu_{\mathcal{G}}$ distributed. Then*

$$\det(\mathrm{Id} - g) \overset{\mathrm{law}}{=} \prod_{k=1}^{n} (1 - x_k \mathrm{X}_k)\,,$$

with $x_1,\dots,x_n,\mathrm{X}_1,\dots,\mathrm{X}_n$ independent random variables, the x_k's μ_{H} distributed, $\mathbb{P}(\mathrm{X}_k = 1) = 1/k$, $\mathbb{P}(\mathrm{X}_k = 0) = 1 - 1/k$.

Remark. Let k_σ be the number of cycles of a random permutation of size n, with respect to the (probability) Haar measure. Corollary 2.12 allows us to recover the law of k_σ :

$$k_\sigma \overset{\mathrm{law}}{=} \mathrm{X}_1 + \cdots + \mathrm{X}_n,$$

with the X_k's Bernoulli variables as previously. Indeed, take for example $\mathrm{H} = \{-1,1\}$ in the Corollary. If a permutation $\sigma \in \mathcal{S}_n$ has k_σ cycles with lengths $l_1,\dots,l_{k_\sigma}$ ($\sum_k l_k = n$), then it is easy to see that

$$\det(x\mathrm{Id} - g) = \prod_{k=1}^{k_\sigma} (x^{l_k} - \eta_k)$$

with the η_k's independent and uniform on $\{-1,1\}$. Using this relation and the result of Corollary 2.12 we get

$$\prod_{k=1}^{n} (1 - x_k \mathrm{X}_k) \overset{\mathrm{law}}{=} \prod_{k=1}^{k_\sigma} (1 - \eta_k),$$

the x_k's being also independent and uniform on $\{-1,1\}$. The equality of the Mellin transforms yields, after conditioning by the X_k's and k_σ,

$$\mathbb{E}\left(e^{\lambda(\mathrm{X}_1+\cdots+\mathrm{X}_k)}\right) = \mathbb{E}\left(e^{\lambda k_\sigma}\right)$$

for any $\lambda \in \mathbb{R}$, giving the expected result. Note that conversely Corollary 2.12 follows from the law of k_σ.

Remark. The above Corollary (2.12) deals with the permutation group $\mathrm{H}^n \cdot \mathcal{S}_n$. For central limit theorems concerning the permutation group itself (i.e. limit theorems for $\log\det(e^{\mathrm{i}\theta} - g)$ with $g \in \mathcal{S}_n$), the reader should read [64]

In this section we have shown that for $g \in \mathcal{G}$, a general compact group endowed with its Haar measure, $\det(\mathrm{Id} - g)$ can be decomposed as a product of independent random variables. This can be generalized to some *h-sampling* of the Haar measure. This will lead us in Section 4 to a generalization of the Ewens sampling formula, well known for the symmetric group.

2. Limit theorems

The decomposition of the characteristic polynomial as a product of independent random variables for the groups $\mathrm{SO}(2n)$, $\mathrm{US}p(2n)$ and

$$(\partial\mathbb{D})^n \cdot \mathcal{S}_n = \{(e^{\mathrm{i}\theta_j}\delta^j_{\sigma(i)})_{1\leqslant i,j\leqslant n} \mid \sigma \in \mathcal{S}_n, (\theta_1, \dots, \theta_n) \in (-\pi,\pi)^n\},$$

implies central limit theorems for its logarithm, together with an estimate for the rate of convergence.

The proof follows from Corollaries 2.8, 2.11, 2.12, and from the Berry-Esseen inequality, already stated in Chapter 1 as Theorem 1.9.

Corollary 2.13. *Let g_n be $\mu_{\mathrm{SO}(2n)}$ distributed. Then as $n \to \infty$*

$$\frac{\log\det(\mathrm{Id} - g_n) + \frac{1}{2}\log n}{\sqrt{\log n}} \xrightarrow{\text{law}} \mathcal{N},$$

where $\mathcal{N}$ is a standard real normal variable. Moreover, this convergence holds with speed $1/(\log n)^{3/2}$: there is a universal constant $c > 0$ such that for any $n \geqslant 1$ and $x \in \mathbb{R}$

$$\left|\mathbb{P}\left(\frac{\log\det(\mathrm{Id} - g_n) + \frac{1}{2}\log n}{\sqrt{\log n}} \leqslant x\right) - \Phi(x)\right| \leqslant \frac{c}{(\log n)^{3/2}(1+|x|)^3}.$$

The same result holds if g_n is $\mu_{\mathrm{US}p(2n)}$ distributed, but with drift $-\frac{1}{2}\log n$ instead of $\frac{1}{2}\log n$ in the two above formulae.

Remark. The above Corollary admits a generalization to any Jacobi ensemble, shown by a different method in Chapter 5.

Now let us consider the case of $(\partial\mathbb{D})^n \cdot \mathcal{S}_n$. Let the $\mathrm{X}_k's$ and ω_k's be all independent, $\mathbb{P}(\mathrm{X}_k = 1) = 1/k$, $\mathbb{P}(\mathrm{X}_k = 0) = 1 - 1/k$, and the ω_k's uniform on $(-\pi,\pi)$. Then the exotic normalization in the following result comes from Corollary 2.12 and the calculation

$$\sum_{k=1}^n \mathrm{var}(|\log(1 - e^{\mathrm{i}\omega_k}\mathrm{X}_k)|^2) \underset{n\to\infty}{\sim} \mathbb{E}(|\log(1 - e^{\mathrm{i}\omega_1})|^2)\sum_{k=1}^n \frac{1}{k},$$

and $\mathbb{E}(|\log(1 - e^{\mathrm{i}\omega_1})|^2) = \sum_{\ell=1}^\infty 1/\ell^2 = \pi^2/6$, as shows the Taylor expansion $\log(1 - e^{\mathrm{i}\omega_1}) = -\sum_{\ell\geqslant 1} e^{\mathrm{i}\ell\omega_1}/\ell$.

Note also that an application of the Berry-Esseen inequality gives a slower rate of convergence, only $1/\sqrt{\log n}$ in this case.

Corollary 2.14. *Let g_n have distribution the Haar measure on $(\partial\mathbb{D})^n \cdot \mathcal{S}_n$. Then as $n \to \infty$*

$$\frac{\log\det(\mathrm{Id} - g_n)}{\sqrt{\frac{\pi^2}{12}\log n}} \xrightarrow{\text{law}} \mathcal{N}_1 + \mathrm{i}\mathcal{N}_2,$$

where $\mathcal{N}_1$ and $\mathcal{N}_2$ are independent standard real normal variables. Moreover, this convergence holds with speed $1/\sqrt{\log n}$, for the real and imaginary parts.

Note that the decompositions as products of independent random variables can also be used to get iterated logarithm laws in the same manner as Proposition 1.12, or large and moderate deviations estimates.

3. The Ewens sampling formula on general compact groups

Here again, a permutation $\sigma \in \mathcal{S}_n$ is identified with the matrix $(\delta^j_{\sigma(i)})_{1\leqslant i,j\leqslant n}$. The Haar measure on $\mathcal{S}_n$ can be generated by induction. Indeed, let $\tau_1,\dots,\tau_n$ be independent transpositions respectively in $\mathcal{S}_1,\dots,\mathcal{S}_n$, with $\mathbb{P}(\tau_k(1)=j)=1/k$ for any $1\leqslant j\leqslant k$. Theorem 2.4 shows that if $\sigma\sim\mu_{\mathcal{S}_n}$ then

$$\sigma \overset{\text{law}}{=} \tau_n \begin{pmatrix} \mathrm{Id}_1 & 0 \\ 0 & \tau_{n-1} \end{pmatrix} \dots \begin{pmatrix} \mathrm{Id}_{n-2} & 0 \\ 0 & \tau_2 \end{pmatrix} \begin{pmatrix} \mathrm{Id}_{n-1} & 0 \\ 0 & \tau_1 \end{pmatrix}. \tag{2.6}$$

Read from right to left, the RHS of (2.6) corresponds to the so-called Chinese restaurant process, while from left to right this is the Feller decomposition of the symmetric group (see e.g. [4]).

What if the independent distributions of the $\tau_k(1)$'s are not uniform anymore? Let $\theta>0$. If for all $k\geqslant 1$

$$\mathbb{P}(\tau_k(1)=j)=\begin{cases} \frac{\theta}{\theta+k-1} & \text{if} \quad j=1 \\ \frac{1}{\theta+k-1} & \text{if} \quad j\neq 1 \end{cases}, \tag{2.7}$$

then the distribution $\mu^{(\theta)}_{\mathcal{S}_n}$ of

$$\sigma = \tau_n \begin{pmatrix} \mathrm{Id}_1 & 0 \\ 0 & \tau_{n-1} \end{pmatrix} \dots \begin{pmatrix} \mathrm{Id}_{n-2} & 0 \\ 0 & \tau_2 \end{pmatrix} \begin{pmatrix} \mathrm{Id}_{n-1} & 0 \\ 0 & \tau_1 \end{pmatrix}.$$

can be expressed as a deformation of the Haar measure $\mu_{\mathcal{S}_n}=\mu^{(1)}_{\mathcal{S}_n}$: for a fixed $\Sigma\in\mathcal{S}_n$

$$\mathbb{P}_{\mu^{(\theta)}_{\mathcal{S}_n}}(\sigma=\Sigma)=\frac{\theta^{k_\Sigma}}{\mathbb{E}_{\mu_{\mathcal{S}_n}}(\theta^{k_\sigma})}\,\mathbb{P}_{\mu_{\mathcal{S}_n}}(\sigma=\Sigma),$$

with k_Σ the number of cycles of the permutation Σ. This is the Ewens sampling formula (for a direct proof, see e.g. [4]), which can also be formulated this way : for any bounded or positive function f from $\mathcal{S}_n$ to $\mathbb{R}$

$$\mathbb{E}_{\mu^{(\theta)}_{\mathcal{S}_n}}(f(\sigma))=\frac{\mathbb{E}_{\mu_{\mathcal{S}_n}}\left(f(\sigma)\theta^{k_\sigma}\right)}{\mathbb{E}_{\mu_{\mathcal{S}_n}}(\theta^{k_\sigma})}, \tag{2.8}$$

which means that $\mu^{(\theta)}_{\mathcal{S}_n}$ is the θ^{k_σ}-sampling of $\mu_{\mathcal{S}_n}$. Our purpose here is to generalize the non-uniform measure (2.7) to any compact group, and to derive the corresponding equivalent of the Ewens sampling formula (2.8).

As usual, in the following, $\mathcal{G}$ is any subgroup of $\mathrm{U}(n,\mathrm{K})$. Take $\delta\in\mathbb{C}$ such that

$$0<\mathbb{E}_{\mu_{\mathcal{G}}}\left(\det(\mathrm{Id}-g)^{\overline{\delta}}\det(\mathrm{Id}-\overline{g})^{\delta}\right)<\infty. \tag{2.9}$$

For $0\leqslant k\leqslant n-1$ we note

$$\exp^{(k)}_\delta : \begin{cases} \mathcal{G} & \to \quad \mathbb{R}^+ \\ g & \mapsto \quad (1-\langle g(e_{k+1}),e_{k+1}\rangle)^{\overline{\delta}}(1-\overline{\langle g(e_{k+1}),e_{k+1}\rangle})^{\delta} \end{cases}.$$

Moreover, define $\det_\delta$ as the function

$$\det\nolimits_\delta : \begin{cases} \mathcal{G} & \to \quad \mathbb{R}^+ \\ g & \mapsto \quad \det(\mathrm{Id}-g)^{\overline{\delta}}\det(\mathrm{Id}-\overline{g})^{\delta} \end{cases}.$$

Then the following generalization of Theorem 2.4 (which corresponds to the case $\delta=0$) holds. However, note that, contrary to Theorem 2.4, in the following result we need that the coherent measures $\nu_0,\dots,\nu_{n-1}$ be supported by the set of reflections.

Theorem 2.15. Generalized Ewens sampling formula. *Let $\mathcal{G}$ be a subgroup of* $\mathrm{U}(n,\mathrm{K})$ *checking condition (R) and (2.9). Let $(\nu_0,\dots,\nu_{n-1})$ be a sequence of measures coherent with $\mu_{\mathcal{G}}$, with $\nu_k(\mathcal{R}_k)=1$. We note $\mu_{\mathcal{G}}^{(\delta)}$ the $\det_\delta$-sampling of $\mu_{\mathcal{G}}$ and $\nu_k^{(\delta)}$ the $\exp_\delta^{(k)}$-sampling of ν_k. Then*

$$\nu_0^{(\delta)}\times\nu_1^{(\delta)}\times\dots\times\nu_n^{(\delta)}=\mu_{\mathcal{G}}^{(\delta)},$$

that is to say, for all bounded measurable functions f on $\mathcal{G}$,

$$\mathbb{E}_{\nu_0^{(\delta)}\times\dots\times\nu_{n-1}^{(\delta)}}\left(f(r_0r_1\dots r_{n-1})\right)=\frac{\mathbb{E}_{\mu_{\mathcal{G}}}\left(f(g)\det(\mathrm{Id}-g)^{\overline{\delta}}\det(\mathrm{Id}-\overline{g})^{\delta}\right)}{\mathbb{E}_{\mu_{\mathcal{G}}}\left(\det(\mathrm{Id}-g)^{\overline{\delta}}\det(\mathrm{Id}-\overline{g})^{\delta}\right)}.$$

Proof. From Theorem 2.4,

$$\frac{\mathbb{E}_{\mu_{\mathcal{G}}}\left(f(g)\det(\mathrm{Id}-g)^{\overline{\delta}}\det(\mathrm{Id}-\overline{g})^{\delta}\right)}{\mathbb{E}_{\mu_{\mathcal{G}}}\left(\det(\mathrm{Id}-g)^{\overline{\delta}}\det(\mathrm{Id}-\overline{g})^{\delta}\right)}$$
$$=\frac{\mathbb{E}_{\nu_0\times\dots\times\nu_{n-1}}\left(f(r_0\dots r_{n-1})\det(\mathrm{Id}-r_0\dots r_{n-1})^{\overline{\delta}}\det(\mathrm{Id}-\overline{r_0\dots r_{n-1}})^{\delta}\right)}{\mathbb{E}_{\nu_0\times\dots\times\nu_{n-1}}\left(\det(\mathrm{Id}-r_0\dots r_{n-1})^{\overline{\delta}}\det(\mathrm{Id}-\overline{r_0\dots r_{n-1}})^{\delta}\right)}.$$

As r_k is almost surely a reflection, we know from the proof of Theorem 2.7 that $\det(\mathrm{Id}-r_0\dots r_{n-1})=\prod_{k=0}^{n-1}(1-r_k)$ a.s. where $r_k=\langle r_k(e_{k+1}),e_{k+1}\rangle$. So thanks to the independence of the r_k's

$$\frac{\mathbb{E}_{\mu_{\mathcal{G}}}\left(f(g)\det(\mathrm{Id}-g)^{\overline{\delta}}\det(\mathrm{Id}-\overline{g})^{\delta}\right)}{\mathbb{E}_{\mu_{\mathcal{G}}}\left(\det(\mathrm{Id}-g)^{\overline{\delta}}\det(\mathrm{Id}-\overline{g})^{\delta}\right)}$$
$$=\mathbb{E}_{\nu_0\times\dots\times\nu_{n-1}}\left(f(r_0\dots r_{n-1})\prod_{k=0}^{n-1}\frac{(1-r_k)^{\overline{\delta}}(1-\overline{r_k})^{\delta}}{\mathbb{E}_{\nu_k}\left((1-r_k)^{\overline{\delta}}(1-\overline{r_k})^{\delta}\right)}\right).$$

By the definition of the measures $\nu_k^{(\delta)}$, this is the desired result. ☐

Remark. A generalized Ewens sampling formula could also be stated for $\Phi(\mathcal{G})$, with $\mathcal{G}$ checking condition (R) and Φ the ring morphism previously defined. For simplicity it is stated in the restricted case when $\mathcal{G}$ directly checks condition (R).

For $\mathcal{G}=\mathrm{U}(n,\mathbb{C})$, Borodin and Olshanski [16] call $\mu_{\mathcal{G}}^{(\delta)}$ a Hua-Pickrell measure (see the introduction for an explanation). We keep this name for all groups checking condition (R).

Definition 2.16. *The measures $\mu_{\mathcal{G}}^{(\delta)}$ are called the Hua-Pickrell measures on the group $\mathcal{G}$ (which must satisfy the conditions of Theorem 2.15).*

The general Theorem 2.15 gives a proof of formula (2.8), although $\det(\mathrm{Id}-g)=0$ in the case of the symmetric group. Indeed, we can consider the semidirect product $\mathcal{G}=\{-1,1\}\cdot\mathcal{S}_n$, consisting of all matrices $(\varepsilon_j\delta^i_{\sigma(j)})$ with $\varepsilon_j=\pm1$, $\sigma\in\mathcal{S}_n$. The group $\mathcal{G}$ checks all conditions of Theorem 2.15 for $\delta\in\mathbb{R}^+$. Moreover a sampling by the function $\exp_\delta^{(k)}$ corresponds to a sampling by a parameter $\theta=2^{2\delta-1}$ in (2.8). Consequently (the first equality follows from Theorem 2.15),

$$\mathbb{E}_{\mu_{\mathcal{S}_n}^{(\theta)}}(f(g))=\mathbb{E}_{\mu_{\mathcal{G}}^{\delta}}f(|g|)=\frac{\mathbb{E}_{\mu_{\mathcal{G}}}\left(f(|g|)|\det(\mathrm{Id}-g)|^{2\delta}\right)}{\mathbb{E}_{\mu_{\mathcal{G}}}\left(|\det(\mathrm{Id}-g)|^{2\delta}\right)}.$$

By conditioning on the permutation and integrating on the ε_j's, we have

$$\begin{aligned}\mathbb{E}_{\mu_{\mathcal{G}}}\left(f(|g|)|\det(\mathrm{Id}-g)|^{2\delta}\right) &= \mathbb{E}_{\mu_{\mathcal{G}}}\left(\mathbb{E}\left(f(|g|)|\det(\mathrm{Id}-g)|^{2\delta} \mid \sigma\right)\right)\\ &= \mathbb{E}_{\mu_{\mathcal{G}}}\left(f(|g|)2^{(2\delta-1)k_\sigma}\right) = \mathbb{E}_{\mu_{\mathcal{S}_n}}\left(f(g)2^{(2\delta-1)k_\sigma}\right).\end{aligned}$$

We thus get the desired result :

$$\mathbb{E}_{\mu^{(\theta)}_{\mathcal{S}_n}}(f(g)) = \frac{\mathbb{E}_{\mu_{\mathcal{S}_n}}\left(f(g)\theta^{k_\sigma}\right)}{\mathrm{cst}}.$$

Chapter 3

The hypergeometric kernel

This chapter is extracted from *Ewens measures on compact groups and hypergeometric kernels* [21], with A. Nikeghbali, A. Rouault, to appear in Séminaire de Probabilités.

In the previous chapter, we considered the Hua-Pickrell measures $\mu^{(\delta)}_{\mathrm{U}(n)}$ on $\mathrm{U}(n)$, the unitary group over the field of complex numbers, which generalizes the Haar measure $\mu_{(\mathrm{U}(n))}$ on $\mathrm{U}(n)$ and which is defined by

$$\mathbb{E}_{\mu^{(\delta)}_{\mathrm{U}(n)}}\left(f(u)\right)=\frac{\mathbb{E}_{\mu_{\mathrm{U}(n)}}\left(f(u)\det(\mathrm{Id}-u)^{\overline{\delta}}\det(\mathrm{Id}-\overline{u})^{\delta}\right)}{\mathbb{E}_{\mu_{\mathrm{U}(n)}}\left(\det(\mathrm{Id}-u)^{\overline{\delta}}\det(\mathrm{Id}-\overline{u})^{\delta}\right)} \tag{3.1}$$

for any continuous function f, for $\mathfrak{Re}(\delta)>-1/2$. In this chapter we are interested in the spectral properties induced by the Hua-Pickrell measures, i.e. by the probability distribution function

$$c_n(\delta)\left|\Delta\left(e^{\mathrm{i}\theta_1},\dots,e^{\mathrm{i}\theta_n}\right)\right|^2\prod_{k=1}^{n}\left(1-e^{\mathrm{i}\theta_k}\right)^{\overline{\delta}}\left(1-e^{-\mathrm{i}\theta_k}\right)^{\delta}, \tag{3.2}$$

on the set of eigenvalues $(e^{\mathrm{i}\theta_1},\dots,e^{\mathrm{i}\theta_n})$ of unitary matrices from $\mathrm{U}(n)$ endowed with the probability measure (3.1). In the above formula, $c_n(\delta)$ is a normalizing constant and Δ denotes the Vandermonde determinant.

We show that these measures give raise to a limit kernel at the edge of the spectrum, generalizing the Bessel kernel. The universality of this *hypergeometric* kernel is then proven using Lubinsky's method.

1. The reproducing kernel

In the above potential (3.2), a non-zero imaginary part b of $\delta=a+\mathrm{i}b$ yields an asymmetric singularity at 1 :

$$\left(1-e^{\mathrm{i}\theta}\right)^{\overline{\delta}}\left(1-e^{-\mathrm{i}\theta}\right)^{\delta}=(2-2\cos\theta)^a e^{-b(\pi\,\mathrm{sgn}\,\theta-\theta)}. \tag{3.3}$$

We note $\lambda^{(\delta)}\mathrm{d}\theta/2\pi$ the probability measure on the unit circle having a density proportional to (3.3) with respect to the Lebesgue measure $\mathrm{d}\theta$. The statistical properties of the θ_k's depend on the successive orthonormal polynomials $(p_k^{(\delta)})$ associated to $\lambda^{(\delta)}$:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\overline{p_k^{(\delta)}(e^{\mathrm{i}\theta})}p_\ell^{(\delta)}(e^{\mathrm{i}\theta})\lambda^{(\delta)}(\mathrm{d}\theta)=\delta_k^\ell,$$

where $p_k^{(\delta)}(\mathrm{X})=a_k^{(\delta)}\mathrm{X}^k+\dots+a_0^{(\delta)}$ with $a_k^{(\delta)}>0$.

More generally, for a positive finite measure μ on the unit circle, which admits positive integer moments of all orders, we write $p_k^{(\mu)}$ for the successive orthonormal polynomials associated to μ, μ_a for the purely absolutely continuous part of μ with respect to the Lebesgue measure $\mathrm{d}\theta$, and use the notation $\mu_a(\alpha) = 2\pi(\mathrm{d}\mu_a/\mathrm{d}\theta)(\alpha)$. The polynomials $p_k^{(\delta)}$ were obtained by Askey (see p. 304 of [5]) and also derived by Basor and Chen [7], thanks to a difference equation. We will give an alternative proof.

Of special interest in random matrix theory is the reproducing kernel

$$\mathrm{K}_n^{(\mu)}(e^{\mathrm{i}\alpha}, e^{\mathrm{i}\beta}) = \sum_{k=0}^{n-1} p_k^{(\mu)}(e^{\mathrm{i}\alpha})\overline{p_k^{(\mu)}(e^{\mathrm{i}\beta})}$$

and the normalized reproducing kernel

$$\tilde{\mathrm{K}}_n^{(\mu)}(e^{\mathrm{i}\alpha}, e^{\mathrm{i}\beta}) = \sqrt{\mu_a(\alpha)\mu_a(\beta)}\mathrm{K}_n^{(\mu)}(e^{\mathrm{i}\alpha}, e^{\mathrm{i}\beta}).$$

The first result of this chapter is an explicit expression of $\mathrm{K}_n^{(\delta)}$, the reproducing kernel associated to the Hua-Pickrell measure with parameter δ (Theorem 3.6) and the following limit at the edge of the spectrum.

Theorem 3.1. *Let $\delta \in \mathbb{C}$, $\mathfrak{Re}(\delta) > -\frac{1}{2}$. Then, uniformly on any compact set of $\mathbb{R}_*^2$,*

$$\frac{1}{n}\tilde{\mathrm{K}}_n^{(\delta)}(e^{\mathrm{i}\frac{\alpha}{n}}, e^{\mathrm{i}\frac{\beta}{n}}) \underset{n\to\infty}{\longrightarrow} \tilde{\mathrm{K}}_\infty^{(\delta)}(\alpha,\beta)$$

with an explicit kernel $\tilde{\mathrm{K}}_\infty^{(\delta)}$ given in Theorem 3.8 further down. If $\mathfrak{Re}(\delta) \geqslant 0$, this convergence is uniform in compact subsets of $\mathbb{R}^2$.

For graphical examples of the hypergeometric kernel, which reflect the main characteristics, see figures 3.3 to 3.6 at the end of this chapter. Due to the asymmetry of $\lambda^{(\delta)}$, $\tilde{\mathrm{K}}_\infty^{(\delta)}(\alpha,\beta) \neq \tilde{\mathrm{K}}_\infty^{(\delta)}(\alpha,\beta)$. For $\delta = 0$ it is naturally the sine kernel, and for real δ it can be expressed in terms of Bessel functions. In all generality it depends on ${}_1\mathrm{F}_1$ functions, so we refer to $\tilde{\mathrm{K}}_\infty^{(\delta)}$ as the *hypergeometric kernel*.

The measure $\lambda^{(\delta)}$ is a generic example leading to a singularity

$$c^{(+)}|\theta|^{2a}\mathbb{1}_{\theta>0} + c^{(-)}|\theta|^{2a}\mathbb{1}_{\theta<0}$$

at $\theta = 0$, with distinct positive constants $c^{(+)}$ and $c^{(-)}$. The hypergeometric kernel, depending on the two parameters a and $b = \frac{1}{2\pi}\log(c^{(-)}/c^{(+)})$, is actually universal for the measures presenting the above singularity.

The most classical kernel appearing in different scaling limits is the *sine kernel*, appearing in the *bulk* of the spectrum (i.e. $\mu_a(\theta) > 0$),

$$\frac{1}{n}\tilde{\mathrm{K}}_n^{(\mu)}(e^{\mathrm{i}(\theta+\alpha/n)}, e^{\mathrm{i}(\theta+\beta/n)}) \underset{n\to\infty}{\longrightarrow} \frac{\sin((\beta-\alpha)/2)}{(\beta-\alpha)/2}. \tag{3.4}$$

For θ at the edge of the spectrum, with a singularity

$$\mu_a(\alpha) \underset{\alpha\to\theta}{\sim} c\,|\theta-\alpha|^{2a},\ a > -1/2,$$

the *Bessel kernel* appears in the above scaling limit :

$$\frac{\mathrm{J}_{a-1/2}(\alpha)\mathrm{J}_{a+1/2}(\beta) - \mathrm{J}_{a+1/2}(\alpha)\mathrm{J}_{a-1/2}(\beta)}{\alpha-\beta}.$$

Its universality was shown, among other results, on the segment by Kuijlaars and Vanlessen [87] and on the circle by Martínez-Finkelshtein & al [94], both relying on the Deift-Zhou steepest descent method for Riemann-Hilbert problems [37]. Some analyticity hypothesis is required in this method, and stronger results were shown by Lubinsky in a series of papers ([90], [91], [92], [93]) Lubinsky showed that the local behavior of the measure near the singularity is sufficient to obtain universality type results. In particular, he showed that (3.4) holds for general measures μ.

We directly use his method to prove the universality of the hypergeometric kernel, which generalizes the Bessel kernel (corresponding to a symmetric singularity) but also the sine kernel, introducing the possibility to be in the bulk of the spectrum ($a=0$) but with a discontinuity of the underlying measure ($b \neq 0$).

Theorem 3.2. *Let μ be a measure on $\partial\mathbb{D}$, such that the set of points with $\mu_a = 0$ has Lebesgue measure 0. Suppose that μ is absolutely continuous in a neighborhood of 1 and $\mu_a(\theta) = h(\theta)\lambda^{(\delta)}(\theta)$ in this neighborhood, with h continuous at 0 and $h(0) > 0$. Then, uniformly in compact subsets of $\mathbb{R}_*^2$,*

$$\frac{1}{n}\tilde{\mathrm{K}}_n^{(\mu)}(e^{\mathrm{i}\frac{\alpha}{n}}, e^{\mathrm{i}\frac{\beta}{n}}) \underset{n\to\infty}{\longrightarrow} \tilde{\mathrm{K}}_\infty^{(\delta)}(\alpha,\beta).$$

If $\mathfrak{Re}(\delta) \geqslant 0$, this convergence is uniform in compact subsets of $\mathbb{R}^2$.

The proof of Theorem 3.2 strongly relies on the particular example given in Theorem 3.1 : we use Lubinsky's method, which allows to compare kernels associated to distinct measures having the same singularity, and this comparison only requires estimates on the kernel evaluated on the diagonal : Lubinsky's inequality states that if $\mu \leqslant \mu^*$ on $\partial\mathbb{D}$, than for any α and β

$$\frac{|\mathrm{K}_n^{(\mu)}(e^{\mathrm{i}\alpha}, e^{\mathrm{i}\beta}) - \mathrm{K}_n^{(\mu^*)}(e^{\mathrm{i}\alpha}, e^{\mathrm{i}\beta})|}{\mathrm{K}_n^{(\mu)}(e^{\mathrm{i}\alpha}, e^{\mathrm{i}\alpha})} \leqslant \left(\frac{\mathrm{K}_n^{(\mu)}(e^{\mathrm{i}\beta}, e^{\mathrm{i}\beta})}{\mathrm{K}_n^{(\mu)}(e^{\mathrm{i}\alpha}, e^{\mathrm{i}\alpha})}\right)^{1/2} \left(1 - \frac{\mathrm{K}_n^{(\mu^*)}(e^{\mathrm{i}\alpha}, e^{\mathrm{i}\alpha})}{\mathrm{K}_n^{(\mu)}(e^{\mathrm{i}\alpha}, e^{\mathrm{i}\alpha})}\right)^{1/2}.$$

In the next section, we prove Theorem 3.1, by first explicitly working out the family of orthogonal polynomials associated to the measure $\lambda^{(\delta)}$ (here the main ingredient is the Pfaff-Saalschutz identity) and then by studying their asymptotics. Section 3 proves Theorem 3.2 : this is made easy thanks to Lubinsky's localization principle once Theorem 3.1 is established.

Remark. By means of the Cayley transform, the universality of the hypergeometric kernel can be translated as the universality of a kernel for measures on the real line with an asymmetric singularity.

2. Determinantal point process for the Hua-Pickrell measure

Let $(e^{i\theta_1}, \dots, e^{i\theta_n})$ be the eigenvalues of a generic element in $\mathrm{U}(n)$. This section explains that Hua-Pickrell measures on $\mathrm{U}(n)$ induce determinantal point processes for the spectrum. There are many motivations to undertake a deeper study of these determinantal processes.

First, Olshanski [106] has shown that the Hua-Pickrell measures are closely linked to the natural analogue on the infinite unitary group of the biregular representations. Such representations can be described by the spectral measure of their characters. This spectral measure is characterized by the determinantal process appearing in Theorem 3.8. More details about these links between Hua-Pickrell measures and representations of infinite dimensional groups can be found in [106].

Moreover, many limit theorems about the Hua-Pickrell measures can be derived from the determinantal expression in Theorem 3.8 further down. For instance, the number of eigenangles on any compact set of $(-\pi,\pi)$ satisfies a central limit theorem. Such results are easy applications of the general theory of determinantal point processes (see [130]).

Finally, and maybe most importantly, this hypergeometric kernel may appear in statistical physics : the limit given in Theorem 3.6 is relevant for a large class of measures presenting the same asymmetric singularity as $\lambda^{(\delta)}$, as shown in Section 3.

2.1. Orthogonal polynomials associated to the Hua-Pickrell measures.

Consider a point process χ on the unit circle $\partial\mathbb{D}$, with successive correlation functions $\rho_1, \rho_2, \dots$ (more precisions about correlation functions and determinantal point processes can be found in [76]).

Definition 3.3. *If there exists a function $\tilde{\mathrm{K}} : \partial\mathbb{D} \times \partial\mathbb{D} \to \mathbb{C}$ such that for all $k \geqslant 1$ and $(z_1, \dots, z_k) \in (\partial\mathbb{D})^k$*

$$\rho_k(z_1, \dots, z_k) = \det\left(\tilde{\mathrm{K}}(z_i, z_j)_{i,j=1}^k\right)$$

then χ is said to be a determinantal point process with correlation kernel $\tilde{\mathrm{K}}$.

An example of determinantal point process related to the Hua-Pickrell measure is explained below. Let $\mathrm{H}(n)$ be the set of $n \times n$ complex Hermitian matrices. Consider the Cayley transform

$$\left\{\begin{array}{ccc} \mathrm{H}(n) & \to & \mathrm{U}(n) \\ \mathrm{X} & \mapsto & \frac{\mathrm{i}-\mathrm{X}}{\mathrm{i}+\mathrm{X}} \end{array}\right. .$$

Its reciprocal is defined almost everywhere and transforms the Hua-Pickrell measure $\mu_{\mathrm{U}(n)}^{(\delta)}$ in a measure $\mu_{\mathrm{H}(n)}^{(\delta)}$. A. Borodin and G. Olshanski [16] studied $\mu_{\mathrm{H}(n)}^{(\delta)}$: they exhibit a determinantal form for the eigenvalues correlation functions, involving hypergeometric functions. Moreover, they suggest that such a form may exist for $\mu_{\mathrm{U}(n)}^{(\delta)}$ itself. Theorem 3.6 gives the determinantal kernel for the probability distribution function

$$c_n(\delta)\left|\Delta\left(e^{\mathrm{i}\theta_1}, \dots, e^{\mathrm{i}\theta_n}\right)\right|^2 \prod_{k=1}^n \left(1 - e^{\mathrm{i}\theta_k}\right)^{\overline{\delta}} \left(1 - e^{-\mathrm{i}\theta_k}\right)^{\delta}$$

induced on the eigenvalues by the Hua-Pickrell measures.

Before stating this theorem, we need the following proposition, which exhibits the sequence of orthogonal polynomials on the unit circle for the measure

$$\begin{aligned} \lambda^{(\delta)}(\mathrm{d}\theta) &= c(\delta)(1 - e^{\mathrm{i}\theta})^{\overline{\delta}}(1 - e^{-\mathrm{i}\theta})^{\delta}\mathrm{d}\theta \\ &= c(\delta)(2 - 2\cos\theta)^a e^{-b(\pi \operatorname{sgn}\theta - \theta)}\mathrm{d}\theta \end{aligned} \tag{3.5}$$

with

$$\delta = a + \mathrm{i}b \quad , \quad c(\delta) = \frac{\Gamma(1+\delta)\Gamma(1+\overline{\delta})}{\Gamma(1+\delta+\overline{\delta})},$$

$\mathrm{d}\theta$ the Lebesgue measure on $(-\pi, \pi)$. Orthogonality and norm here are with respect to the Hermitian product

$$\langle \varphi, \psi \rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} \overline{\varphi(e^{\mathrm{i}\theta})}\psi(e^{\mathrm{i}\theta})\lambda^{(\delta)}(\theta)\mathrm{d}\theta. \tag{3.6}$$

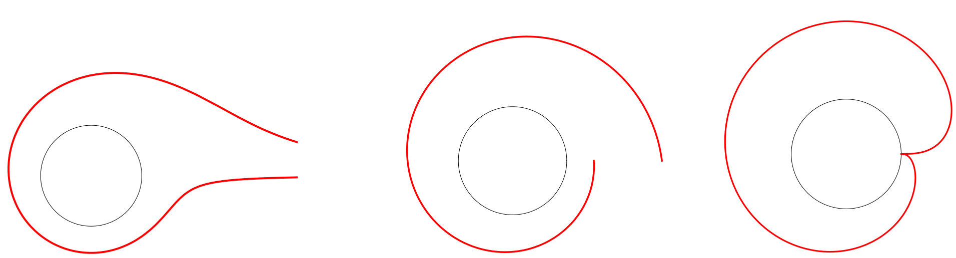


Figure 3.1. Examples of densities $\lambda^{(\delta)}$, for $\mathfrak{Re}(\delta)$ negative, zero and positive.

Proposition 3.4. *Let $\delta \in \mathbb{C}$ with $\mathfrak{Re}(\delta) > -1/2$. Then $\lambda^{(\delta)}(\theta)\frac{\mathrm{d}\theta}{2\pi}$ is a probability measure with successive monic orthogonal polynomials*

$$\mathrm{P}_n^{(\delta)}(\mathrm{X}) = \mathrm{X}^n {}_2\mathrm{F}_1(\delta, -n; -n-\overline{\delta}; \mathrm{X}^{-1}),\ n \geqslant 0.$$

Moreover,

$$\left\|\mathrm{P}_n^{(\delta)}\right\|^2 = \frac{(\overline{\delta}+\delta+1)_n n!}{(\overline{\delta}+1)_n(\delta+1)_n}.$$

Remark. In this proposition and in the following, $(x)_n$ stands for the Pochhammer symbol : if $n \geqslant 0$, $(x)_n = x(x+1)\dots(x+n-1)$, and if $n \leqslant 0$ $(x)_n = 1/(x+n)_{-n}$.

The hypergeometric function ${}_2\mathrm{F}_1(\delta, -n; -n-\overline{\delta}; \mathrm{X}^{-1})$ is stricly speaking not well defined : ${}_2\mathrm{F}_1(a, b; c; z)$ is generally not convergent for $|z| = 1$ if $\mathfrak{Re}(c-a-b) < 0$. However, in our case, as $-b \in \mathbb{N}$, the hypergeometric series contains a finite number of terms and therefore converges. Actually

$$\mathrm{P}_n^{(\delta)}(z) = \sum_{k=0}^{n} \binom{n}{k} \frac{\Gamma(n-k+\delta)\Gamma(k+1+\overline{\delta})}{\Gamma(\delta)\Gamma(n+1+\overline{\delta})} z^k .$$

Proof. Let $n \geqslant 0$. We first calculate the moment of order n

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-\mathrm{i}n\theta}\lambda^{(\delta)}(\theta)\mathrm{d}\theta = \frac{c(\delta)}{2\pi}\int_{-\pi}^{\pi} e^{-\mathrm{i}n\theta}(1-e^{\mathrm{i}\theta})^{\overline{\delta}}(1-e^{-\mathrm{i}\theta})^{\delta}\mathrm{d}\theta.$$

As the Taylor series

$$(1-e^{\mathrm{i}\theta})^{\overline{\delta}}(1-e^{-\mathrm{i}\theta})^{\delta} = \left(\sum_{k\geqslant 0}\frac{(-\overline{\delta})_k}{k!}e^{\mathrm{i}k\theta}\right)\left(\sum_{l\geqslant 0}\frac{(-\delta)_l}{l!}e^{-\mathrm{i}l\theta}\right)$$

agrees with formula (3.5), after an expansion of the double series all terms with $n+l \neq k$ cancel (the exchange of order between integral and sum requires some attention) and we get

$$c_n = c(\delta)\sum_{l\geqslant 0}\frac{(-\delta)_l}{l!}\frac{(-\overline{\delta})_{l+n}}{(l+n)!} = c(\delta)\frac{(-\overline{\delta})_n}{n!}{}_2\mathrm{F}_1(-\delta, -\overline{\delta}+n; n+1; 1).$$

Combining our choice for $c(\delta)$ and Gauss formula (see [3])

$${}_2\mathrm{F}_1(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\quad,\quad \mathfrak{Re}(c-a-b) > 0, \tag{3.7}$$

leads to

$$c_n = \frac{(-\overline{\delta})_n}{(\delta+1)_n}. \tag{3.8}$$

The same method shows that formula (3.8) stands also for $n \leqslant 0$ ($(x)_n$ is defined in the preceding remark for a negative n). Note that $c_0 = 1$ so $\lambda^{(\delta)}(\theta)\frac{\mathrm{d}\theta}{2\pi}$ is a probability measure.

The polynomials $\mathrm{P}_n^{(\delta)}$ ($n \geqslant 0$) are clearly monic, and they are orthogonal if and only if

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathrm{P}_n^{(\delta)}(e^{\mathrm{i}\theta})e^{-\mathrm{i}l\theta}\lambda^{(\delta)}(\theta)\mathrm{d}\theta = 0,\ 0 \leqslant l \leqslant n-1,$$

for all $n \geqslant 1$. Note that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{P}_n^{(\delta)}(e^{\mathrm{i}\theta})e^{-\mathrm{i}l\theta}\lambda^{(\delta)}(\theta)\mathrm{d}\theta$$
$$= \sum_{k=0}^{n}\frac{(\delta)_k(-n)_k}{(-n-\overline{\delta})_k k!}c_{l+k-n} = \sum_{k=0}^{n}\frac{(\delta)_k(-n)_k}{(-n-\overline{\delta})_k k!}\frac{(-\overline{\delta})_{l+k-n}}{(\delta+1)_{l+k-n}}$$
$$= \frac{(-\overline{\delta})_{l-n}}{(\delta+1)_{l-n}}\,{}_3\mathrm{F}_2(\delta,-\overline{\delta}+l-n,-n;1+\delta+l-n,-n-\overline{\delta};1).$$

The Pfaff-Saalschutz identity (see [3]) states that if $-c\in\mathbb{N}$ and $d+e=a+b+c+1$ then
$${}_3\mathrm{F}_2(a,b,c;d,e;1) = \frac{(d-a)_{-c}(d-b)_{-c}}{(d)_{-c}(d-a-b)_{-c}}.$$
Consequently, as $l<n$, $\langle \mathrm{P}_n^{(\delta)},\mathrm{X}^l\rangle = 0$ and so the $\mathrm{P}_n^{(\delta)}$'s are orthogonal. Moreover, as they are monic, we get
$$\left\|\mathrm{P}_n^{(\delta)}\right\|^2 = \langle \mathrm{P}_n^{(\delta)},\mathrm{X}^n\rangle = {}_3\mathrm{F}_2(\delta,-\overline{\delta},-n;-n-\overline{\delta},1+\delta;1),$$
which is $\frac{(\overline{\delta}+\delta+1)_n n!}{(\overline{\delta}+1)_n(\delta+1)_n}$ once again thanks to the Pfaff-Saalschutz identity. □

Note that the orthogonal polynomials in the above Proposition 3.4 necessarily follow the Szegö's recursion formula
$$\mathrm{P}_{n+1}^{(\delta)}(z) = z\mathrm{P}_n^{(\delta)}(z) - \bar{\alpha}_n \mathrm{P}_n^{(\delta)^*}(z)$$
where $\mathrm{P}_n^{(\delta)^*}(z) = z^n\,\overline{\mathrm{P}_n^{(\delta)}(\bar{z}^{-1})}$. The coefficients α_n are called Verblunsky coefficients and satisfy the condition $\alpha_n\in\mathbb{D}$: they play a central role in the theory of orthogonal polynomials on the unit circle (see [122], [123]). In particular, they can be used as a set of coordinates for the probability measure $\lambda^{(\delta)}/2\pi$ in the manifold of probability measures on the unit circle. Hence it is of interest to identify them : it follows from Proposition 3.4 that the Verblunsky coefficients associated to the measure $\lambda^{(\delta)}$ are
$$\alpha_n = -\frac{(\overline{\delta})_{n+1}}{(\delta+1)_{n+1}} \underset{n\to\infty}{\sim} -\frac{1}{n}\frac{\Gamma(\delta+1)}{\Gamma(\overline{\delta})}e^{-2\mathrm{i}\mathfrak{Im}(\delta)\log n}$$
as shown by the asymptotics $\Gamma(n+a)/\Gamma(n)\sim n^a$.

Another quantity of interest in this theory is the Caratheodory function, easily calculated thanks to the exact expression of the moments c_n previously obtained :
$$\mathrm{F}(z) = \int_{-\pi}^{\pi}\frac{e^{\mathrm{i}\theta}+z}{e^{\mathrm{i}\theta}-z}\frac{\lambda^{(\delta)}(\theta)\mathrm{d}\theta}{2\pi} = 1+2\sum_{n=1}^{\infty}c_n z^n$$
$$= 1+2\sum_{n=1}^{\infty}\frac{(-\overline{\delta})_n}{(\delta+1)_n}z^n = 2\,{}_2\mathrm{F}_1(-\overline{\delta},1,\delta+1,z)-1$$

The main results concerning the Hua-Pickrell measures are summarized below.

Hua-Pickrell density	$\lambda^{(\delta)}(\theta) = c(\delta)(2-2\cos\theta)^a e^{-b(\pi\,\mathrm{sgn}\,\theta-\theta)}$ $\delta=a+\mathrm{i}b$, $c(\delta)=\frac{\Gamma(1+\delta)\Gamma(1+\overline{\delta})}{\Gamma(1+\delta+\overline{\delta})}$
Moments	$c_n = \frac{(-\overline{\delta})_n}{(\delta+1)_n}$
Orthogonal polynomials L^2 norm	$\mathrm{P}_n^{(\delta)}(\mathrm{X}) = \mathrm{X}^n{}_2\mathrm{F}_1(\delta,-n;-n-\overline{\delta};\mathrm{X}^{-1})$ $\left\|\mathrm{P}_n^{(\delta)}\right\|^2 = \frac{(\overline{\delta}+\delta+1)_n n!}{(\overline{\delta}+1)_n(\delta+1)_n}$
Verblunsky coefficients Asymptotics	$\alpha_n = -\frac{(\overline{\delta})_{n+1}}{(\delta+1)_{n+1}}$ $\alpha_n \underset{n\to\infty}{\sim} -\frac{1}{n}\frac{\Gamma(\delta+1)}{\Gamma(\overline{\delta})}e^{-2\mathrm{i}\mathfrak{Im}(\delta)\log n}$
Caratheodory function	$\mathrm{F}(z) = 2\,{}_2\mathrm{F}_1(-\overline{\delta},1,\delta+1,z)-1$

2.2. Consequences for fixed n.

In fact, the second claim of the above Proposition 3.4 gives a new proof of the Keating-Snaith formula [80] for the average of the characteristic polynomial $\mathrm{Z}_n = \det(\mathrm{Id}-u)$ of a random unitary matrix, originally derived with Selberg's integrals :

Corollary 3.5. *Let* $\mathrm{Z}_n = \det(\mathrm{Id}-u)$, *with* $u \sim \mu_{\mathrm{U}(n)}$. *Then*

$$\mathbb{E}_{\mu_{\mathrm{U}(n)}}\left(|\mathrm{Z}_n|^t e^{s\arg \mathrm{Z}_n}\right) = \prod_{k=1}^{n} \frac{\Gamma(k)\,\Gamma(k+t)}{\Gamma\left(k+\frac{t+\mathrm{i}s}{2}\right)\Gamma\left(k+\frac{t-\mathrm{i}s}{2}\right)}. \tag{3.9}$$

Proof. An application of Heine's formula (see e.g. [133]) yields :

$$\mathbb{E}_{\mu_{\mathrm{U}(n)}}\left(|\mathrm{Z}_n|^t e^{s\arg \mathrm{Z}_n}\right) = \frac{1}{c(\delta)^n}\det\left(\langle \mathrm{X}^k, \mathrm{X}^l\rangle_{k,l=0}^{n-1}\right),$$

where $t > -1$, $s \in \mathbb{R}$ and where the Hermitian product is defined in (3.6) with $\delta = \frac{t+\mathrm{i}s}{2}$. It is well known that this determinant of a Gram matrix is the square of the volume of the parallelepiped determined by the vectors $1, \mathrm{X}, \dots, \mathrm{X}^{n-1}$. The "base times height" formula implies that this volume is the product of the norms of the successive monic orthogonal polynomials :

$$\mathbb{E}_{\mu_{\mathrm{U}(n)}}\left(|\mathrm{Z}_n|^t e^{s\arg \mathrm{Z}_n}\right) = \frac{1}{c(\delta)^n}\prod_{k=0}^{n-1}\left\|\mathrm{P}_k^{(\delta)}\right\|^2 = \prod_{k=1}^{n} \frac{\Gamma(k)\,\Gamma(k+t)}{\Gamma\left(k+\frac{t+\mathrm{i}s}{2}\right)\Gamma\left(k+\frac{t-\mathrm{i}s}{2}\right)}.$$

This agrees with (3.9). □

Note that Chapter 1 contains still another proof of (3.9), relying on a decomposition of Z_n as a product of n independent random variables.

Theorem 3.6. *Let* $\delta \in \mathbb{C}$, $\mathfrak{Re}(\delta) > -\frac{1}{2}$. *For* $u \sim \mu_{\mathrm{U}(n)}^{(\delta)}$, *consider* $\chi = \{e^{\mathrm{i}\theta_1}, \dots, e^{\mathrm{i}\theta_n}\}$ *the set of the eigenvalues of* u. *Then* χ *is a determinantal point process with correlation kernel*

$$\tilde{\mathrm{K}}_n^{(\delta)}(e^{i\alpha}, e^{i\beta}) = d_n(\delta)\sqrt{\lambda^{(\delta)}(\alpha)\lambda^{(\delta)}(\beta)}$$
$$\frac{e^{\mathrm{i}\frac{n(\alpha-\beta)}{2}}\mathrm{Q}_n^{(\delta)}(e^{-\mathrm{i}\alpha})\mathrm{Q}_n^{(\overline{\delta})}(e^{\mathrm{i}\beta}) - e^{-\mathrm{i}\frac{n(\alpha-\beta)}{2}}\mathrm{Q}_n^{(\overline{\delta})}(e^{\mathrm{i}\alpha})\mathrm{Q}_n^{(\delta)}(e^{-\mathrm{i}\beta})}{e^{\mathrm{i}\frac{\alpha-\beta}{2}} - e^{-\mathrm{i}\frac{\alpha-\beta}{2}}}.$$

Here $d_n(\delta) = \frac{(\overline{\delta}+1)_n(\delta+1)_n}{(\overline{\delta}+\delta+1)_n n!}$, $\mathrm{Q}_n^{(\delta)}(x) = {}_2\mathrm{F}_1(\delta, -n; -n-\overline{\delta}; x)$ *and* $\lambda^{(\delta)}(\alpha)$ *is defined by (3.5).*

Remark. First, substituting $\delta = 0$ leads to the kernel

$$\mathrm{K}_n^{(0)}(e^{i\alpha}, e^{i\beta}) = \frac{\sin\frac{n(\alpha-\beta)}{2}}{\sin\frac{\alpha-\beta}{2}}. \tag{3.10}$$

Moreover, in the above theorem and in the following, the values of the kernels on the diagonal are defined by continuity, the limit being easily derived from L'Hospital's rule (examples of these diagonal values are given in figure 3.2) :

$$\tilde{\mathrm{K}}_n^{(\delta)}(e^{\mathrm{i}\alpha}, e^{\mathrm{i}\alpha}) = d_n(\delta)\lambda^{(\delta)}(\alpha)2\mathfrak{Re}\Big({}_2\mathrm{F}_1(\overline{\delta}, -n, -n-\delta, e^{\mathrm{i}\alpha})$$
$$\left(\frac{n}{2}{}_2\mathrm{F}_1(\delta, -n, -n-\overline{\delta}, e^{-\mathrm{i}\alpha}) - \frac{n\delta}{n+\overline{\delta}}e^{-\mathrm{i}\alpha}{}_2\mathrm{F}_1(\delta+1, -n+1, -n-\overline{\delta}+1, e^{-\mathrm{i}\alpha})\right)\Big)$$

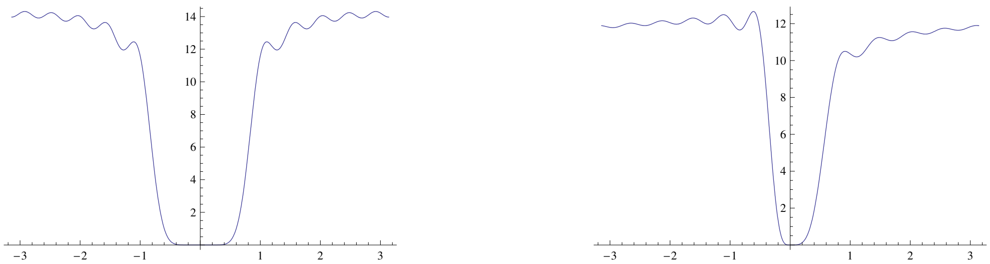


Figure 3.2. $\tilde{\mathrm{K}}_n^{(\delta)}(e^{\mathrm{i}\alpha}, e^{\mathrm{i}\alpha})$ for $n = 10$ and $\delta = 5$ (left), $\delta = 3 + \frac{\mathrm{i}}{2}$ (right)

Proof. This is straightforward once we know the orthogonal polynomials from Proposition 3.4 : the following arguments are standard in Random Matrix Theory, and more details can be found in [96]. Let $f(e^{\mathrm{i}\theta})\mathrm{d}\theta$ be a probability measure on $(-\pi, \pi)$. Consider the probability distribution

$$\mathrm{F}(e^{\mathrm{i}\theta_1}, \dots, e^{\mathrm{i}\theta_n})\mathrm{d}\theta_1 \dots \mathrm{d}\theta_n = c(n, f) \prod_j f(e^{\mathrm{i}\theta_j}) \prod_{k<l} |e^{\mathrm{i}\theta_l} - e^{\mathrm{i}\theta_k}|^2 \mathrm{d}\theta_1 \dots \mathrm{d}\theta_n$$

on $(-\pi, \pi)^n$, with $c(n, f)$ the normalization constant. Let P_k $(0 \leqslant k \leqslant n-1)$ be monic polynomials with degree k. Thanks to Vandermonde's formula and multilinearity of the determinant

$$\prod_j \sqrt{f(e^{\mathrm{i}\theta_j})} \prod_{k<l} (e^{\mathrm{i}\theta_l} - e^{\mathrm{i}\theta_k}) = \sqrt[n]{\prod_{k=0}^{n-1} \|\mathrm{P}_k\|_{\mathrm{L}^2(f)}} \det \left(\sqrt{f(e^{\mathrm{i}\theta_j})} \frac{\mathrm{P}_k(e^{\mathrm{i}\theta_j})}{\|\mathrm{P}_k\|_{\mathrm{L}^2(f)}} \right)_{k,j=1}^n .$$

Multiplying this identity with its conjugate and using $\det(\mathrm{AB}) = \det\mathrm{A}\det\mathrm{B}$ gives

$$\mathrm{F}(e^{\mathrm{i}\theta_1}, \dots, e^{\mathrm{i}\theta_n}) = \det \left(\tilde{\mathrm{K}}_n(e^{\mathrm{i}\theta_j}, e^{\mathrm{i}\theta_k})_{j,k=1}^n \right)$$

with $\tilde{\mathrm{K}}_n(x, y) = c\sqrt{f(x)f(y)} \sum_{k=0}^{n-1} \frac{\mathrm{P}_k(x)\overline{\mathrm{P}_k(y)}}{\|\mathrm{P}_k\|^2_{\mathrm{L}^2(f)}}$, the constant c depending on f, n and the P_i's. This shows that the correlation ρ_n has the desired determinantal form. Gaudin's lemma (see [96]) implies that if the polynomials P_k's are orthogonal in $\mathrm{L}^2(f)$, then

$$\rho_l(e^{\mathrm{i}\theta_1}, \dots, e^{\mathrm{i}\theta_l}) = \det \left(\tilde{\mathrm{K}}_n(e^{\mathrm{i}\theta_j}, e^{\mathrm{i}\theta_k})_{j,k=1}^l \right)$$

for all $1 \leqslant l \leqslant n$. As $\frac{1}{2\pi}\int_{-\pi}^{\pi} \rho_1(e^{\mathrm{i}\theta})\mathrm{d}\theta = n$, then $c = 2\pi$, so the Christoffel-Darboux formula gives

$$\begin{aligned} \tilde{\mathrm{K}}_n(x, y) &= 2\pi\sqrt{f(x)f(y)} \sum_{k=0}^{n-1} \frac{\mathrm{P}_k(x)\overline{\mathrm{P}_k(y)}}{\|\mathrm{P}_k\|^2_{\mathrm{L}^2(f)}} \\ &= 2\pi \frac{\sqrt{f(x)f(y)}}{\|\mathrm{P}_n\|^2_{\mathrm{L}^2(f)}} \frac{\mathrm{P}_n^*(x)\overline{\mathrm{P}_n^*(y)} - \mathrm{P}_n(x)\overline{\mathrm{P}_n(y)}}{x - y} . \end{aligned}$$

where $\mathrm{P}_n^*(x) = x^n\, \overline{\mathrm{P}_n(1/\overline{x})}$.

Concerning the Hua-Pickrell measure, taking in the above discussion $\lambda^{(\delta)}$ for f, replacing the kernel $\tilde{\mathrm{K}}_n(e^{\mathrm{i}\alpha}, e^{\mathrm{i}\beta})$ by $e^{\mathrm{i}\frac{n\alpha}{2}}\tilde{\mathrm{K}}_n(e^{\mathrm{i}\alpha}, e^{\mathrm{i}\beta})e^{-\mathrm{i}\frac{n\beta}{2}}$ (this doesn't change the determinant), we get directly the result of Theorem 3.6. $\square$

2.3. Asymptotics.

The asymptotics of the kernel (3.10) is given by

$$\tilde{\mathrm{K}}_\infty^{(0)}(\alpha, \beta) = \lim_{n\to\infty} \frac{1}{n} \tilde{\mathrm{K}}_n^{(0)}(e^{\mathrm{i}\frac{\alpha}{n}}, e^{\mathrm{i}\frac{\beta}{n}}) = \frac{\sin\left(\frac{\alpha-\beta}{2}\right)}{\frac{\alpha-\beta}{2}}.$$

A similar limit holds for the Hua-Pickrell determinantal kernel, given in Theorem 3.8. To this end, we shall need the following asymptotics :

Proposition 3.7. *Let $\delta \in \mathbb{C}$, $\mathfrak{Re}(\delta) \geqslant 1/2$. Then*

$$\lim_{n\to\infty} n^{-\overline{\delta}}\ {}_2\mathrm{F}_1(\overline{\delta}, -n; -n-\delta; e^{\mathrm{i}\frac{\theta}{n}}) = \frac{\Gamma(\delta+1)}{\Gamma(\delta+\overline{\delta}+1)}\ {}_1\mathrm{F}_1(\overline{\delta}, \delta+\overline{\delta}+1, \mathrm{i}\theta),$$

uniformly on $\{\theta \in \mathcal{K}\}$, with $\mathcal{K}$ any compact set of $\mathbb{R}$.

Proof. The function $g^{(n)} : \theta \mapsto n^{-\overline{\delta}}\ {}_2\mathrm{F}_1(\overline{\delta}, -n; -n-\delta; e^{\mathrm{i}\frac{\theta}{n}})$ satisfies the ordinary differential equation

$$\begin{aligned}&\left[n(1-e^{-\mathrm{i}\frac{\theta}{n}})\right]\partial_{\theta\theta}g^{(n)}\\ &\qquad+\left[-\mathrm{i}(1-e^{-\mathrm{i}\frac{\theta}{n}})+\mathrm{i}e^{-\mathrm{i}\frac{\theta}{n}}(n+\delta)-\mathrm{i}(n-\overline{\delta}-1)\right]\partial_\theta g^{(n)}+\overline{\delta}g^{(n)}=0 \quad (3.11)\end{aligned}$$

with initial conditions (here we use the Chu-Vandermonde identity)

$$\left\{\begin{array}{ccccc} g^{(n)}(0) & = & \dfrac{(-n-\delta-\overline{\delta})_n}{n^{\overline{\delta}}(-n-\delta)_n} & = & \dfrac{(\delta+\overline{\delta}+1)_n}{n^{\overline{\delta}}(\delta+1)_n} \\ g^{(n)\prime}(0) & = & \dfrac{\mathrm{i}\overline{\delta}(-n-\delta-\overline{\delta})_{n-1}}{n^{\overline{\delta}}(n+\delta)(-n-\delta+1)_{n-1}} & = & \dfrac{\mathrm{i}\overline{\delta}(\delta+\overline{\delta}+2)_{n-1}}{n^{\overline{\delta}}(n+\delta)(\delta+1)_{n-1}} \end{array}\right. .$$

Taking $n \to \infty$ in (3.11) and using the classical theory of differential equations (i.e. the Cauchy-Lipschitz theorem with a parameter), we can conclude that $g^{(n)}$ converges uniformly on any compact set to the solution g of the differential equation

$$(\mathrm{i}\theta)\partial_{\theta\theta}g + (\mathrm{i}(\delta+\overline{\delta}+1)+\theta)\partial_\theta g + \overline{\delta}g = 0 \tag{3.12}$$

with initial values $g(0) = \lim_n g^{(n)}(0)$ and $g'(0) = \lim_n g^{(n)\prime}(0)$ i.e.

$$g(0) = \frac{\Gamma(\delta+1)}{\Gamma(\delta+\overline{\delta}+1)} \quad , \quad g'(0) = \frac{\mathrm{i}\overline{\delta}\Gamma(\delta+1)}{\Gamma(\delta+\overline{\delta}+2)} .$$

The unique solution of (3.12) is

$$g(\theta) = \frac{\Gamma(\delta+1)}{\Gamma(\delta+\overline{\delta}+1)}\ {}_1\mathrm{F}_1(\overline{\delta}, \delta+\overline{\delta}+1, \mathrm{i}\theta),$$

concluding the proof. □

Consequently, using this proposition and Theorem 3.6 the re-scaled correlation function associated to the Hua-Pickrell measure can be written in a determinantal form. Note that in the following theorem, if $\mathfrak{Re}(\delta) < 0$, the convergence is uniform only on compact subsets of $\mathbb{R}_*^2$ because of the pole of $\lambda^{(\delta)}(\theta)$ at $\theta = 0$. If $\mathfrak{Re}(\delta) \geqslant 0$, no such problem appears and the convergence is uniform on compacts of $\mathbb{R}^2$.

Theorem 3.8. *Let $\delta \in \mathbb{C}$, $\mathfrak{Re}(\delta) > -\frac{1}{2}$. Uniformly on any compact set of $\mathbb{R}_*^2$,*

$$\frac{1}{n}\tilde{\mathrm{K}}_n^{(\delta)}(e^{\mathrm{i}\frac{\alpha}{n}}, e^{\mathrm{i}\frac{\beta}{n}}) \underset{n\to\infty}{\longrightarrow} \tilde{\mathrm{K}}_\infty^{(\delta)}(\alpha,\beta)$$

with

$$\begin{aligned}\tilde{\mathrm{K}}_\infty^{(\delta)}(\alpha,\beta) &= e(\delta)\,|\alpha\beta|^{\mathfrak{Re}\delta}\, e^{-\frac{\pi}{2}(\mathfrak{Im}\delta)(\operatorname{sgn}\alpha+\operatorname{sgn}\beta)}\\ &\qquad\qquad \frac{e^{\mathrm{i}\frac{\alpha-\beta}{2}}\mathrm{Q}^{(\delta)}(-\mathrm{i}\alpha)\mathrm{Q}^{(\overline{\delta})}(\mathrm{i}\beta) - e^{-\mathrm{i}\frac{\alpha-\beta}{2}}\mathrm{Q}^{(\overline{\delta})}(\mathrm{i}\alpha)\mathrm{Q}^{(\delta)}(-\mathrm{i}\beta)}{\alpha-\beta}.\end{aligned}$$

Here $e(\delta) = \frac{1}{2\mathrm{i}\pi}\frac{\Gamma(\delta+1)\Gamma(\overline{\delta}+1)}{\Gamma(\delta+\overline{\delta}+1)^2}$ and $\mathrm{Q}^{(\delta)}(x) = {}_1\mathrm{F}_1(\delta, \delta+\overline{\delta}+1; x)$. If $\mathfrak{Re}(\delta) \geqslant 0$, this convergence is uniform on compact subsets of $\mathbb{R}^2$.

Remark. As expected, the kernel $\tilde{\mathrm{K}}_\infty^{(\delta)}$ coincides with the sine kernel for $\delta = 0$:

$$\tilde{\mathrm{K}}^{(0)}(e^{\mathrm{i}\alpha}, e^{\mathrm{i}\beta}) = \frac{\sin\left(\frac{\alpha-\beta}{2}\right)}{\frac{\alpha-\beta}{2}}.$$

For $\mathfrak{Re}(\delta) > -1/2$, ${}_1\mathrm{F}_1(\delta, \delta + \overline{\delta} + 1, x)$ (and then $\tilde{\mathrm{K}}_\infty^{(\delta)}$) can be expressed in terms of Whittaker functions, and for $\delta \in \mathbb{R}$ as Bessel functions (see [3]).

Remark. In [16] the determinantal kernel on the real line associated to Hua-Pickrell measures on the Hermitian matrices $\mathrm{H}(n)$ is given. Its asymptotics, after the scaling $x \mapsto nx$ on the eigenvalues, is noted $\tilde{\mathrm{K}}_\infty^{(\delta,\mathrm{H})}(x, y)$. With no surprise, the expression given by Borodin and Olshanski coincides with ours after a suitable change of variables :

$$\tilde{\mathrm{K}}_\infty^{(\delta,\mathrm{H})}(x, y) = f(\delta)\tilde{\mathrm{K}}_\infty^{(\delta,\mathrm{U})}\left(\frac{2}{x}, \frac{2}{y}\right) \tag{3.13}$$

for a constant $f(\delta)$. This was observed by Borodin and Olshanski thanks to the following argument, linking the unitary and Hermitian ensembles via the Cayley transform

$$\left\{ \begin{array}{ccc} \mathrm{H}(n) & \to & \mathrm{U}(n) \\ \mathrm{X} & \mapsto & \frac{\mathrm{i}-\mathrm{X}}{\mathrm{i}+\mathrm{X}} \end{array} \right. .$$

Indeed, this function transforms the Hua-Pickrell measure on $\mathrm{H}(n)$ into the Hua-Pickrell measure on $\mathrm{U}(n)$. Therefore, a scaling $x \mapsto nx$ of the eigenvalues on $\mathrm{H}(n)$ corresponds to a scaling $\alpha \mapsto \frac{\alpha}{n}$ for the eigenangles on $\mathrm{U}(n)$:

$$\frac{\mathrm{i} - nx}{\mathrm{i} + nx} = -e^{-\frac{2\mathrm{i}}{nx}} + \mathrm{O}\left(\frac{1}{n^2}\right), \tag{3.14}$$

leading to the correspondence $\alpha = \frac{2}{x}$. Theorem 3.8 gives an alternative proof of (3.13), by direct calculation. Note that for fixed n we do not have an identity like (3.13), because of the error term in (3.14).

3. Universality of the hypergeometric kernel

Theorem 3.2 is a direct application of Lubinsky's localization method, that we recall here for completeness.

Lubinsky's inequality (see [90], [91], [92], [93] or the review [126] by Barry Simon) states that if $\mu \leqslant \mu^*$ on $\partial\mathbb{D}$, then for any α and β

$$\frac{|\mathrm{K}_n^{(\mu)}(e^{\mathrm{i}\alpha}, e^{\mathrm{i}\beta}) - \mathrm{K}_n^{(\mu^*)}(e^{\mathrm{i}\alpha}, e^{\mathrm{i}\beta})|}{\mathrm{K}_n^{(\mu)}(e^{\mathrm{i}\alpha}, e^{\mathrm{i}\alpha})} \leqslant \left(\frac{\mathrm{K}_n^{(\mu)}(e^{\mathrm{i}\beta}, e^{\mathrm{i}\beta})}{\mathrm{K}_n^{(\mu)}(e^{\mathrm{i}\alpha}, e^{\mathrm{i}\alpha})}\right)^{1/2} \left(1 - \frac{\mathrm{K}_n^{(\mu^*)}(e^{\mathrm{i}\alpha}, e^{\mathrm{i}\alpha})}{\mathrm{K}_n^{(\mu)}(e^{\mathrm{i}\alpha}, e^{\mathrm{i}\alpha})}\right)^{1/2}$$

Therefore, with the diagonal control of the Christoffel-Darboux kernel and some localization work, the off-diagonal limits of a general kernel can be obtained from a special one, here $\mathrm{K}_n^{(\delta)}$. Lubinsky evaluates the diagonal kernel using the variational formulation

$$\frac{1}{\mathrm{K}_n(e^{\mathrm{i}\theta}, e^{\mathrm{i}\theta})} = \min_{\deg(\mathrm{P}) \leqslant n-1} \frac{\frac{1}{2\pi}\int_{-\pi}^{\pi} |\mathrm{P}(e^{\mathrm{i}s})|^2 \mathrm{d}\mu(s)}{|\mathrm{P}(e^{\mathrm{i}\theta})|^2}.$$

To state his result, we need to introduce the concept of mutual regularity.

Definition 3.9. *Let μ and ν be measures on $\partial\mathbb{D}$. We say that they are mutually regular if as $n \to \infty$*

$$\sup_{\deg \mathrm{P} \leqslant n} \left(\frac{\int |\mathrm{P}|^2 \mathrm{d}\mu}{\int |\mathrm{P}|^2 \mathrm{d}\nu}\right)^{1/n} \to 1 \quad \textit{and} \quad \sup_{\deg \mathrm{P} \leqslant n} \left(\frac{\int |\mathrm{P}|^2 \mathrm{d}\nu}{\int |\mathrm{P}|^2 \mathrm{d}\mu}\right)^{1/n} \to 1.$$

Moreover, mutual regularity is linked to the concept of regularity in the sense of Stahl and Totik, defined for a measure μ on the unit circle by the condition

$$(a_n^{(\mu)})^{1/n} \underset{n\to\infty}{\longrightarrow} 1,$$

where $a_n^{(\mu)}$ is the coefficient of X^n in $p_n^{(\mu)}$, the n^{th} orthonormal polynomial for the measure μ. If μ_a is almost everywhere positive then it is regular, and it is mutually regular with the weight $\nu' = 1$ with the same support as μ (see [127]). In particular, an almost everywhere positive measure is mutually regular with $\lambda^{(\delta)}$.

As indicated by the title of his paper [91], Lubinsky has shown that *mutually regular measures have similar universality limits* in the context of measures on the real line. His Theorem 4 in [91] has the following strict analogue on the unit circle, the proof being the same as Lubinsky's, replacing everywhere the scalar product by the Hermitian one.

Theorem 3.10. *Let μ and ν be mutually regular measures on $\partial\mathbb{D}$. Let J be a compact subset of the support $\mathrm{supp}(\mu)$ of μ. Assume that I is an open set containing J, such that in $\mathrm{I}\cap\mathrm{supp}(\mu)$, μ and ν are mutually absolutely continuous. Assume moreover that at each point of J, the Radon-Nikodym derivative $\mathrm{d}\mu/\mathrm{d}\nu$ is positive and continuous. Assume that (ε_n) is a sequence of positive numbers with limit 0, such that for any $\mathrm{A} > 0$*

$$\lim_{\eta\to 0^+} \limsup_{n\to\infty} \frac{\mathrm{K}_n^{(\nu)}(e^{\mathrm{i}(\theta+\alpha\varepsilon_n)}, e^{\mathrm{i}(\theta+\alpha\varepsilon_n)})}{\mathrm{K}_{n-\lfloor \eta n\rfloor}^{(\nu)}(e^{\mathrm{i}(\theta+\alpha\varepsilon_n)}, e^{\mathrm{i}(\theta+\alpha\varepsilon_n)})} = 1 \tag{3.15}$$

uniformly for $\theta \in \mathrm{J}$ and $|\alpha| \leqslant \mathrm{A}$. Then for any $\mathrm{A} > 0$, uniformly in α, β in $[-\mathrm{A}, \mathrm{A}]$ and $\theta \in \mathrm{J}$, with $\theta + \alpha\varepsilon_n$, $\theta + \beta\varepsilon_n$ restricted to $\mathrm{supp}(\mu)$,

$$\lim_{n\to\infty} \frac{\left|\frac{\mathrm{d}\mu}{\mathrm{d}\nu}(\theta)\mathrm{K}_n^{(\mu)}(e^{\mathrm{i}(\theta+\alpha\varepsilon_n)}, e^{\mathrm{i}(\theta+\beta\varepsilon_n)}) - \mathrm{K}_n^{(\nu)}(e^{\mathrm{i}(\theta+\alpha\varepsilon_n)}, e^{\mathrm{i}(\theta+\beta\varepsilon_n)})\right|}{\mathrm{K}_n^{(\mu)}(e^{\mathrm{i}(\theta+\alpha\varepsilon_n)}, e^{\mathrm{i}(\theta+\alpha\varepsilon_n)})} = 0. \tag{3.16}$$

Remark. If J consists of a single point in the interior of the support, this Theorem shows easily the universality of the sine kernel by taking for ν the uniform measure on $\partial\mathbb{D}$. We will use the above result when $\mathrm{J} = \{1\}$ is a point at the edge of the spectrum, and $\nu = \lambda^{(\delta)}(\theta)\mathrm{d}\theta$. Note that the continuity of $\mathrm{d}\mu/\mathrm{d}\nu$ in J is in the sense of approaching points in J from all points of the support of μ.

Moreover, in Theorem 4 in [91], there is a technical condition on the diagonal kernel (condition (4) in [91]) which aims at replacing $\mathrm{K}_n^{(\mu)}(e^{\mathrm{i}(\theta+\alpha\varepsilon_n)}, e^{\mathrm{i}(\theta+\alpha\varepsilon_n)})$ by $\mathrm{K}_n^{(\mu)}(e^{\mathrm{i}(\theta)}, e^{\mathrm{i}(\theta)})$ in (3.16). We do not need this replacement here, hence we omit the analogue of Lubinsky's technical condition (4).

Proof of Theorem 3.2. We choose $\nu = \lambda^{(\delta)}(\theta)\mathrm{d}\theta$, $\varepsilon_n = 1/n$ and μ as in Theorem 3.2 to apply Theorem 3.10, . The measure μ is almost everywhere strictly positive, so as previously mentioned it is mutually regular with ν.

Moreover, for our choice of ν, the technical condition (3.15) follows directly from the calculations of the previous section, in particular Theorem 3.8.

The conclusion of Theorem 3.10 holds, so the kernels $h(0)\mathrm{K}_n^{(\mu)}$ and $\mathrm{K}_n^{(\delta)}$ have the same asymptotics uniformly on $\mathbb{R}^2$. This implies that the normalized reproducing kernels $\tilde{\mathrm{K}}_n^{(\mu)}$ and $\tilde{\mathrm{K}}_n^{(\delta)}$ have the same asymptotics uniformly on $\mathbb{R}^2$. The asymptotics of $\tilde{\mathrm{K}}_n^{(\delta)}$ are given in Theorem 3.8, and yield the expected result, with distinct uniformity domains whether $\mathfrak{Re}(\delta) \geqslant 0$ or not. $\square$

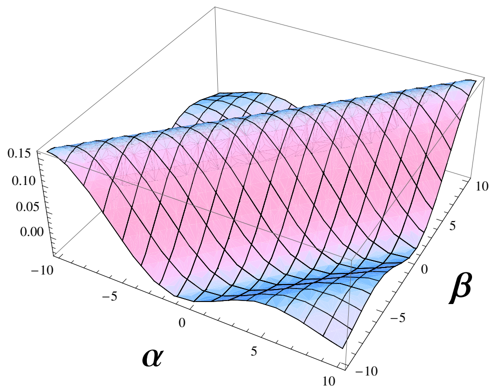


Figure 3.3. The sine kernel : $\delta = 0$. Depends only on the difference $\alpha - \beta$.

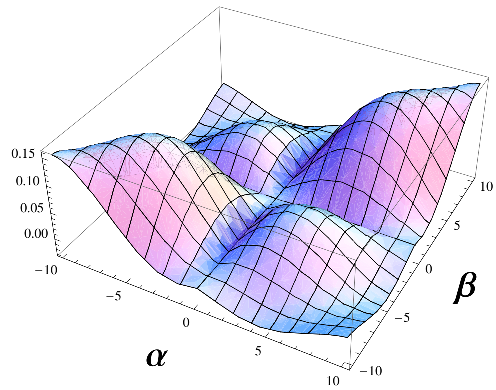


Figure 3.4. The Bessel kernel : $\delta = 1$. Symmetric with respect to $(0, 0)$.

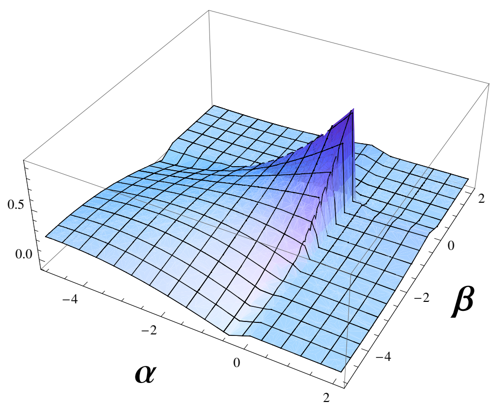


Figure 3.5. The pure discontinuity for the hypergeometric kernel : $\delta = \mathrm{i}$. No symmetry except $(\alpha, \beta) \to (\beta, \alpha)$, discontinuous at $\alpha = 0$ or $\beta = 0$.

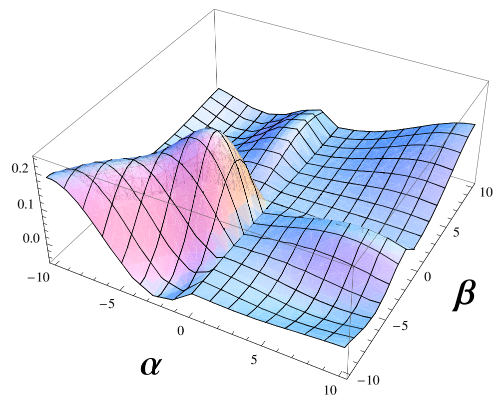


Figure 3.6. The hypergeometric kernel : $\delta = 1 + \mathrm{i}$. Continuous if $\mathfrak{Re}(\delta) \geqslant 0$, discontinuous otherwise.

Chapter 4

Random orthogonal polynomials on the unit circle

This chapter corresponds to *Circular Jacobi ensembles and deformed Verblunsky coefficients*, International Mathematics Research Notices, 4357-4394 (2009), and *The characteristic polynomial on compact groups with Haar measure : some equalities in law* [22], C. R. Acad. Sci. Paris, Ser. I 345, 229-232, (2007), joint works with A. Nikeghbali and A. Rouault.

This chapter combines the decomposition of the characteristic polynomial as product of independent random variables, obtained in the first two chapters, with the theory of orthogonal polynomials on the unit circle (OPUC).

In particular, writing in a generic manner $\det(\mathrm{Id}-u)=\prod_{k=1}^{n}(1-\gamma_k)$ with independent γ_k's, we give a geometric meaning to the γ_k's in terms of *deformed Verblunsky coefficients*, which are closely related to the Verblunsky coefficients appearing in the OPUC theory.

This point of view allows us to propose a simple matrix model for the following circular analogue of the Jacobi ensemble :

$$c_{n,\beta,\delta}\prod_{1\leqslant k<l\leqslant n}|e^{\mathrm{i}\theta_k}-e^{\mathrm{i}\theta_l}|^{\beta}\prod_{j=1}^{n}(1-e^{-\mathrm{i}\theta_j})^{\delta}(1-e^{\mathrm{i}\theta_j})^{\overline{\delta}}$$

with $\mathfrak{Re}\,\delta>-1/2$. In the case $\delta=0$, the construction is due to Killip and Nenciu and is based on the classical Verblunsky coefficients.

The introduction of these deformed Verblunsky coefficients also allows to give limit theorems for the empirical spectral distribution

$$\frac{1}{n}\sum_{k=1}^{n}\delta_{e^{\mathrm{i}\theta_k}}$$

in the regime $\delta=\delta(n)$ with $\delta(n)/n\to \mathrm{d}$ as $n\to\infty$: we prove its weak convergence in probability towards a measure supported on an arc of the unit circle, and a large deviations principle with explicit rate function.

1. Introduction

1.1. The Jacobi circular ensemble.

The theory of random unitary matrices was developed using the existence of a natural uniform probability measure on compact Lie groups, namely the Haar measure. The statistical properties of the eigenvalues as well as the characteristic polynomial of these random matrices have played a crucial role both in physics (see [96] for an historical account) and in analytic number theory to model L-functions (see [80] and [81] where Keating and Snaith predict moments of L-functions on the critical line using knowledge on the moments of the characteristic polynomial of random unitary matrices).

The circular unitary ensemble (CUE) is $\mathrm{U}(n)$, the unitary group over $\mathbb{C}^n$, equipped with its Haar measure $\mu_{\mathrm{U}(n)}$. The Weyl integration formula allows one to average any (bounded measurable) function on $\mathrm{U}(n)$ which is conjugation-invariant

$$\int f \mathrm{d}\mu_{\mathrm{U}(n)} = \frac{1}{n!} \int \cdots \int |\Delta(e^{\mathrm{i}\theta_1}, \dots, e^{\mathrm{i}\theta_n})|^2 f(\mathrm{diag}\,(e^{\mathrm{i}\theta_1}, \dots, e^{\mathrm{i}\theta_n})) \frac{\mathrm{d}\theta_1}{2\pi} \cdots \frac{\mathrm{d}\theta_n}{2\pi}, \quad (4.1)$$

where $\Delta(e^{\mathrm{i}\theta_1}, \dots, e^{\mathrm{i}\theta_n}) = \prod_{1 \leqslant j < k \leqslant n} (e^{\mathrm{i}\theta_k} - e^{\mathrm{i}\theta_j})$ denotes the Vandermonde determinant (see Theorem 1.18 in Chapter 1 for a probabilistic proof of (4.1)).

The circular orthogonal ensemble (COE) is the subset of $\mathrm{U}(n)$ consisting of symmetric matrices, i.e. $\mathrm{U}(n)/\mathrm{O}(n) = \{v\,{}^{\mathrm{t}}v \mid v \in \mathrm{U}(n)\}$ equipped with the measure obtained by pushing forward $\mu_{\mathrm{U}(n)}$ by the mapping $v \mapsto v\,{}^{\mathrm{t}}v$. The integration formula is similar to (4.1) but with $|\Delta(e^{\mathrm{i}\theta_1}, \dots, e^{\mathrm{i}\theta_n})|^2$ replaced by $|\Delta(e^{\mathrm{i}\theta_1}, \dots, e^{\mathrm{i}\theta_n})|$ and with the normalizing constant changed accordingly.

For the circular symplectic ensemble (CSE), which will not be recalled here, the integration formula uses $|\Delta(e^{\mathrm{i}\theta_1}, \dots, e^{\mathrm{i}\theta_n})|^4$.

Dyson observed that the induced eigenvalues distributions correspond to the Gibbs distribution for the classical Coulomb gas on the circle at three different temperatures. More generally, n identically charged particles confined to move on the unit circle, each interacting with the others through the usual Coulomb potential $-\log|z_i - z_j|$, give rise to the Gibbs measure with parameters n, the number of particles, and β, the inverse temperature (see the discussion and references in [84] and in [48] chap. 2) :

$$\mathbb{E}_n^{\beta}(f) = c_{0,\beta}^{(n)} \int f(e^{\mathrm{i}\theta_1}, \dots, e^{\mathrm{i}\theta_n}) |\Delta(e^{\mathrm{i}\theta_1}, \dots, e^{\mathrm{i}\theta_n})|^{\beta} \mathrm{d}\theta_1 \dots \mathrm{d}\theta_n,$$

where $c_{0,\beta}^{(n)}$ is a normalizing constant chosen so that

$$h_{0,\beta}^{(n)}(\theta_1, \dots, \theta_n) = c_{0,\beta}^{(n)} |\Delta(e^{\mathrm{i}\theta_1}, \dots, e^{\mathrm{i}\theta_n})|^{\beta} \quad (4.2)$$

is a probability density on $(-\pi, \pi)^n$ and where f is any symmetric function. The unitary, orthogonal and symplectic circular ensembles correspond to matrix models for the Coulomb gas at three different temperatures, but are there matrix models for general inverse temperature $\beta > 0$ for Dyson's circular eigenvalue statistics ?

Killip and Nenciu [84] provided matrix models for Dyson's circular ensemble, using the theory of orthogonal polynomials on the unit circle. In particular, they obtained a sparse matrix model which is pentadiagonal, called CMV (after the names of the authors Cantero, Moral, Velásquez [26]). In this framework, there is not a natural underlying measure such as the Haar measure ; the matrix ensemble is characterized by the laws of its elements.

There is an analogue of Dyson's circular ensembles on the real line : the probability density function of the eigenvalues $(x_1, \dots, x_n)$ for such ensembles with inverse temperature parameter β is proportional to

$$|\Delta(x_1, \dots, x_n)|^{\beta} \prod_{j=1}^{n} e^{-x_j^2/2} \mathrm{d}x_1 \dots \mathrm{d}x_n. \quad (4.3)$$

For $\beta = 1, 2$ or 4, this corresponds to the classical Gaussian ensembles. Dumitriu and Edelman [44] gave a simple tridiagonal matrix model for (4.3). Killip and Nenciu [84], gave an analogue matrix model for the Jacobi measure on the segment $(-2, 2)$, which is up to a normalizing constant,

$$|\Delta(x_1, \dots, x_n)|^{\beta} \prod_{j=1}^{n} (2 - x_j)^a (2 + x_j)^b \mathrm{d}x_1 \dots \mathrm{d}x_n, \quad (4.4)$$

where $a, b > 0$, relying on the theory of orthogonal polynomials on the unit circle and its links with orthogonal polynomials on the segment. When a and b are strictly positive integers, the Jacobi measure (4.4) can be interpreted as the potential $|\Delta(x_1, \dots, x_{n+a+b})|^\beta$ on $(-2, 2)^{n+a+b}$ conditioned to have a elements on 2 and b elements on -2. Consequently, the Jacobi measure on the unit circle should be a two parameters extension of (4.2), corresponding to conditioning to have specific given eigenvalues. Such an analogue was defined as the *Jacobi circular ensemble* in [48] and [52].

Definition 4.1. *Throughout this chapter, we note $h^{(n)}_{\delta,\beta}$ the probability density function on $(-\pi, \pi)^n$ given by :*

$$h^{(n)}_{\delta,\beta}(\theta_1, \dots, \theta_n) = c^{(n)}_{\delta,\beta} |\Delta(e^{\mathrm{i}\theta_1}, \dots, e^{\mathrm{i}\theta_n})|^\beta \prod_{j=1}^{n} (1 - e^{-\mathrm{i}\theta_j})^\delta (1 - e^{\mathrm{i}\theta_j})^{\overline{\delta}} \tag{4.5}$$

with $\delta \in \mathbb{C}$, $\mathfrak{Re}(\delta) > -\frac{1}{2}$.

If $\delta \in \frac{\beta}{2}\mathbb{N}$, this measure coincides with (4.2) conditioned to have $2\delta/\beta$ eigenvalues at 1. For $\beta = 2$, such spectral measures were first considered by Hua [70] and Pickrell [111], [112]. This case was also widely studied in [103] and [16] for its connections with the theory of representations, and in Chapter 2 for its analogies with the Ewens measure on permutations group.

One of our goals in this chapter is to provide a matrix model for the Jacobi circular ensemble, i.e. a distribution on $\mathrm{U}(n)$ such that the arguments of the eigenvalues $(e^{\mathrm{i}\theta_1}, \dots, e^{\mathrm{i}\theta_n})$ are distributed as in (4.5). One can guess that additional problems may appear because the distribution of the eigenvalues is not rotation invariant anymore. Nevertheless, some statistical information for the Jacobi circular ensemble can be obtained from Dyson's circular ensemble by a sampling (or a change of probability measure) with the help of the determinant.

If we consider a matrix model for $h^{(n)}_{0,\beta}$, we can define[1] a matrix model for $h^{(n)}_{\delta,\beta}$ by the means of a sampling (in the sense of Definition 1.16), noticing that when the charges are actually the eigenvalues of a matrix u, then (4.2) differs from (4.5) by a factor which is a function of $\det(\mathrm{Id} - u)$. Actually $\det_\delta$ is defined as in the previous chapters :

$$\det_\delta(u) = \det(\mathrm{Id} - u)^{\overline{\delta}} \det(\mathrm{Id} - \overline{u})^\delta ,$$

and we will use this $\det_\delta$ sampling.

Actually we look for an effective construction of a random matrix, for instance starting from a reduced number of independent random variables with known distributions. Notice that in the particular case $\beta = 2$, the density $h_{0,2}$ corresponds to eigenvalues of a matrix under the Haar measure on $\mathrm{U}(n)$ and the $\det_\delta$ sampling of this measure is the Hua-Pickrell measure studied in Chapter 2.

1.2. Orthogonal polynomials on the unit circle.

We now wish to outline the main ingredients which are needed from the theory of orthogonal polynomials on the unit circle to construct matrix models for the general Dyson's circular ensemble. The reader can refer to [122] and [123] for more results and references ; in particular, all the results about orthogonal polynomials on the unit circle (named OPUC) can be found in these volumes.

Let us explain why OPUC play a prominent role in these constructions. Throughout this chapter, $\mathbb{D}$ denotes the open unit disk $\{z \in \mathbb{C} : |z| < 1\}$ and $\partial\mathbb{D}$ the unit

1. for $\mathfrak{Re}(\delta) > -1/2$, due to an integrability constraint

circle $\{z \in \mathbb{C} : |z| = 1\}$. Let $(\mathcal{H}, u, e)$ be a triple where $\mathcal{H}$ is a Hilbert space, u a unitary operator and e a cyclic unit vector, i.e. $\{u^j e\}_{j=-\infty}^{\infty}$ is total in $\mathcal{H}$. We say that two triples $(\mathcal{H}, u, e)$ and $(\mathcal{K}, v, e')$ are equivalent if and only if there exists an isometry $k : \mathcal{H} \to \mathcal{K}$ such that $v = kuk^{-1}$ and $e' = ke$. The spectral theorem says that for each equivalence class, there exists a unique probability measure μ on $\partial\mathbb{D}$ such that

$$\langle e, u^k e\rangle_{\mathcal{H}} = \int_{\partial\mathbb{D}} z^k \mathrm{d}\mu(z) \quad , \quad k = 0, \pm 1, \dots$$

Conversely, such a probability measure μ gives rise to a triple consisting of the Hilbert space $\mathrm{L}^2(\mu)$, the operator of multiplication by z, i.e. $h \mapsto (z \mapsto zh(z))$ and the vector $\mathbf{1}$, i.e. the constant function 1. When the space $\mathcal{H}$ is fixed, the probability measure μ associated with the triple $(\mathcal{H}, u, e)$ is called the *spectral measure of the pair* (u, e).

Let us consider the finite n-dimensional case. Assume that u is unitary and e is cyclic. It is classical that u has n different eigenvalues $(e^{\mathrm{i}\theta_j}, j = 1, \dots, n)$. In any orthonormal basis whose first vector is e say $(e_1 = e, \dots, e_n)$, u is represented by a matrix u and there is a unitary matrix Π diagonalizing u. It is then straightforward that

$$\mu = \sum_{j=1}^{n} \pi_j \, \delta_{e^{\mathrm{i}\theta_j}} \tag{4.6}$$

where the weights are defined as $\pi_j = |\langle e_1, \Pi e_j\rangle|^2$. Note that $\pi_j > 0$ because a cyclic vector cannot be orthogonal to any eigenvector (and we also have $\sum_{j=1}^{n} \pi_j = 1$ because Π is unitary). The eigenvalues $(e^{\mathrm{i}\theta_j}, j = 1, \dots n)$ and the vector $(\pi_1, \dots, \pi_n)$ can then be used as coordinates for the probability measure μ.

Keeping in mind our purpose, we see that the construction of a matrix model from a vector $(e^{\mathrm{i}\theta_j}, j = 1, \dots, n)$ may be achieved in two steps : first give a vector of weights $(\pi_1, \dots, \pi_n)$, then find a matricial representative of the equivalence class with a rather simple form. The key tool for the second task is the sequence of orthogonal polynomials associated with the measure μ. In $\mathrm{L}^2(\partial\mathbb{D}, \mathrm{d}\mu)$ equipped with the natural basis $\{1, z, z^2, \dots, z^{n-1}\}$, the Gram-Schmidt procedure provides the family of monic orthogonal polynomials $\Phi_0, \dots, \Phi_{n-1}$. We can still define Φ_n as the unique monic polynomial of degree n with $\| \Phi_n \|_{\mathrm{L}^2(\mu)} = 0$:

$$\Phi_n(z) = \prod_{j=1}^{n} (z - e^{\mathrm{i}\theta_j}) \, . \tag{4.7}$$

The Φ_k's $(k = 0, \dots, n)$ obey the Szegö recursion relation :

$$\Phi_{j+1}(z) = z\Phi_j(z) - \bar{\alpha}_j \Phi_j^*(z) \tag{4.8}$$

where

$$\Phi_j^*(z) = z^j \, \overline{\Phi_j(\bar{z}^{-1})} \, .$$

The coefficients α_j's $(0 \leqslant j \leqslant n-1)$ are called Verblunsky coefficients and satisfy the condition $\alpha_0, \dots, \alpha_{n-2} \in \mathbb{D}$ and $\alpha_{n-1} \in \partial\mathbb{D}$.

When the measure μ has infinite support, one can define the family of orthogonal polynomials $(\Phi_n)_{n\geqslant 0}$ associated with μ for all n. Then there are infinitely many Verblunsky coefficients (α_n) which all lie in $\mathbb{D}$.

Verblunsky's Theorem (see e.g. Theorem 1.7.11 in [122]) states that there is a bijection between probability measures on the unit circle and sequences of Verblunsky coefficients.

The matrix of the multiplication by z in $\mathrm{L}^2(\partial\mathbb{D}, \mu)$, in the basis of orthonormal polynomials, has received much attention. This unitary matrix is called GGT by B. Simon [122] (for Geronimus [58], Gragg [62], and Teplyaev [134])

It is noted $\mathcal{G}(\alpha_0,\dots,\alpha_{n-2},\alpha_{n-1})$ and is in the Hessenberg form : all entries below the subdiagonal are zero, whereas the entries above the subdiagonal are nonzero and the subdiagonal is nonnegative. Formulae (4.1.5) and (4.1.6) in [122] give an explicit expression for the entries in terms of the Verblunsky coefficients.

Moreover, there is an explicit useful decomposition of these matrices into product of block matrices, called the AGR decomposition by Simon [125], after the paper [2]. For $0 \leqslant k \leqslant n-2$, let

$$\Theta^{(k)}(\alpha) = \mathrm{Id}_k \oplus \begin{pmatrix} \overline{\alpha}_k & \rho_k \\ \rho_k & -\alpha_k \end{pmatrix} \oplus \mathrm{Id}_{n-k-2}. \tag{4.9}$$

and set $\Theta^{(n-1)}(\alpha_{n-1}) = \mathrm{Id}_{n-1} \oplus (\overline{\alpha}_{n-1})$, with $|\alpha_{n-1}| = 1$. Then the AGR decomposition states that ([125] Theorem 10.1)

$$\mathcal{G}(\alpha_0,\dots,\alpha_{n-1}) = \Theta^{(0)}(\alpha_0)\Theta^{(1)}(\alpha_1)\dots\Theta^{(n-1)}(\alpha_{n-1})\,. \tag{4.10}$$

Now we state a crucial result of Killip and Nenciu which enabled them to obtain a matrix model in the Hessenberg form for Dyson's circular ensemble.

Theorem 4.2 (Killip-Nenciu [84])**.** *The following formulae express the same measure on the manifold of probability distributions on $\partial\mathbb{D}$ supported on n points :*

$$\frac{2^{1-n}}{n!}|\Delta(e^{\mathrm{i}\theta_1},\dots,e^{\mathrm{i}\theta_n})|^\beta \prod_{j=1}^n \pi_j^{\beta/2-1}\mathrm{d}\theta_1\dots\mathrm{d}\theta_n\mathrm{d}\pi_1\dots\mathrm{d}\pi_{n-1}$$

in the (θ,π) coordinates and

$$\prod_{k=0}^{n-2}(1-|\alpha_k|^2)^{(\beta/2)(n-k-1)-1}\mathrm{d}^2\alpha_0\dots\mathrm{d}^2\alpha_{n-2}\frac{\mathrm{d}\varphi}{2\pi}$$

in terms of Verblunsky coefficients.

This result is highly non-trivial because there is no simple change of variables between the Verblunsky coefficients and the eigenvalues/weights. To comment on the above theorem, we need to introduce a notation and definition.

Definition 4.3. *For $s>1$ let ν_s be the probability measure on $\mathbb{D}$ with density*

$$\frac{s-1}{2\pi}(1-|z|^2)^{(s-3)/2}.$$

It is the law of $re^{\mathrm{i}\psi}$ where r and ψ are independent, ψ is uniformly distributed on $(-\pi,\pi)$ and $r \overset{\text{law}}{=} \sqrt{\mathrm{B}_{1,\frac{s-1}{2}}}$, the square root of a beta variable with the indicated parameters. We adopt the convention that ν_1 is the uniform distribution on the unit circle. We denote by $\eta_{0,\beta}^{(n)}$ the distribution on $\mathbb{D}^{n-1}\times\partial\mathbb{D}$ given by

$$\eta_{0,\beta}^{(n)} = \otimes_{k=0}^{n-1}\nu_{\beta(n-k-1)+1}\,.$$

The Dirichlet distribution of order $n\geqslant 2$ with parameter $a>0$, denoted by $Dir_n(a)$, is the probability distribution on the simplex $\{(x_1,\dots,x_n)\in(0,1)^n : \sum_{i=1}^n x_i = 1\}$ with density

$$\frac{\Gamma(na)}{\Gamma(a)^n}\prod_{k=1}^n x_k^{a-1}\,.$$

Theorem 4.2 may be restated as follows : to pick at random a measure μ such that $(\alpha_0,\dots,\alpha_{n-1})$ is $\eta_{0,\beta}^{(n)}$ distributed is equivalent to pick the support $(\theta_1,\dots,\theta_n)$

according to $h_{0,\beta}^{(n)}$ (see (4.2)) and independently pick the weights $(\pi_1, \dots, \pi_n)$ according to $\mathrm{Dir}_n(\beta/2)$.

As a consequence, if one takes independent coefficients $(\alpha_0, \dots, \alpha_{n-2}, \alpha_{n-1})$ such that α_k is $\nu_{\beta(n-k-1)+1}$ distributed for $0 \leqslant k \leqslant n-1$, then the GGT matrix $\mathcal{G}(\alpha_0, \dots, \alpha_{n-2}, \alpha_{n-1})$ will be a matrix model for Dyson's circular ensemble with inverse temperature β (see also Proposition 2.11 in [48]). Actually in [84], Killip and Nenciu provide a matrix model which is much sparser (pentadiagonal) as shall be explained in Section 4.

Let us now define the laws on $\mathrm{U}(n)$ which we will consider in the sequel.

Definition 4.4. *We denote by $CJ_{0,\beta}^{(n)}$ the probability distribution on $\mathrm{U}(n)$ supported by the set of matrices of the GGT form (4.10) whenever the parameters $\alpha_0, \dots, \alpha_{n-1}$ are defined as above. We denote by $CJ_{\delta,\beta}^{(n)}$ the probability distribution on $\mathrm{U}(n)$ which is the $\det_\delta$ sampling of $CJ_{0,\beta}^{(n)}$.*

The above approach is not sufficient to produce matrix ensembles for the Jacobi circular ensemble because, as we shall see in Section 3, under the measure $\mathrm{CJ}_{\delta,\beta}^{(n)}$, the Verblunsky coefficients are not independent anymore. To overcome this difficulty, we associate to a measure on the unit circle, or equivalently to its Verblunsky coefficients, a new sequence of coefficients $(\gamma_k)_{0\leqslant k\leqslant n-1}$, which we shall call deformed Verblunsky coefficients. There is a simple bijection between the original sequence $(\alpha_k)_{0\leqslant k\leqslant n-1}$ and the new one $(\gamma_k)_{0\leqslant k\leqslant n-1}$. These coefficients satisfy among several properties that $|\alpha_k| = |\gamma_k|$, and they are independent under $\mathrm{CJ}_{\beta,\delta}^{(n)}$. Moreover, for $\delta = 0$ the α_k's and the γ_k's have the same distribution. These deformed Verblunsky coefficients have a geometric interpretation in terms of reflections : this leads to a decomposition of the GGT matrix $\mathcal{G}(\alpha_0, \dots \alpha_{n-1})$ as a product of independent elementary reflections constructed from the γ_k's. The explicit expression of the densities allows an asymptotic study, as $n \to \infty$, of the γ_k's, and consequently it also gives the asymptotic behavior of the (empirical) spectral measure.

1.3. Organization of the chapter.

In Section 2, after recalling basic facts about the reflections introduced in Chapters 1 and 2, we define the deformed Verblunsky coefficients $(\gamma_k)_{0\leqslant k\leqslant n-1}$ and give some of its basic properties. In particular we prove that the GGT matrix $\mathcal{G}(\alpha_0, \dots \alpha_{n-1})$ can be decomposed into a product of elementary complex reflections (Theorem 4.12).

In Section 3, we derive the law of the γ_k's under $\mathrm{CJ}_{\delta,\beta,}^{(n)}$ (Thorem 4.14) ; in particular we show that they are independent and that the actual Verblunsky coefficients are dependent if $\delta \neq 0$. We then prove an analogue of the above Theorem 4.2 on the (θ, π) coordinates of μ (Theorem 4.15).

In Section 4, we propose our matrix model (Theorem 4.16). It is a modification of the AGR factorization, where we transform the Θ_k's so that they become reflections :

$$\Xi^{(k)}(\alpha) = \mathrm{Id}_k \oplus \begin{pmatrix} \overline{\alpha} & e^{\mathrm{i}\varphi}\rho \\ \rho & -e^{\mathrm{i}\varphi}\alpha \end{pmatrix} \oplus \mathrm{Id}_{n-k-2},$$

with $e^{\mathrm{i}\varphi} = \frac{1-\overline{\alpha}}{1-\alpha}$. Of course the CMV model, which is five diagonal, is also available, but this time the α_k's are not independent. Using the following elementary fact proven in Section 2,

$$\Phi_n(1) = \det(\mathrm{Id} - u) = \prod_{k=0}^{n-1}(1 - \gamma_k),$$

we are able to generalize our previous results in Chapters 1 and 2 about the decomposition of the characteristic polynomial evaluated at 1 as a product of independent complex variables (Corollary 4.18).

In Section 5, we study asymptotic properties of our model as $n \to \infty$, when $\delta = \beta n \mathrm{d}/2$, with $\mathfrak{Re}\,\mathrm{d} \geqslant 0$. We first prove that the Verblunsky coefficients have deterministic limits in probability. This entails that the spectral measure converges weakly in probability to the same deterministic measure (denoted by $\mu_{\mathtt{esd}}^{\infty}$) which is supported by an arc of the unit circle (Theorem 4.19). Besides, we consider the empirical spectral distribution (ESD), where the Dirac masses have same weight $1/n$. Bounding the distances between both random measures, we proved that the ESD has the same limit (Theorem 4.21). Moreover starting from the explicit joint distribution (4.5), we prove also that the ESD satisfies a large deviation principle at scale $(\beta/2)n^2$ whose rate function reaches its minimum at $\mu_{\mathtt{esd}}^{\infty}$ (Theorem 4.22).

2. Deformed Verblunsky coefficients and reflections

In this section, we introduce the deformed Verblunsky coefficients and we establish some of their relevant properties, in particular a geometric interpretation in terms of reflections. One remarkable property of the Verblunsky coefficients, as it appears in Theorem 4.2, is that they are independent under $\mathrm{CJ}_{0,\beta}^{(n)}$. As we shall see in Section 3, this does not hold anymore under $\mathrm{CJ}_{\delta,\beta}^{(n)}$. This motivated us to introduce a new set of coefficients, $(\gamma_0, \dots, \gamma_{n-2}, \gamma_{n-1})$, called deformed Verblunsky coefficients, which are uniquely associated with a set of Verblunsky coefficients. In particular, $\gamma_k \in \mathbb{D}$ for $0 \leqslant k \leqslant n-2$, $\gamma_{n-1} \in \partial\mathbb{D}$ and the map $(\gamma_0, \dots, \gamma_{n-2}, \gamma_{n-1}) \mapsto (\alpha_0, \dots, \alpha_{n-2}, \alpha_{n-1})$ is a bijection. Moreover, the characteristic polynomial at 1 can be expressed simply in terms of $(\gamma_0, \dots, \gamma_{n-2}, \gamma_{n-1})$.

2.1. Analytical properties

Let μ be a probability measure on the unit circle supported at n points. Keeping the notations of the introduction, we let $(\Phi_k(z))_{0\leqslant k\leqslant n}$ denote the monic orthogonal polynomials associated with μ and $(\alpha_k)_{0\leqslant k\leqslant n-1}$ its corresponding set of Verblunsky coefficients through Szegö's recursion formula (4.8). The functions

$$b_k(z) = \frac{\Phi_k(z)}{\Phi_k^*(z)}\ ,\ k \leqslant n-1 \tag{4.11}$$

are known as the inverse Schur iterates ([123] p.476, after Khrushchev [83] p.273). They are analytic in a neighborhood of $\bar{\mathbb{D}}$ and meromorphic in $\mathbb{C}$. Each b_k is a finite Blashke product

$$b_k(z) = \prod_{j=1}^{n} \left(\frac{z - z_j}{1 - \bar{z}_j z} \right)$$

where $z_1, \dots, z_k$ are the zeros of Φ_k. Let us now explain the term *inverse Schur iterate*.

The Schur function is a fundamental object in the study of the orthogonal polynomials on the unit circle. Let us briefly recall its definition (see [122] or [124] for more details and proofs) : if μ is a probability measure on the unit circle (whether it is supported on finitely many points or not), its Schur function $f : \mathbb{D} \to \mathbb{D}$ is defined as :

$$f(z) = \frac{1}{z}\frac{\mathrm{F}(z) - 1}{\mathrm{F}(z) + 1} \text{ where } \mathrm{F}(z) = \int \frac{e^{\mathrm{i}\theta} + z}{e^{\mathrm{i}\theta} - z}\,\mathrm{d}\mu(e^{\mathrm{i}\theta}).$$

It is a bijection between the set of probability measures on the unit circle and analytic functions mapping $\mathbb{D}$ to $\bar{\mathbb{D}}$. The Schur algorithm (which is described in [122] or [124] p.438) allows to parametrize the Schur function f by a sequence of so-called Schur parameters, which are actually the Verblunsky coefficients associated to μ (Geronimus theorem). In particular, there are finitely many Verblunsky coefficients (or equivalently the measure μ is supported at n points) if and only if f is a finite Blaschke product. The name *inverse Schur iterate* [82] for b_k comes from the result (1.12) of the latter paper where b_k is identified as the Schur function corresponding to the *reversed sequence* $(-\bar{\alpha}_{k-1}, \dots, -\bar{\alpha}_0, 1)$ (see also [123] Prop. 9.2.3).

Let us define our sequence of functions, which shall lead us the deformed coefficients.

Definition 4.5. *If μ is supported at n points and with the notation above, define $\gamma_k(z)$ for $0 \leqslant k \leqslant n-1$, as :*

$$\gamma_k(z) = z - \frac{\Phi_{k+1}(z)}{\Phi_k(z)}, \tag{4.12}$$

From the Szegö's recursion formula (4.8) and notation (4.11), this is equivalent to

$$\gamma_k(z) = \frac{\bar{\alpha}_k}{b_k(z)}\,, \tag{4.13}$$

so that γ_k is meromorphic, with poles in $\mathbb{D}$ and zeros lying outside $\bar{\mathbb{D}}$.

The next proposition shows how the functions $\gamma_k(z)$ can be defined recursively with the help of the coefficients α_k. As a consequence, we shall see that the $\gamma_k(z)$ are closely related to a fundamental object in the theory of random matrices : the characteristic polynomial.

Proposition 4.6. *For any $z \in \mathbb{C}$, $\gamma_0(z) = \bar{\alpha}_0$ and the following decomposition for $\Phi_k(z)$ holds :*

$$\Phi_k(z) = \prod_{j=0}^{k-1}(z - \gamma_j(z))\ , \quad k = 1, \dots, n\,.$$

The $\gamma_k(z)$'s may be also defined by means of the α's through the recursion :

$$\begin{aligned} \gamma_k(z) &= \bar{\alpha}_k \prod_{j=0}^{k-1} \frac{1 - z\widetilde{\gamma}_j(z)}{z - \gamma_j(z)}, \\ \widetilde{\gamma}_k(z) &= \overline{\gamma_k(\bar{z}^{-1})}\,. \end{aligned}$$

Proof. The first claim is an immediate consequence of (4.12). Now, using $\Phi_k(z) = \prod_{j=0}^{k-1}(z - \gamma_j(z))$, we obtain

$$\Phi_k^*(z) = \prod_{j=0}^{k-1}(1 - z\widetilde{\gamma}_j(z)),$$

and hence (we use (4.13))

$$\gamma_k(z) = \bar{\alpha}_k \prod_{j=0}^{k-1} \frac{1 - z\widetilde{\gamma}_j(z)}{z - \gamma_j(z)}.$$

□

Note that when $|z| = 1$, $|\gamma_k(z)| = |\alpha_k|$. Combined with the above proposition, this leads us to introduce the following set of coefficients.

Definition 4.7. *Define the coefficients $(\gamma_k)_{0 \leqslant k \leqslant n-1}$ by*

$$\gamma_k = \gamma_k(1), \quad k = 0, \dots, n-1\,.$$

We shall refer to the γ_k's as the deformed Verblunsky coefficients.

Proposition 4.8. *The following properties hold for the deformed Verblunsky coefficients :*

a) For all $0 \leqslant k \leqslant n-1$, $|\gamma_k| = |\alpha_k|$, *and in particular* $\gamma_{n-1} \in \partial\mathbb{D}$ *;*

b) $\gamma_0 = \bar{\alpha_0}$ *and*

$$\gamma_k = \bar{\alpha}_k e^{\mathrm{i}\varphi_{k-1}} \quad , \quad e^{\mathrm{i}\varphi_{k-1}} = \prod_{j=0}^{k-1} \frac{1-\bar{\gamma}_j}{1-\gamma_j} \quad , \quad (k = 1, \ldots, n-1)\,. \tag{4.14}$$

The last term is particular. Since $|\alpha_{n-1}| = 1$, *we set* $\alpha_{n-1} = e^{\mathrm{i}\psi_{n-1}}$, *so that*

$$\gamma_{n-1} = e^{\mathrm{i}(-\psi_{n-1}+\varphi_{n-2})} := e^{\mathrm{i}\theta_{n-1}}\,. \tag{4.15}$$

c) Let μ *be the spectral measure associated to* (u, e_1), $u \in \mathrm{U}(n)$. *Then* $\Phi_n(z)$ *is its characteristic polynomial,*

$$\Phi_n(1) = \det(\mathrm{Id} - u) = \prod_{k=0}^{n-1}(1-\gamma_k). \tag{4.16}$$

Proof. All the results are direct consequences of the definition 4.7 and the formulae in Proposition 4.6 evaluated at 1. □

Remark. In [85], Killip and Stoiciu have already considered variables which are the complex conjugate of our deformed Verblunsky coefficients as auxiliary variables in the study of the Prüfer phase (Lemma 2.1 in [85]). Nevertheless, the way we define them as well as the use we make of them are different.

Remark. The formula (4.14) shows that the γ_k's can be obtained from the α_k's recursively. Hence starting from a spectral measure associated to a unitary matrix, one can associate with it the Verblunsky coefficients and then the deformed Verblunsky coefficients. Conversely, one can translate any property of the deformed ones into properties for the spectral measure associated with it by inverting the transformations (4.14).

Remark. The distribution of the characteristic polynomial of random unitary matrices evaluated at 1, through its Mellin-Fourier transform, plays a key role in the theory of random matrices, especially through its links with analytic number theory (see [98] for an account). In Chapter 1 it is proven that it can be decomposed in law into a product of independent random variables when working on the unitary and orthogonal groups endowed with the Haar measure; since we will prove in Section 3 that the γ_k's are independent under $\mathrm{CJ}_{\delta,\beta}^{(n)}$, then we can conclude that this latter result holds for any Jacobi circular ensemble.

2.2. Geometric interpretation

We give a connection between the coefficients $(\gamma_k)_{0\leqslant k\leqslant n-1}$ and reflections defined just below. This allows us to obtain a new decomposition of the GGT matrix associated with a measure μ supported at n points on the unit circle as a product of n elementary reflections.

Many distinct definitions of reflections on the unitary group exist, the most well-known may be the Householder reflections. The transformations which will be relevant to us are the following ones.

Definition 4.9. *An element* r *in* $\mathrm{U}(n)$ *will be referred to as a reflection if* $r - \mathrm{Id}$ *has rank 0 or 1.*

If $v \in \mathbb{C}^n$, we denote by $\langle v|$ the linear form $w \mapsto \langle v, w\rangle$. The reflections can also be described in the following way. If e and $m \neq e$ are unit vectors of $\mathbb{C}^n$, there is a unique reflection r such that $r(e) = m$, and

$$r = \mathrm{Id} - \frac{1}{1 - \langle m, e\rangle}(m-e)\,\langle (m-e)| \;. \tag{4.17}$$

Let $\mathrm{F} = \mathrm{span}\{e, m\}$ be the 2-dimensional vector space which is spanned by the vectors e and m. It is clear that the reflection given by formula (4.17) leaves $\mathrm{F}^\perp$ invariant. Now set

$$\gamma = \langle m, e\rangle \;,\; \rho = \sqrt{1 - |\gamma|^2} \;,\; e^{\mathrm{i}\varphi} = \frac{1-\gamma}{1-\bar\gamma}\,,$$

and let $g \in \mathrm{F}$ be the unit vector orthogonal to e obtained by the Gram-Schmidt procedure. Then in the basis (e, g) of F, the matrix of the restriction of r is

$$\Xi(\gamma) = \begin{pmatrix} \gamma & \rho e^{\mathrm{i}\varphi} \\ \rho & -\bar\gamma e^{\mathrm{i}\varphi} \end{pmatrix} . \tag{4.18}$$

Conversely, for $\gamma \in \mathbb{D}$, such a matrix represents the unique reflection in $\mathbb{C}^2$ provided with its canonical basis, mapping e_1 onto $\gamma e_1 + \sqrt{1-|\gamma|^2}e_2$. The eigenvalues of r are 1 and $-e^{\mathrm{i}\varphi}$.

Let u be a unitary operator in $\mathbb{C}^n$ and e a cyclic vector for u. We define n reflections $r_1, \dots, r_n$ recursively as follows. Let $(\varepsilon_1, \dots, \varepsilon_n)$ be the orthonormal basis obtained from the Gram-Schmidt procedure applied to $(e, ue, \dots, u^{n-1}e)$.

Let r_1 be the reflection, mapping $e = \varepsilon_1$ onto $u\,e = u\varepsilon_1$. More generally, for $k \geqslant 2$ let r_k be the reflection mapping ε_k onto $r_{k-1}^{-1}r_{k-2}^{-1}\dots r_1^{-1}u\varepsilon_k$. We will identify these reflections and establish the decomposition of u. Following the basics recalled about GGT matrices in the introduction, we note that the matrix of u in the basis $(\varepsilon_1, \dots, \varepsilon_n)$ is the GGT matrix associated to the measure μ, i.e. the matrix $\mathcal{G}(\alpha_0, \cdots, \alpha_{n-2}, \alpha_{n-1})$, where $(\alpha_0, \cdots, \alpha_{n-2}, \alpha_{n-1})$ are the Verblunsky coefficients associated with the measure μ. We will use formula (4.1.6) of [122] for the identification of scalar products.

Proposition 4.10. *a. For every $1 \leqslant k \leqslant n-1$, the reflection r_k leaves invariant the $n-2$-dimensional space* $\mathrm{Span}\{\varepsilon_1, \dots, \varepsilon_{k-1}, \varepsilon_{k+2}, \dots, \varepsilon_n\}$. *The reflection r_n leaves invariant Span$\{\varepsilon_1, \dots, \varepsilon_{n-1}\}$.*

b. The following decomposition holds :

$$u = r_1 \cdots r_n \,.$$

Proof. (1) In view of Section 2.1, it is enough to prove that for $j \notin \{k, k+1\}$, the vectors ε_j and $r_k\varepsilon_k$ are orthogonal.

For $k = 1$, $\langle \varepsilon_j, r_1\varepsilon_1\rangle = \langle \varepsilon_j, u\varepsilon_1\rangle = 0$ as soon as $j \geqslant 3$ from (4.10).

Assume that for every $\ell \leqslant k-1$, r_ℓ leaves invariant $\mathrm{span}\{\varepsilon_1, \dots, \varepsilon_{\ell-1}, \varepsilon_{\ell+2}, \dots, \varepsilon_n\}$. For every $j = 1, \dots, n$, we have

$$\langle \varepsilon_j, r_k\varepsilon_k\rangle = \langle \varepsilon_j, r_{k-1}^{-1}r_{k-2}^{-1}\dots r_1^{-1}u\varepsilon_k\rangle = \langle r_1\cdots r_{k-1}\varepsilon_j, u\varepsilon_k\rangle \,. \tag{4.19}$$

For $j \geqslant k+2$, by assumption, the reflections $r_1, \dots, r_{k-1}$ leave invariant ε_j, so that the above expression reduces to $\langle \varepsilon_j, u\varepsilon_k\rangle$ which is 0 again by (4.10).

For $j = k-1$, we have $r_1\cdots r_{k-1}\varepsilon_{k-1} = u\varepsilon_{k-1}$ by definition of r_{k-1}, so that (4.19) gives $\langle \varepsilon_{k-1}, r_k\varepsilon_k\rangle = \langle u\varepsilon_{k-1}, u\varepsilon_k\rangle$, which is 0 since u is unitary.

For $j < k-1$, by assumption, the reflections $r_{j+1}, \dots, r_{k-1}$ leave invariant ε_j, so that the right hand side of (4.19) reduces to $\langle r_1\cdots r_j\varepsilon_j, u\varepsilon_k\rangle$. By definition of r_j, it is $\langle u\varepsilon_j, u\varepsilon_k\rangle$ which is 0.

(2) For k fixed, it is clear from (1) that $r_1\cdots r_n\varepsilon_k = r_1\cdots r_k\varepsilon_k$ which is $u\varepsilon_k$ by definition of r_k. □

Proposition 4.11. *For $k = 1, \dots, n-1$, the matrix of the restriction of r_k in the basis $(\varepsilon_k, \varepsilon_{k+1})$ is $\Xi(\gamma_{k-1})$. In particular*

$$\langle \varepsilon_k, r_k \varepsilon_k \rangle = \gamma_{k-1} \, . \tag{4.20}$$

The restriction of r_n to $\mathbb{C}\varepsilon_n$ is the multiplication by γ_{n-1}.

Proof. Note that for every $k \leqslant n-1$

$$\langle \varepsilon_{k+1}, r_k \varepsilon_k \rangle = \langle r_1 \cdots r_{k-1} \varepsilon_{k+1}, u\varepsilon_k \rangle = \langle \varepsilon_{k+1}, u\varepsilon_k \rangle = \rho_{k-1} \, . \tag{4.21}$$

Since r_k is a reflection acting on the subspace $\{\varepsilon_k, \varepsilon_{k+1}\}$, identities (4.21) and (4.20) entail that the matrix representing r_k in the basis $(\varepsilon_1, \dots, \varepsilon_n)$ is precisely $\Xi(\gamma_{k-1})$ (see (4.18)). It is then enough to prove (4.20).

For $k = 1$ it is immediate that :

$$\langle \varepsilon_1, r_1 \varepsilon_1 \rangle = \langle \varepsilon_1, u\varepsilon_1 \rangle = \bar{\alpha}_0 = \gamma_0 \, .$$

Let us proceed by induction. For $j \geqslant 1$ set $q_j = \langle \varepsilon_j, r_j \varepsilon_j \rangle$. Assume that $q_j = \gamma_{j-1}$ for $j \leqslant k$. We have $q_{k+1} = \langle \varepsilon_{k+1}, r_{k+1} \varepsilon_{k+1} \rangle = \langle r_1 \dots r_k \varepsilon_{k+1}, u\varepsilon_{k+1} \rangle$. Equation (4.17) implies

$$r_k \varepsilon_{k+1} = \varepsilon_{k+1} - \frac{1}{1 - \bar{\gamma}_{k-1}} \big(r_k \varepsilon_k - \varepsilon_k \big) \langle r_k \varepsilon_k, \varepsilon_{k+1} \rangle,$$

and since $r_j \varepsilon_\ell = \varepsilon_\ell$ for $\ell \geqslant j+2$, we get :

$$r_1 \dots r_k \varepsilon_{k+1} = \varepsilon_{k+1} - \frac{1}{1 - \bar{\gamma}_{k-1}} \big(r_1 \dots r_k \varepsilon_k - r_1 \dots r_{k-1} \varepsilon_k \big) \langle u\varepsilon_k, \varepsilon_{k+1} \rangle.$$

Now, it is known that $\langle u\varepsilon_k, \varepsilon_{k+1} \rangle = \overline{\langle \varepsilon_{k+1}, u\varepsilon_k \rangle} = \rho_k$. If we set $v_1 = \varepsilon_1$,

$$v_j = r_1 \dots r_{j-1} \varepsilon_j \ , \ a_j = \frac{\rho_{j-1}}{1 - \bar{\gamma}_{j-1}} \ , \ w_{j+1} = \varepsilon_{j+1} - a_j u \varepsilon_j$$

we get the recursion

$$v_{j+1} = a_j v_j + w_{j+1} \ , \ (j \leqslant k) \, ,$$

which we solve in :

$$v_{k+1} = \Big(\prod_{j=1}^{k} a_j \Big) \varepsilon_1 + \sum_{\ell=2}^{k+1} \Big(\prod_{j=\ell}^{k} a_j \Big) w_\ell.$$

Taking the scalar product with $u\varepsilon_{k+1}$ yields

$$q_{k+1} = \Big(\prod_{j=1}^{k} \bar{a}_j \Big) \langle \varepsilon_1, u\varepsilon_{k+1} \rangle + \sum_{\ell=2}^{k+1} \Big(\prod_{j=\ell}^{k} \bar{a}_j \Big) \langle w_\ell, u\varepsilon_{k+1} \rangle.$$

But $\langle w_\ell, u\varepsilon_{k+1} \rangle = \langle \varepsilon_\ell, u\varepsilon_{k+1} \rangle - \bar{a}_{\ell-1} \langle u\varepsilon_{\ell-1}, u\varepsilon_{k+1} \rangle$, and since $\ell \leqslant k+1$, we have

$$\langle w_\ell, u\varepsilon_{k+1} \rangle = \langle \varepsilon_\ell, u\varepsilon_{k+1} \rangle = -\bar{\alpha}_k \alpha_{\ell-2} \prod_{m=\ell-1}^{k-1} \rho_m,$$

which yields (with $\alpha_{-1} = -1$)

$$\begin{aligned}
-\frac{q_{k+1}}{\bar{\alpha}_k} &= \sum_{\ell=1}^{k+1} \Big(\prod_{j=\ell}^{k} \bar{a}_j \Big) \alpha_{\ell-2} \prod_{m=\ell-1}^{k-1} \rho_m \\
&= \sum_{\ell=1}^{k+1} \prod_{m=\ell-1}^{k-1} \rho_m^2 \prod_{j=0}^{\ell-3} (1 - \bar{\gamma}_j) \frac{\bar{\gamma}_{\ell-2}}{\prod_{s=0}^{k-1} (1 - \gamma_s)} (1 - \gamma_{\ell-2}) \\
&= \frac{1}{\prod_{s=0}^{k-1} (1 - \gamma_s)} \sum_{\ell=1}^{k+1} \left(\prod_{m=\ell-2}^{k-1} \rho_m^2 \prod_{j=0}^{\ell-3} (1 - \bar{\gamma}_j) - \prod_{m=\ell-1}^{k-1} \rho_m^2 \prod_{j=0}^{\ell-2} (1 - \bar{\gamma}_j) \right) \\
&= - \prod_{s=0}^{k-1} \frac{(1 - \bar{\gamma}_s)}{(1 - \gamma_s)},
\end{aligned}$$

and eventually $q_{k+1} = \gamma_k$. □

Now, we can summarize the above results in the following theorem.

Theorem 4.12. *Let $u \in \mathrm{U}(n)$ and e a cyclic vector for u. Let μ be the spectral measure of the pair (u, e), and $(\alpha_0, \dots \alpha_{n-2}, \alpha_{n-1})$ its Verblunsky coefficients. Let $(\varepsilon_1, \dots, \varepsilon_n)$ be the orthonormal basis obtained from the Gram-Schmidt procedure applied to $(e, ue, \dots, u^{n-1}e)$. Then, u can be decomposed as a product of n reflections $(r_k)_{1 \leqslant k \leqslant n}$:*

$$u = r_1 \dots r_n$$

where r_1 is the reflection mapping ε_1 onto $u\varepsilon_1$ and by induction for each $2 \leqslant k \leqslant n$, r_k maps ε_k onto $r_{k-1}^{-1} r_{k-2}^{-1} \dots r_1^{-1} u \varepsilon_k$.

This decomposition can also be restated in terms of the GGT matrix :

$$\mathcal{G}(\alpha_0, \cdots, \alpha_{n-2}, \alpha_{n-1}) = \Xi^{(0)}(\gamma_0)\Xi^{(1)}(\gamma_1) \dots \Xi^{(n-1)}(\gamma_{n-1}),$$

where for $0 \leqslant k \leqslant n-2$, the matrix $\Xi^{(k)}$ is given by

$$\Xi^{(k)}(\gamma_{k-1}) = \mathrm{Id}_k \oplus \Xi(\gamma_{k-1}) \oplus \mathrm{Id}_{n-k-2}, \tag{4.22}$$

with $\Xi(\gamma)$ defined in (4.18). For $k = n-1$,

$$\Xi^{(n-1)}(\gamma_{n-1}) = \mathrm{Id}_{n-1} \oplus (\gamma_{n-1}). \tag{4.23}$$

Remark. The above geometric interpretation of the deformed Verblunsky coefficients and Theorem 1.2 in Chapter 1 proves the Killip-Nenciu Theorem 4.2 for the unitary group ($\beta = 2$).

3. Deformed Verblunsky coefficients and independence

We now use the point of view of sampling (or change of probability measure) to compute the distribution of the deformed Verblunsky coefficients under $\mathrm{CJ}^{(n)}_{\delta,\beta}$. Let us first remark that, if the α_k's are independent and with rotational invariant distribution, then from (4.14)

$$(\alpha_0, \dots, \alpha_{n-1}) \overset{\text{law}}{=} (\gamma_0, \dots, \gamma_{n-1}).$$

This is the case under $\mathrm{CJ}^{(n)}_{0,\beta}$.

We first prove that when $\delta \neq 0$ the Verblunsky coefficients are not independent anymore studying the simple case $n = 2, \beta = 2$, and then we compute the distribution of $(\gamma_0, \dots, \gamma_{n-1})$ under $\mathrm{CJ}^{(n)}_{\delta,\beta}$. We then show that under this distribution, the weights of the measure associated to the Verblunsky coefficients $(\alpha_0, \dots, \alpha_{n-1})$ are independent from the points at which the measure is supported and follow a Dirichlet distribution.

In the sequel, we shall need the following notation, identical to (3.3) :

$$\begin{aligned} \lambda^{(\delta)}(\theta)\mathrm{d}\theta &= c(\delta)(1 - e^{\mathrm{i}\theta})^{\overline{\delta}}(1 - e^{-\mathrm{i}\theta})^{\delta}\mathrm{d}\theta \\ &= c(\delta)(2 - 2\cos\theta)^a e^{-b(\pi \operatorname{sgn}\theta - \theta)}\mathrm{d}\theta \end{aligned}$$

with

$$\delta = a + \mathrm{i}b \quad , \quad c(\delta) = \frac{\Gamma(1+\delta)\Gamma(1+\overline{\delta})}{\Gamma(1+\delta+\overline{\delta})}.$$

When $\beta = 2$ and $\delta \neq 0$, the Verblunsky coefficients are dependent. Indeed, let $u \in \mathrm{U}(2)$ with Verblunsky coefficients α_0 and α_1. Then

$$\det(\mathrm{Id} - u) = (1 - \bar{\alpha}_0 - \bar{\alpha}_1(1 - \alpha_0)),$$

with $|\alpha_0| < 1$ and $|\alpha_1| = 1$. To simplify the notations, let us simply write $\alpha_0 = \alpha$ and $\alpha_1 = e^{\mathrm{i}\varphi}$. Under $\mathrm{CJ}_{0,2}^{(n)}$, the variables α and $e^{\mathrm{i}\varphi}$ are independent, with density $\frac{1}{2\pi^2}$ with respect to the measure $\mathrm{d}^2\alpha \otimes \mathrm{d}\varphi$ on $\mathbb{D} \times \partial\mathbb{D}$ (see Theorem 4.2 above with $\beta = 2$ and $n = 2$). Hence, under $\mathrm{CJ}_{\delta,2}^{(2)}$, with $\delta = a + \mathrm{i}b$, $a, b \in \mathbb{R}$, which is a $\det_\delta$ sampling of $\mathrm{CJ}_{0,2}^{(n)}$ (see the introduction for the definition and notation for the $\det_\delta$ sampling), the joint density of (α, φ) with respect to the measure $\mathrm{d}^2\alpha \otimes \mathrm{d}\varphi$ is (we omit the normalization constant)

$$f(\alpha, \varphi) = (1 - \bar{\alpha} - e^{-\mathrm{i}\varphi}(1 - \alpha))^{\bar{\delta}}(1 - \alpha - e^{\mathrm{i}\varphi}(1 - \bar{\alpha}))^{\delta}\mathbb{1}_{|\alpha|<1},$$

that is to say

$$f(\alpha, \varphi) = (1 - \bar{\alpha})^{\bar{\delta}}(1 - \alpha)^{\delta}(1 - \gamma e^{-\mathrm{i}\varphi})^{\bar{\delta}}(1 - \bar{\gamma} e^{\mathrm{i}\varphi})^{\delta}\mathbb{1}_{|\alpha|<1},$$

where

$$\gamma = \frac{1 - \alpha}{1 - \bar{\alpha}}.$$

Since $|\gamma| = 1$, we can set $\gamma = e^{\mathrm{i}\varphi_0(\alpha)}$ for some $\varphi_0(\alpha) \in (-\pi, \pi)$ which is a continuous function of α. With this notation,

$$f(\alpha, \varphi) = \frac{1}{c(\delta)}(1 - \bar{\alpha})^{\bar{\delta}}(1 - \alpha)^{\delta}\lambda^{(\delta)}(\varphi_0(\alpha) - \varphi)\mathbb{1}_{|\alpha|<1}.$$

Consequently, the marginal probability distribution of α is proportional to

$$\frac{1}{\pi c(\delta)}(1 - \bar{\alpha})^{\bar{\delta}}(1 - \alpha)^{\delta}\mathbb{1}_{|\alpha|<1},$$

whereas the conditional probability density function of φ given α is proportional to

$$\frac{1}{2\pi}\lambda^{(\delta)}(\varphi_0(\alpha) - \varphi).$$

It is clear that this last quantity depends on α (unless $\delta = 0$) and consequently the original Verblunsky coefficients α_0 and α_1 are dependent.

The next Theorem 4.14 illustrates our interest in the deformed Verblunsky coefficients : under $\mathrm{CJ}_{\delta,\beta}^{(n)}$, they are independent. For the proof of this theorem, we shall need the following lemma which will also be useful when we study limit theorems :

Lemma 4.13. *Let s, t, ℓ be complex numbers such that : $\mathfrak{Re}(s + \ell + 1) > 0, \mathfrak{Re}(t + \ell + 1) > 0$. Then, the following identity holds :*

$$\int_{\mathbb{D}} (1 - |z|^2)^{\ell-1}(1 - z)^s(1 - \bar{z})^t \mathrm{d}^2 z = \frac{\pi\Gamma(\ell)\Gamma(\ell + 1 + s + t)}{\Gamma(\ell + 1 + s)\Gamma(\ell + 1 + t)}\,. \tag{4.24}$$

Proof. A Taylor development yields

$$(1 - z)^s(1 - \bar{z})^t = \sum_{m,n\geqslant 0} \rho^{m+n}\frac{(-s)_n(-t)_m}{n!m!}e^{i(m-n)\theta},$$

and by integration

$$\begin{aligned}\int_{\mathbb{D}} (1 - |z|^2)^{\ell-1}(1 - z)^s(1 - \bar{z})^t d^2 z &= 2\pi\sum_{n\geqslant 0}\frac{(-s)_n(-t)_n}{n!n!}\int_0^1 (1 - \rho^2)^{\ell-1}\rho^{2n+1}d\rho \\ &= \pi\sum_{n\geqslant 0}\frac{(-s)_n(-t)_n}{n!}\frac{(\ell - 1)!}{(n + \ell)!} \\ &= \frac{\pi}{\ell}\ {}_2\mathrm{F}_1(-s, -t; \ell + 1; 1)\,,\end{aligned}$$

where ${}_2\mathrm{F}_1$ is the classical hypergeometric function and an application of the Gauss formula (see [3]) shows that the last expression is exactly the right hand side of (4.24). $\square$

Theorem 4.14. *Let $\delta \in \mathbb{C}$ with $\mathfrak{Re}\,\delta > -1/2$ and $\beta > 0$. Set $\beta' = \beta/2$. Under $CJ^{(n)}_{\delta,\beta}$, the distribution of $(\gamma_0, \dots, \gamma_{n-1})$, denoted hereafter $\eta^{(n)}_{\delta,\beta}$, is the following :*

a. *the variables $\gamma_0, \dots, \gamma_{n-2}, \gamma_{n-1} = e^{i\theta_{n-1}}$ are independent ;*

b. *for $k = 0, \dots, n-2$ the density of γ_k with respect to the Lebesgue measure d^2z on $\mathbb{C}$ is*
$$c_{k,n}(\delta)\left(1 - |z|^2\right)^{\beta'(n-k-1)-1}(1-z)^{\overline{\delta}}(1-\bar{z})^{\delta}\mathbb{1}_{\mathbb{D}}(z)\,,$$
where
$$c_{k,n}(\delta) = \frac{\Gamma\big(\beta'(n-k-1)+1+\delta\big)\Gamma\big(\beta'(n-k-1)+1+\overline{\delta}\big)}{\pi\Gamma\big(\beta'(n-k-1)\big)\Gamma\big(\beta'(n-k-1)+1+\delta+\overline{\delta}\big)}\,; \tag{4.25}$$

c. *the density of θ_{n-1} on $(-\pi, \pi)$ with respect to the Lebesgue measure is given by $\frac{1}{2\pi}\lambda^{(\delta)}(\theta)$.*

Proof. The distribution of the α's in the β-circular unitary ensemble is $\eta^{(n)}_{0,\beta}$. More precisely, as seen in Definition 4.3 they are independent and if $\alpha_k = \mathrm{r}_k e^{i\psi_k}$, for $0 \leqslant k \leqslant n-2$, then r_k and ψ_k are independent, ψ_k is uniformly distributed and r_k^2 has the Beta$(1, \beta'(n-k-1)$ distribution. Moreover $\alpha_{n-1} = e^{i\psi_{n-1}}$ where ψ_{n-1} is uniformly distributed on $(-\pi, \pi)$.

From (4.16), the sampling factor is
$$\det(\mathrm{Id} - u)^{\overline{\delta}}\det(\mathrm{Id} - \bar{u})^{\delta} = (1-\gamma_{n-1})^{\overline{\delta}}(1-\bar{\gamma}_{n-1})^{\delta}\prod_{k=0}^{n-2}(1-\gamma_k)^{\overline{\delta}}(1-\bar{\gamma}_k)^{\delta}\,.$$
so that, under $\mathrm{CJ}^{(n)}_{\delta,\beta}$, the density of $(\mathrm{r}_0, \dots, \mathrm{r}_{n-2}, \psi_0, \dots, \psi_{n-2}, \psi_{n-1})$ is proportional to
$$\lambda^{(\delta)}\big(\varphi_{n-2} - \psi_{n-1}\big)\prod_{k=0}^{n-2}(1-\mathrm{r}_k^2)^{\beta'(n-1-k)-1}\mathrm{r}_k(1-\gamma_k)^{\overline{\delta}}(1-\bar{\gamma}_k)^{\delta}\mathbb{1}_{(0,1)}(\mathrm{r}_k)\,,$$
with $\gamma_k = \mathrm{r}_k e^{i\theta_k}$. Thanks to the relations (4.14) and (4.15), the Jacobian matrix of the mapping
$$(\mathrm{r}_0, \dots, \mathrm{r}_{n-2}, \psi_0, \dots, \psi_{n-2}, \psi_{n-1}) \to (\mathrm{r}_0, \dots, \mathrm{r}_{n-2}, \theta_0, \dots, \theta_{n-2}, \theta_{n-1})$$
is lower triangular with diagonal elements ± 1, so that, under $\mathrm{CJ}^{(n)}_{\delta,\beta}$, the density of $(\mathrm{r}_0, \dots, \mathrm{r}_{n-2}, \theta_0, \dots, \theta_{n-1})$, is proportional to
$$\lambda^{(\delta)}(\theta_{n-1})\prod_{k=0}^{n-2}(1-\mathrm{r}_k^2)^{\beta'(n-1-k)-1}\mathrm{r}_k(1-\gamma_k)^{\overline{\delta}}(1-\bar{\gamma}_k)^{\delta}\mathbb{1}_{(0,1)}(\mathrm{r}_k)\,,$$
which proves the required independence and the expression of the distributions, up to the determination of $c_{k,n}(\delta)$. This quantity is obtained by taking $\ell = \beta'(n-k-1)$, $s = \overline{\delta}$, $t = \delta$ in (4.24), which gives (4.25) and completes the proof of the Theorem. $\square$

Starting with a set of deformed Verblunsky coefficients, with distribution $\eta_{\delta,\beta}$, we obtain the coefficients $(\alpha_0, \dots, \alpha_{n-2}, \alpha_{n-1})$ by inverting formula (4.16). These are the coordinates of some probability measure μ supported at n points on the unit circle :
$$\mu = \sum_{k=1}^{n}\pi_k\delta_{e^{i\theta_k}}\,,$$

with $\pi_k > 0$ and $\sum_{k=1}^{n} \pi_k = 1$. The next theorem gives the distribution induced on the vector $(\pi_1, \dots, \pi_n, \theta_1, \dots, \theta_n)$ by $(\gamma_0, \dots, \gamma_{n-2}, \gamma_{n-1})$.

Theorem 4.15. *The following formulae express the same measure on the manifold of probability distribution on $\partial\mathbb{D}$ supported at n points :*

$$\mathrm{K}_{\delta,\beta}^{(n)} |\Delta(e^{\mathrm{i}\theta_1}, \dots, e^{\mathrm{i}\theta_n})|^{\beta} \prod_{k=1}^{n} (1 - e^{-\mathrm{i}\theta_k})^{\delta} (1 - e^{\mathrm{i}\theta_k})^{\bar{\delta}} \prod_{k=1}^{n} \pi_k^{\beta'-1} \mathrm{d}\theta_1 \dots \mathrm{d}\theta_n \mathrm{d}\pi_1 \dots \mathrm{d}\pi_{n-1}$$

in the (θ, π) coordinates and

$$\mathrm{K}_{\delta,\beta}^{(n)} \prod_{k=0}^{n-2} (1 - |\gamma_k|^2)^{\beta'(n-k-1)-1} \prod_{k=0}^{n-1} (1 - \gamma_k)^{\bar{\delta}} (1 - \bar{\gamma}_k)^{\delta} \mathrm{d}^2\gamma_0 \dots \mathrm{d}^2\gamma_{n-2} \mathrm{d}\varphi$$

in terms of the deformed Verblunsky coefficients, with $\gamma_{n-1} = e^{\mathrm{i}\varphi}$. Here, $\mathrm{K}_{\delta,\beta}^{(n)}$ is a constant :

$$\mathrm{K}_{\delta,\beta}^{(n)} = \frac{\Gamma(1+\delta)\Gamma(1+\bar{\delta})}{2^{n-1}\pi\Gamma(1+\delta+\bar{\delta})} \prod_{k=0}^{n-2} c_{k,n}(\delta),$$

with $c_{k,n}(\delta)$ given in Theorem 4.14. Consequently, if $(\gamma_0, \dots, \gamma_{n-1})$ is $\eta_{\delta,\beta}^{(n)}$ distributed, then $(\pi_1, \dots, \pi_n)$ and $(\theta_1, \dots, \theta_n)$ are independent ; the vector of weights $(\pi_1, \dots, \pi_n)$ follows the $Dir_n(\beta')$ distribution and the vector $(\theta_1, \dots, \theta_n)$ has the density $h_{\delta,\beta}^{(n)}$.

Proof. In the course of this proof, we shall adopt the following point of view. Starting with a measure supported at n points on the unit circle, we associate with it its Verblunsky coefficients $(\alpha_0, \dots, \alpha_{n-2}, \alpha_{n-1})$ and then the corresponding GGT matrix which we note g for simplicity. Then e_1 is a cyclic vector for g and μ is the spectral measure of (g, e_1). Conversely, starting with the set of deformed Verblunsky coefficients with $\eta_{\delta,\beta}$ distribution, we construct the coefficients $(\alpha_0, \dots, \alpha_{n-2}, \alpha_{n-1})$ with the transformations (4.14), then the GGT matrix associated with it and finally μ the spectral measure associated with this matrix and e_1.

We use the following well known identity (see [122] or [84] Lemma 4.1) :

$$|\Delta(e^{\mathrm{i}\theta_1}, \dots, e^{\mathrm{i}\theta_n})|^2 \prod_{k=1}^{n} \pi_k = \prod_{k=0}^{n-2} (1 - |\alpha_k|^2)^{n-k-1}.$$

Since $|\gamma_k| = |\alpha_k|$, we can also write

$$|\Delta(e^{\mathrm{i}\theta_1}, \dots, e^{\mathrm{i}\theta_n})|^2 \prod_{k=1}^{n} \pi_k = \prod_{k=0}^{n-2} (1 - |\gamma_k|^2)^{n-k-1}. \tag{4.26}$$

Moreover, from (4.16),

$$\det(\mathrm{Id} - g) = \prod_{k=1}^{n} (1 - e^{\mathrm{i}\theta_k}) = \prod_{k=0}^{n-1} (1 - \gamma_k). \tag{4.27}$$

In our setting, π is modulus squared of the first component of the i-th eigenvector of the matrix g. If Π diagonalizes g, define $q_k = |\langle e_1, \Pi e_k \rangle| = \sqrt{\pi_k}$ for $k = 1, \dots, n$. It is known (see for example Forrester [48], Chapter 2 and [51] Theorem 2) that the Jacobian of the map $(\alpha_0, \dots, \alpha_{n-2}, \alpha_{n-1}) \mapsto (\theta_1, \dots \theta_n, q_1, \dots, q_{n-1})$ is given by

$$\frac{\prod_{k=0}^{n-2} (1 - |\alpha_k|^2)}{q_n \prod_{k=1}^{n} q_k}.$$

Moreover, the map $(\gamma_0, \dots, \gamma_{n-2}, \gamma_{n-1}) \mapsto (\alpha_0, \dots, \alpha_{n-2}, \alpha_{n-1})$ is invertible and its Jacobian is 1, as already seen. The result now follows from simple integral manipulations combined with the identities (4.26) and (4.27). $\square$

4. A matrix model for the Jacobi circular ensemble

The results of the previous sections now can be used to propose a simple matrix model for the Jacobi circular ensemble. There are mainly two ways to generate matrix models for a given spectral measure encoded by its Verblunsky coefficients.

The AGR decomposition : if $u = \Theta^{(0)}(\alpha_0)\Theta^{(1)}(\alpha_1)\dots\Theta^{(n-1)}(\alpha_{n-1})$ with the notation (4.9), the Verblunsky coefficients for the spectral measure associated to (u, e_1) are precisely $(\alpha_0,\dots,\alpha_{n-1})$ (see [2] or [125] Section 10). Therefore, taking independent α_k's with law $\eta_{0,\beta}^{(n)}$, the density of the eigenvalues of

$$u = \Theta^{(0)}(\alpha_0)\Theta^{(1)}(\alpha_1)\dots\Theta^{(n-1)}(\alpha_{n-1})$$

is proportional to $|\Delta(e^{\mathrm{i}\theta_1},\dots,e^{\mathrm{i}\theta_n})|^\beta$. The matrix u obtained above is the GGT matrix associated with the α_k's. It is in Hessenberg form.

The CMV form : Set

$$\begin{cases} \mathcal{L} &= \Theta^{(0)}(\alpha_0)\Theta^{(2)}(\alpha_2)\dots \\ \mathcal{M} &= \Theta^{(1)}(\alpha_1)\Theta^{(3)}(\alpha_3)\dots \end{cases}$$

Cantero, Moral, and Velazquez [26] proved that the Verblunsky coefficients associated to $(\mathcal{LM}, e_1)$ are precisely $(\alpha_0,\dots,\alpha_{n-1})$. Therefore, taking as previously independent α_k's with distribution $\eta_{0,\beta}^{(n)}$, the density of the eigenvalues of the spectral law of $\mathcal{LM}$ is proportional to $|\Delta(e^{\mathrm{i}\theta_1},\dots,e^{\mathrm{i}\theta_n})|^\beta$ [84]. This matrix model is sparse : it is pentadiagonal.

We now propose a matrix model for the Jacobi circular ensemble : it is reminiscent of the AGR factorization with the noticeable difference that it is based on the deformed Verblunsky coefficients and actual reflections as defined in Section 2.

Theorem 4.16. *If $(\gamma_0,\dots,\gamma_{n-1})$ is $\eta_{\delta,\beta}^{(n)}$ distributed, then with the notation of (4.22) and (4.23),*

$$\Xi^{(0)}(\gamma_0)\Xi^{(1)}(\gamma_1)\dots\Xi^{(n-1)}(\gamma_{n-1})$$

is a matrix model for the Jacobi circular ensemble, i.e. the density of the eigenvalues is $h_{\delta,\beta}$ (see (4.5)).

Proof. We know from Theorem 4.12 that

$$\mathcal{G}(\alpha_0,\cdots,\alpha_{n-2},\alpha_{n-1}) = \Xi^{(0)}(\gamma_0)\Xi^{(1)}(\gamma_1)\dots\Xi^{(n-1)}(\gamma_{n-1}).$$

We also proved in Theorem 4.15 that the set of deformed Verblunsky coefficients with probability distribution $\eta_{\delta,\beta}^{(n)}$ induces a distribution on the eigenvalues of the GGT matrix $\mathcal{G}(\alpha_0,\cdots,\alpha_{n-2},\alpha_{n-1})$ which has exactly the density $h_{\delta,\beta}$. This completes the proof of the Theorem. □

Remark. We now say a few extra words on the CMV form obtained by Killip and Nenciu in [84]. Cantero, Moral and Velazquez [26] introduced the basis $\chi_0,\dots,\chi_{n-1}$ obtained by orthogonalizing the sequence $1, z, z^{-1},\dots$. They prove that in this basis the matrix is pentadiagonal. We name this matrix $\mathcal{C}(\alpha_0,\cdots,\alpha_{n-2},\alpha_{n-1})$. It turns out that there exists a unitary P such that :

$$\mathrm{P}\mathcal{G}(\alpha_0,\cdots,\alpha_{n-2},\alpha_{n-1})\mathrm{P}^\star = \mathcal{C}(\alpha_0,\cdots,\alpha_{n-2},\alpha_{n-1})\ ,\ \mathrm{P}\varphi_0 = \chi_0.$$

The two pairs $(\mathcal{G}(\alpha_0,\cdots,\alpha_{n-2},\alpha_{n-1}),\varphi_0)$ and $(\mathcal{C}(\alpha_0,\cdots,\alpha_{n-2},\alpha_{n-1}),\chi_0)$ are equivalent, they admit the α_k's as Verblunsky coefficients, and have the same spectral measure. We conclude that if we start with the γ_k's distributed as $\eta_{\delta,\beta}^{(n)}$, and build the α_k's by inverting the transformation (4.14), then $\mathcal{C}(\alpha_0,\cdots,\alpha_{n-2},\alpha_{n-1})$ will be a matrix model for the Jacobi circular ensemble. But we do not know how to construct the CMV matrix from the γ_k's directly. We saw at the beginning of this section that Cantero et al. introduced the matrices $\mathcal{L}$ and $\mathcal{M}$, as direct product of small blocks $\Theta^{(k)}(\alpha_k)$ and obtained $\mathcal{C}$ as $\mathcal{C} = \mathcal{L}\mathcal{M}$. It would be interesting to have an analogue construction based on the independent γ_k's.

Theorem 4.12 which is a deterministic result, has also the following consequence :

Proposition 4.17. *Let $(\alpha_0, \dots, \alpha_{n-2}, \alpha_{n-1}) \in \mathbb{D}^{n-1} \times \partial\mathbb{D}$ be independent random variables with rotationally invariant distribution. Then*

$$\Theta^{(0)}(\alpha_0)\Theta^{(1)}(\alpha_1)\dots\Theta^{(n-1)}(\alpha_{n-1}) \overset{\text{law}}{=} \Xi^{(0)}(\alpha_0)\Xi^{(1)}(\alpha_1)\dots\Xi^{(n-1)}(\alpha_{n-1}).$$

Proof. We give two proofs of this result. The first one is a consequence of Theorem 4.12 from which we know that

$$\Theta^{(0)}(\alpha_0)\Theta^{(1)}(\alpha_1)\dots\Theta^{(n-1)}(\alpha_{n-1}) = \Xi^{(0)}(\gamma_0)\Xi^{(1)}(\gamma_1)\dots\Xi^{(n-1)}(\gamma_{n-1}),$$

and the remark at the beginning of Section 3.

For the second proof, we proceed by induction on n. For $n = 1$ the result is obvious. Suppose the result holds at rank $n-1$: thanks to the recurrence hypothesis,

$$\Xi^{(0)}(\alpha_0)\Xi^{(1)}(\alpha_1)\dots\Xi^{(n-1)}(\alpha_{n-1}) \overset{\text{law}}{=} \Xi^{(0)}(\alpha_0)\Theta^{(1)}(\alpha_1)\dots\Theta^{(n-1)}(\alpha_{n-1}).$$

Let $e^{\mathrm{i}\varphi_0} = \frac{1-\overline{\alpha_0}}{1-\alpha_0}$. An elementary calculation gives

$$\begin{aligned}\Xi^{(0)}(\alpha_0)&\Theta^{(1)}(\alpha_1)\dots\Theta^{(n-2)}(\alpha_{n-2})\Theta^{(n-1)}(\alpha_{n-1})\\ &= \Theta^{(0)}(\alpha_0)\Theta^{(1)}(e^{-\mathrm{i}\varphi_0}\alpha_1)\dots\Theta^{(n-2)}(e^{-\mathrm{i}\varphi_0}\alpha_{n-1})\Theta^{(n-1)}(e^{-\mathrm{i}\varphi_0}\alpha_{n-1}).\end{aligned}$$

As the α_k's are independent with law invariant by rotation,

$$(\alpha_0, e^{-\mathrm{i}\varphi_0}\alpha_1, \dots, e^{-\mathrm{i}\varphi_0}\alpha_{n-2}, e^{\mathrm{i}\varphi_0}\alpha_{n-1}) \overset{\text{law}}{=} (\alpha_0, \alpha_1, \dots, \alpha_{n-2}, \alpha_{n-1}),$$

which completes the proof. □

Now that we have a matrix model for the Jacobi circular ensemble, we can study the characteristic polynomial for such matrices : the following corollary is an easy consequence of (4.16) and Theorem 4.14.

Corollary 4.18. *Let u, a unitary matrix of size n and let $\mathrm{Z}_n = \det(\mathrm{Id} - u)$ be its characteristic polynomial evaluated at 1. Then, in the Jacobi circular ensemble, Z_n can be written as a product of n independent complex random variables :*

$$\mathrm{Z}_n = \prod_{k=0}^{n-1}(1-\gamma_k),$$

where the laws of the γ_k's are given in Theorem 4.14.

Remark. The above result associated to Lemma 4.13 immediately implies the following closed form for the Mellin-Fourier transform of Z_n :

$$\mathbb{E}\left(|\mathrm{Z}_n|^t e^{\mathrm{i}s\arg \mathrm{Z}_n}\right) = \prod_{k=0}^{n-1}\frac{\Gamma(\beta'k+1+\delta)\Gamma(\beta'k+1+\overline{\delta})\Gamma(\beta'k+1+\delta+\overline{\delta}+t)}{\Gamma(\beta'k+1+\delta+\overline{\delta})\Gamma(\beta'k+1+\delta+\frac{t-s}{2})\Gamma(\beta'k+1+\overline{\delta}+\frac{t+s}{2})}.$$

5. Limiting spectral measure and large deviations

In (4.6) we defined the spectral measure which is a central tool for the study of our circular ensembles. We are concerned with its asymptotics under $\mathrm{CJ}^{(n)}_{\delta,\beta}$ when $n \to \infty$

with $\delta = \delta(n) = \beta' n\mathrm{d}$, where as usual $\beta' = \beta/2$. Actually we prefer to write the measure on $[0, 2\pi)$ as

$$\mu_{\mathtt{sp}}^{(n)} = \sum_{k=1}^{n} \pi_k^{(n)} \delta_{\theta_k^{(n)}},$$

where the variables $\theta_k^{(n)}$ and $\pi_k^{(n)}$ are distributed as in Theorem 4.15, and where we put a superscript $^{(n)}$ to stress on the dependency on n. Besides, in classical Random Matrix Theory, many authors are in the first place interested in the empirical spectral distribution (ESD) defined by

$$\mu_{\mathtt{esd}}^{(n)} = \frac{1}{n} \sum_{k=1}^{n} \delta_{\theta_k^{(n)}}.$$

In this section we prove that both sequences converge weakly in probability and we establish a large deviation Principle for the ESD.

5.1. Spectral measure

The following theorem provides the explicit limit of the sequence.

Theorem 4.19. *As $n \to \infty$, the sequence of random probability measures $(\mu_{\mathtt{sp}}^{(n)})_n$ converges weakly in probability towards the measure on the unit circle*

$$\mathrm{d}\mu_{\mathtt{sp}}^{\infty}(\theta) = \mathrm{V}_{\mathrm{d}}(\theta) \mathbb{1}_{(\theta_{\mathrm{d}}+\xi_{\mathrm{d}}, 2\pi-\theta_{\mathrm{d}}+\xi_{\mathrm{d}})}(\theta)\, \mathrm{d}\theta\,, \tag{4.28}$$

where $\theta_{\mathrm{d}} = 2 \arcsin\left|\frac{\mathrm{d}}{1+\mathrm{d}}\right|$, $e^{\mathrm{i}\xi_{\mathrm{d}}} = \frac{1+\mathrm{d}}{1+\bar{\mathrm{d}}}$, *and*

$$\mathrm{V}_{\mathrm{d}}(\theta) = \frac{\sqrt{\sin^2\big((\theta-\xi_{\mathrm{d}})/2\big) - \sin^2(\theta_{\mathrm{d}}/2)}}{|1+\alpha_{\mathrm{d}}|\, \sin(\theta/2)}\,.$$

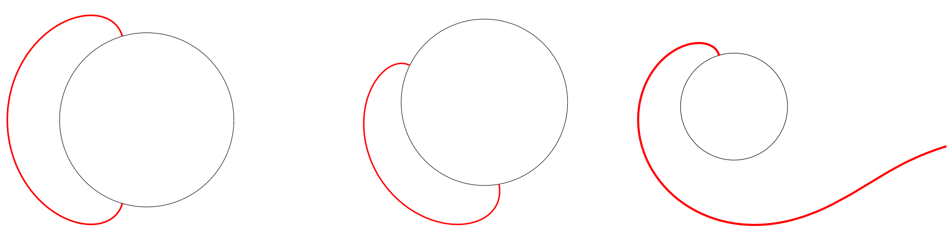


Figure 4.1. Examples of densities $\mu_{\mathtt{sp}}^{\infty}$, respectively for real, complex and purely imaginary d.

To prove this convergence we use the parametrization of measures by their modified Verblunsky coefficients and the following lemma, whose proof is postponed until the end of the section.

Lemma 4.20. *For every fixed $k > 0$, as $n \to \infty$,*

$$\gamma_k^{(n)} \xrightarrow{\mathrm{P}} -\frac{\mathrm{d}}{1+\bar{\mathrm{d}}} \tag{4.29}$$

and consequently

$$\alpha_k^{(n)} \xrightarrow{\mathrm{P}} \alpha_{\mathrm{d}} e^{-\mathrm{i}(k+1)\xi_{\mathrm{d}}} \quad , \quad \alpha_{\mathrm{d}} = -\frac{\bar{\mathrm{d}}}{1+\bar{\mathrm{d}}}\ , \ e^{\mathrm{i}\xi_{\mathrm{d}}} = \frac{1+\mathrm{d}}{1+\bar{\mathrm{d}}}\,. \tag{4.30}$$

Proof of Theorem 4.19. The measure $\mu_{\mathtt{sp}}^{(n)}$ is characterized by its sequence of Verblunsky coefficients $(\alpha_k^{(n)})_{0\leqslant k\leqslant n}$ or by the sequence of its deformed Verblunsky coefficients $(\gamma_k^{(n)})_{0\leqslant k\leqslant n}$. The moments $m_k(\nu)=\int e^{\mathrm{i}k\theta}d\nu(\theta)$ for $k\geqslant 1$ of a probability measure ν on the unit circle are related to the Verblunsky coefficients $(\alpha_k(\nu))_{k\geqslant 0}$ in a continuous way : $m_j(\nu)$ is a continuous function of $(\alpha_0,\dots,\alpha_{j-1})$. The convergence of the Verblunsky coefficients in Lemma 4.20 ensures the convergence of moments of the sequence $(\mu_{\mathtt{sp}}^{(n)})_n$, hence its weak convergence. It remains to identify the limit distribution.

Let $\alpha\in\mathbb{D}$ and ν be the measure on $\partial\mathbb{D}$ with all Verblunsky coefficients equal to α. It is known from [122] p.87 (or [123] formula (11.3.20)) that if $\theta=2\arcsin|\alpha|$ and ξ is defined by $e^{\mathrm{i}\xi}=\frac{1+\alpha}{1+\overline{\alpha}}$, then ν has an absolutely continuous part $w(\varphi)d\varphi$ supported by $(\theta,2\pi-\theta)$ with

$$w(\varphi)=\frac{\sqrt{\sin^2(\varphi/2)-\sin^2(\theta/2)}}{|1+\alpha|\,\sin((\varphi+\xi)/2)},$$

and that it has no additional Dirac mass if $|\alpha+\frac12|\leqslant\frac12$. Here, taking $\alpha=\alpha_{\mathrm{d}}$ we see that

$$\alpha_{\mathrm{d}}+\frac12=\frac{1-\overline{\mathrm{d}}}{2(1+\overline{\mathrm{d}})}\,,$$

so that the above condition is fulfilled if and only if $\mathfrak{Re}\,\mathrm{d}\geqslant 0$, which is the case. When $\alpha=\alpha_{\mathrm{d}}$, we set $\theta=\theta_{\mathrm{d}},\xi=\xi_{\mathrm{d}}$ and $\nu=\nu_{\mathrm{d}}$, $w=w_{\mathrm{d}}$. The orthogonal polynomials are known as *Geronimus polynomials.*

It is known (see [123] p.960) that if $(\alpha_k)_{k\geqslant 0}$ is the sequence of Verblunsky coefficients of some measure μ, then the coefficients $(e^{-\mathrm{i}(k+1)\xi_{\mathrm{d}}}\alpha_k)_{k\geqslant 0}$ are associated with μ rotated by ξ_{d}. Consequently,

$$d\mu_{\mathtt{sp}}^{\infty}(\theta)=d\nu_{\mathrm{d}}(\theta-\xi_{\mathrm{d}})\,,$$

which is precisely (4.28). □

Proof of Lemma 4.20. For $\gamma_k^{(n)}$ we use the Mellin transform

$$\mathbb{E}(1-\gamma_k^{(n)})^s=\frac{\Gamma\big(\beta'(n-k-1)+\delta+\overline{\delta}+s\big)\Gamma\big(\beta'(n-k-1)+\overline{\delta}\big)}{\Gamma\big(\beta'(n-k-1)+\delta+\overline{\delta}\big)\Gamma\big(\beta'(n-k-1)+\overline{\delta}+s\big)},$$

(this follows immediately from (4.24)). Since for fixed $z\in\mathbb{C}$

$$\lim_{n\to\infty}\frac{\Gamma(n+z)}{\Gamma(n)n^z}=1,$$

we get, for fixed s (and k)

$$\lim_{n\to\infty}\mathbb{E}(1-\gamma_k^{(n)})^s=\left(\frac{1+\overline{\mathrm{d}}+\mathrm{d}}{1+\overline{\mathrm{d}}}\right)^s,$$

which implies that

$$1-\gamma_k^{(n)}\xrightarrow{\text{law}}\frac{1+\overline{\mathrm{d}}+\mathrm{d}}{1+\overline{\mathrm{d}}},$$

and this is equivalent to (4.29). The statement (4.30) is a direct consequence of (4.29) and (4.14). □

Remark. The convergences in probability in the above lemma actually imply convergence in L^p, for all $p>0$ because all variables are bounded by 1.

Moreover, as expected, the equilibrium measure for $\mathrm{d}=0$ is the uniform one, and if $\delta(n)$ is real and $n=o(\delta(n))$, $\mu_{\mathtt{sp}}^{(n)}$ weakly converges in probability to the Dirac measure at point -1.

5.2. ESD : convergence and Large Deviations

In a first part, we state the convergence of the ESD. To prove this convergence, instead of the classical way (parametrization of measures by the set of their moments) we use Theorem 4.19 and prove that the two sequences $(\mu_{\mathtt{sp}}^{(n)})$ and $(\mu_{\mathtt{esd}}^{(n)})$ are contiguous. In a second part, we use the explicit form of the density of the eigenvalues, as in the classical models, to prove a Large Deviation Principle.

Theorem 4.21. *The sequence of empirical spectral distributions $(\mu_{\mathtt{esd}}^{(n)})_n$ converges weakly in probability towards $\mu_{\mathtt{sp}}^{\infty}$ given in (4.28).*

Proof. Let us prove the contiguity of the two sequences of measures $(\mu_{\mathtt{sp}}^{(n)})$ and $(\mu_{\mathtt{esd}}^{(n)})$. We have

$$\sup_\theta |\mu_{\mathtt{sp}}^{(n)}((-\pi,\theta)) - \mu_{\mathtt{esd}}^{(n)}((-\pi,\theta))| = \max_k \Big| \sum_{j=1}^k \pi_k^{(n)} - \frac{k}{n} \Big|$$

We showed in Theorem 4.15 that the vector $(\pi_1^{(n)}, \dots, \pi_n^{(n)})$ follows the $\mathrm{Dir}_n(\beta')$ distribution. It entails that the variable $\sum_{j=1}^k \pi_k^{(n)}$ is beta distributed with parameters $\beta' k$ and $\beta'(n-k)$. Its mean is k/n and its fourth central moment is proportional to k^2/n^4. By the Markov inequality and the union bound, we conclude that

$$\mathbb{P}\Big(\max_k \Big| \sum_{j=1}^k \pi_k^{(n)} - \frac{k}{n} \Big| > \delta \Big) = \mathrm{O}\left(\frac{1}{n}\right),$$

so that the sequence $(\mu_{\mathtt{esd}}^{(n)})_n$ converges weakly in probability to the same limit as $(\mu_{\mathtt{sp}}^{(n)})_n$. $\square$

Our large deviations result follows the way initiated by the pioneer paper of Ben Arous and Guionnet [8] and continued by Hiai and Petz ([68] and [69]).

Recall the definition of a large deviation principle [39]. We say that a sequence (P_n) of probability measures on a measurable Hausdorff space $(\mathcal{X}, \mathrm{B}(\mathcal{X}))$ satisfies the LDP at scale s_n (with $s_n \to \infty$), if there exists a lower semicontinous function $\mathrm{I} : \mathcal{X} \to [0,\infty]$ such that

$$\liminf \frac{1}{s_n} \log \mathrm{P}_n(\mathrm{G}) \geqslant -\inf\{\mathrm{I}(x); x \in \mathrm{G}\}$$
$$\limsup \frac{1}{s_n} \log \mathrm{P}_n(\mathrm{F}) \leqslant -\inf\{\mathrm{I}(x); x \in \mathrm{F}\}$$

for every open set $\mathrm{G} \subset \mathcal{X}$ and every closed set $\mathrm{F} \subset \mathcal{X}$. The rate function I is called good if its level sets are compact. More generally, a sequence of $\mathcal{X}$-valued random variables is said to satisfy the LDP if their distributions satisfy the LDP.

We work with the set $\mathcal{M}_1([0,2\pi))$ of probability measures on the torus

$$\Sigma(\mu) = \iint \log |e^{\mathrm{i}\theta} - e^{\mathrm{i}\theta'}| \mathrm{d}\mu(\theta)\mathrm{d}\mu(\theta').$$

We define also the potential

$$\mathrm{Q}(e^{\mathrm{i}\theta}) = \mathrm{Q}(\theta) = -(\mathfrak{Re}\,\mathrm{d}) \log\big(2 \sin\frac{\theta}{2}\big) - (\mathfrak{Im}\,\mathrm{d}) \frac{\theta - \pi}{2}, \quad (\theta \in (0,2\pi)).$$

Theorem 4.22. *For $n \in \mathbb{N}$, consider the distribution*

$$\frac{1}{\mathcal{Z}(n)} \prod_{k=1}^n (1 - e^{\mathrm{i}\theta_k})^{\overline{\delta(n)}} (1 - e^{-\mathrm{i}\theta_k})^{\delta(n)} \prod_{j<k} |e^{\mathrm{i}\theta_j} - e^{\mathrm{i}\theta_k}|^\beta \mathrm{d}\theta_1 \dots \mathrm{d}\theta_n \tag{4.31}$$

where $\mathrm{d}\theta$ is the normalized Lebesgue measure on the $\partial\mathbb{D}$ and $\delta(n)/n \to \beta'\mathrm{d}$ with $\mathfrak{Re}\,\mathrm{d} > 0$. Then *a. We have*

$$\frac{1}{n^2\beta'}\log \mathcal{Z}(n) \to \mathrm{B}(\mathrm{d})$$

where

$$\mathrm{B}(\mathrm{d}) = \int_0^1 x \log \frac{x(x+\mathfrak{Re}\,\mathrm{d})}{|x+\mathrm{d}|^2}\,\mathrm{d}x\,.$$

b. When $(\theta_1, \dots \theta_n)$ is distributed as (4.31), the sequence of empirical measures

$$\mu^{(n)}_{\mathtt{esd}} = \frac{\delta_{\theta_1} + \cdots + \delta_{\theta_n}}{n}$$

satisfies the LDP at scale $\beta' n^2$ with good rate function defined for $\mu \in \mathcal{M}_1(\partial\mathbb{D})$ by

$$\mathrm{I}(\mu) = -\Sigma(\mu) + 2\int \mathrm{Q}(\theta)\mathrm{d}\mu(\theta) + \mathrm{B}(\mathrm{d})\,.$$

c. The rate function vanishes only at $\mu = \mu^{\infty}_{\mathtt{sp}}$.

Proof. We assume for simplicity that $\delta(n) = \beta' n\mathrm{d}$.

(1) An exact expression of $\mathcal{Z}(n)$ is obtained using the following lemma, whose proof is postponed at the end of this subsection.

Lemma 4.23. *The integral*

$$\mathcal{Z}_{s,t}(n) = \int_{(\partial\mathbb{D})^n} \prod_{k=1}^n (1-e^{\mathrm{i}\theta_k})^s(1-e^{-\mathrm{i}\theta_k})^t \prod_{j,k} |e^{\mathrm{i}\theta_j} - e^{\mathrm{i}\theta_k}|^\beta \mathrm{d}\theta_1 \dots \mathrm{d}\theta_n$$

is equal to

$$\mathcal{Z}_{s,t}(n) = \frac{\Gamma(\beta' n+1)}{\big(\Gamma(\beta'+1)\big)^n} \prod_0^{n-1} \frac{\Gamma(\beta' j+1)\Gamma(\beta' j+1+s+t)}{\Gamma(\beta' j+1+s)\Gamma(\beta' j+1+t)}\,. \tag{4.32}$$

We have $\mathcal{Z}(n) = \mathcal{Z}_{\overline{\delta(n)},\delta(n)}(n)$ and then

$$\begin{aligned}\log \mathcal{Z}(n) &= \log\Gamma(\beta' n+1) - n\log\Gamma(\beta'+1) + \sum_{j=0}^{n-1}\log\Gamma(\beta' j+1)\\ &+ \sum_{j=0}^{n-1}\left(\log\Gamma(\beta' j+1+2\mathfrak{Re}\,\mathrm{d}n) - 2\mathfrak{Re}\,\log\Gamma(\beta' j+1+\mathrm{d}n)\right).\end{aligned}$$

From the Binet formula (Abramowitz and Stegun [1] or Erdélyi et al. [45] p.21), we have for $\mathfrak{Re}\,x > 0$

$$\log\Gamma(x) = \left(x-\frac{1}{2}\right)\log x - x + \frac{1}{2}\log(2\pi) + \int_0^\infty f(s)e^{-sx}\,\mathrm{d}s\,. \tag{4.33}$$

where the function f is defined by

$$f(s) = \left(\frac{1}{2} - \frac{1}{s} + \frac{1}{e^s-1}\right)\frac{1}{s} = 2\sum_{k=1}^\infty \frac{1}{s^2+4\pi^2k^2}\,,$$

and satisfies for every $s \geqslant 0$

$$0 < f(s) \leqslant f(0) = 1/12\,,\quad 0 < \left(sf(s) + \frac{1}{2}\right) < 1\,.$$

Using (4.33), a straightforward study of Riemann sums gives

$$\frac{1}{\beta' n^2}\log \mathcal{Z}(n) \to \int_0^1 x\log\frac{x(x+\mathfrak{Re}\,\mathrm{d})}{|x+\mathrm{d}|^2}\,\mathrm{d}x.$$

(2) The proof is based on the explicit form of the joint eigenvalue density. Denote $\mathbb{P}^{(n)}$ the distribution of $\mu^{(n)}_{\mathsf{esd}}$. We follow the lines of Hiai-Petz [68] (since we work on $\partial\mathbb{D}$) and [69] (since the potential is not continuous in $\theta = 0$). Let us summarize the main steps. The LDP is equivalent to the following two inequalities for every $\mu \in \mathcal{M}(\partial\mathbb{D})$:

$$\inf_{\mathrm{G}}\left(\limsup_{n}\frac{1}{n^2}\log\mathbb{P}^{(n)}(\mathrm{G})\right) \leqslant -\int\!\!\int \mathrm{F}(\theta,\theta')\mathrm{d}\mu(\theta)\mathrm{d}\mu(\theta') - \mathrm{B}(\mathrm{d}) \tag{4.34}$$

$$\inf_{\mathrm{G}}\left(\liminf_{n}\frac{1}{n^2}\log\mathbb{P}^{(n)}(\mathrm{G})\right) \geqslant -\int\!\!\int \mathrm{F}(\theta,\theta')\mathrm{d}\mu(\theta)\mathrm{d}\mu(\theta') - \mathrm{B}(\mathrm{d}) \tag{4.35}$$

where G runs over a neighborhood base of μ.

Let for $\theta,\theta' \in [0,2\pi)$

$$\mathrm{F}(\theta,\theta') = -\log|e^{\mathrm{i}\theta} - e^{\mathrm{i}\theta'}| + \frac{1}{2}(\mathrm{Q}(\theta)+\mathrm{Q}(\theta'))\,.$$

and for $\mathrm{R} > 0$, $\mathrm{F}_{\mathrm{R}} = \min(\mathrm{F},\mathrm{R})$. As in [69] (Proof of (2.5)) we have easily

$$\limsup_{n}\frac{1}{n^2}\log\mathbb{P}^{(n)}(\mathrm{G}) \leqslant -\inf_{\mu'\in\mathrm{G}}\int\!\!\int \mathrm{F}_{\mathrm{R}}(\theta,\theta')\mathrm{d}\mu(\theta)\mathrm{d}\mu(\theta') - \mathrm{B}(\mathrm{d})\,.$$

Now the function F_{R} is continuous, bounded above by R and below by $-(1+\mathfrak{Re}\,\mathrm{d})\log 2 - |\mathfrak{Im}\,\mathrm{d}|\pi/2)$, which implies the continuity of $\nu \mapsto \int\!\int \mathrm{F}_{\mathrm{R}}(\theta,\theta')\mathrm{d}\nu(\theta)\mathrm{d}\nu(\theta')$ hence the inequality

$$\inf_{\mathrm{G}}\left(\limsup_{n}\frac{1}{n^2}\log\mathbb{P}^{(n)}(\mathrm{G})\right) \leqslant -\inf_{\nu\in\mathrm{G}}\int\!\!\int \mathrm{F}_{\mathrm{R}}(\theta,\theta')\mathrm{d}\nu(\theta)\mathrm{d}\nu(\theta') - \mathrm{B}(\mathrm{d})\,,$$

Taking the infimum in R yields (4.34).

For (4.35), we follow Hiai-Petz [69] again. The main change is that the only singularity is in 1. We can exclude the case where μ has an atom at 0, for then $\int\!\int \mathrm{F}(\theta,\theta')\mathrm{d}\mu(\theta)\mathrm{d}\mu(\theta')$ would be infinite. Otherwise, it is easy to see that we may assume that μ is supported in $[\theta_0, 2\pi-\theta_0]$ for some $\theta_0 \in [0,\pi)$, by conditioning. Then to make a regularization, we take for $\varepsilon \in (0,\theta_0)$ a C^∞ probability density φ_ε supported in $[-\varepsilon,\varepsilon]$ and we set

$$g_\varepsilon(\theta)\mathrm{d}\theta = \left(\int \varphi_\varepsilon(\theta-s)\mathrm{d}\mu(s)\right)\mathrm{d}\theta = \int \varphi_\varepsilon(s)\mathrm{d}\mu_s(\theta)\mathrm{d}s$$

where $\int f(\theta)\mathrm{d}\mu_s(\theta) = \int f(\theta-s)\mathrm{d}\mu(\theta)$. The measure $g_\varepsilon(\theta)\mathrm{d}\theta$ is then a mixture of the family of measures $(\mu_s, s\in[0,2\pi))$, with the mixing measure $\varphi_\varepsilon(s)\mathrm{d}s$. So by the concavity of Σ we have

$$\Sigma(g_\varepsilon(\theta)\mathrm{d}\theta) \geqslant \int \Sigma(\mu_s)\varphi_\varepsilon(s)\mathrm{d}s$$

but

$$\begin{aligned}
\Sigma(\mu_s) &= \int\!\!\int \log|e^{\mathrm{i}\theta} - e^{\mathrm{i}\theta'}|\mathrm{d}\mu_s(\theta)\mathrm{d}\mu_s(\theta')\\
&= \int\!\!\int \log|e^{\mathrm{i}(\theta-s)} - e^{\mathrm{i}(\theta'-s)}|\mathrm{d}\mu(\theta)\mathrm{d}\mu(\theta')\\
&= \int\!\!\int \log|e^{\mathrm{i}\theta} - e^{\mathrm{i}\theta'}|\mathrm{d}\mu(\theta)\mathrm{d}\mu(\theta') = \Sigma(\mu)
\end{aligned}$$

and then
$$\Sigma(g_\varepsilon(\theta)\mathrm{d}\theta) \geqslant \Sigma(\mu)\,.$$
Moreover
$$\lim_{\varepsilon} \int \mathrm{Q}(\theta) g_\varepsilon(\theta)\mathrm{d}\theta = \int \mathrm{Q}(\theta)\mathrm{d}\mu(\theta)$$
since Q is continous in a neighborhood of the support of μ. So if G is an open set containing μ, for ε small enough, it contains an open set G_ε containing $g_\varepsilon(\theta)\mathrm{d}\theta$. Assuming for a moment that we have proved (4.35) for μ_ε, we get for ε fixed

$$\begin{aligned}
\inf_{\mathrm{G}\ni\mu} \liminf_n \frac{1}{n^2}\log \mathbb{P}^{(n)}(\mathrm{G}) &\geqslant \inf_{\widetilde{\mathrm{G}}\ni\mu_\varepsilon} \liminf_n \frac{1}{n^2}\log \mathbb{P}^{(n)}(\widetilde{\mathrm{G}}) \\
&\geqslant -\int\int \mathrm{F}(\theta,\theta')\mathrm{d}\mu_\varepsilon(\theta)\mathrm{d}\mu_\varepsilon(\theta') \\
&\geqslant -\Sigma(\mu) + \int \mathrm{Q}(\theta)\mathrm{d}\mu_\varepsilon(\theta)
\end{aligned}$$

and taking the limit in ε does the job. Moreover, as in [69] we may assume that the density is bounded below and above. Eventually the proof of (4.35) for $\mu = \mu_\varepsilon$ is exactly the same as in [69] (proof of 2.6) and [68] (proof of 3.4). The uniqueness of the minimizer is a direct consequence of the strict convexity of I which comes from the strict concavity of Σ.

We are not able to give a self contained proof of the explicit value of the minimizer. But on the one hand in Theorem 4.21 we proved that $\mu^{(n)}_{\mathtt{esd}}$ converges weakly in probability to $\mu^{\infty}_{\mathtt{sp}}$, and on the other hand the LDP and the uniqueness of the minimizer imply that $\mu^{(n)}_{\mathtt{esd}}$ converges weakly in probability to this minimizer. This ends the proof. □

Proof of Lemma 4.23. . We have
$$\frac{\mathcal{Z}_{s,t}(n)}{\mathcal{Z}_{0,0}(n)} = \mathbb{E}\big(\det(\mathrm{Id}-u)^s \det(\mathrm{Id}-\bar{u})^t\big)$$
where the mean is taken with respect to the $\mathrm{CJ}^{(n)}_{0,\beta}$ distribution. We know also that under this distribution $\det(\mathrm{Id}-u)$ is distributed as the product of independent variables $1-\bar{\alpha}_k$, where α_k is $\nu_{\beta(n-k-1)+1}$ distributed. Hence
$$\mathbb{E}\big(\det(\mathrm{Id}-u)^s \det(\mathrm{Id}-\bar{u})^t\big) = \prod_{j=0}^{n-1} \mathbb{E}\left((1-\bar{\alpha}_j)^s(1-\alpha_j)^t\right)$$
From (4.24) we get
$$\frac{\mathcal{Z}_{s,t}(n)}{\mathcal{Z}_{0,0}(n)} = \prod_0^{n-1} \frac{\Gamma(\beta' j+1)\Gamma(\beta' j+1+s+t)}{\Gamma(\beta' j+1+s)\Gamma(\beta' j+1+t)}$$
Besides, Lemma 4.4 in [84] gives $\mathcal{Z}_{0,0}(n) = \frac{\Gamma(\beta' n+1)}{\Gamma(\beta'+1)^n}$, which concludes the proof. □

Chapter 5

Derivatives, traces and independence

The first section of this chapter is extracted from *Conditional Haar measures on classical compact groups* [17], Annals of Probability vol 37, Number 4, 1566-1586, (2009). The next sections only appear in this thesis.

This chapter gathers together distinct results about random matrices, whose proofs make use of distinct techniques. Their common point is that independence appears where it was not a priori expected, with respect to the Haar measure.

The first section explains why the first non-zero derivative of the characteristic polynomial can be decomposed as a product of independent random variables, with respect to the Haar measure conditioned to the existence of a stable subspace. This is linked to number theoretical problems thanks to works by Nina Snaith, and gives easy proofs for asymptotic densities of such characteristic polynomials.

The second section concentrates on limit theorems for the trace. Let u have eigenvalues distributed from the circular Jacobi ensemble

$$c_{n,\beta,\delta}\prod_{1\leqslant k<l\leqslant n}|e^{\mathrm{i}\theta_k}-e^{\mathrm{i}\theta_l}|^{\beta}\prod_{j=1}^{n}(1-e^{-\mathrm{i}\theta_j})^{\delta}(1-e^{\mathrm{i}\theta_j})^{\overline{\delta}}. \tag{5.1}$$

Thanks to a result by Widom [139], we show that $\mathrm{Tr}(u),\dots,\mathrm{Tr}(u^{\ell})$ converge to independent normal variables for $\beta=2$, $\mathfrak{Re}(\delta)>-1/2$, and it directly follows from works by Johansson [74] that the same is true if $\delta=0$ for any $\beta>0$. These convergences in law are actually linked to the *almost sure* convergence of a remarkable martingale to a Gaussian limit.

In the third section we consider the joint convergence of the characteristic polynomial and the trace for the circular ensembles : the couple converges to a Gaussian vector with independent entries. The proof relies on the deformed Verblunsky coefficients

The fourth section explores limit theorems for functionals on the special unitary group $\mathrm{SU}(n)$: the Mellin transform of the characteristic polynomial, weak convergence of the trace and central limit theorems are considered.

Finally, the fifth section extends a result by E. Rains : he proved that if $u\sim\mu_{\mathrm{U}(n)}$, then the eigenvalues of u^m, for $m\geqslant n$, are independent and uniform. We give similar results for densities (5.1) with $\beta/2$ and δ integers.

1. Scission in law for the derivatives

Let $\mathrm{Z}_{\mathrm{SO}}^{(2p)}$ be the $(2p)^{th}$ derivative of the characteristic polynomial at point 1, for the Haar measure on $\mathrm{SO}(n+2p)$ conditioned to have $2p$ eigenvalues equal to 1. In her study of moments of L-functions associated to elliptic curves, N. Snaith ([128], [129]) explains that the moments of $\mathrm{Z}_{\mathrm{SO}}^{(2p)}$ are relevant : she conjectures that $\mathrm{Z}_{\mathrm{SO}}^{(2p)}$ is related to averages on L-functions moments and therefore, via the Birch and Swinnerton-Dyer conjecture, on the rank of elliptic curves. For the number theoretic applications of these derivatives, see [99], [128] and [129].

Relying on the Selberg integral, she computed the asymptotics of the density of $Z_{\mathrm{SO}}^{(2p)}$ as $\varepsilon \to 0$, finding

$$\mathbb{P}(Z_{\mathrm{SO}}^{(2p)} < \varepsilon) \underset{\varepsilon \to 0}{\sim} c_{n,p} \varepsilon^{2p+\frac{1}{2}}, \tag{5.2}$$

for an explicit constant $c_{n,p}$. Similar results (and also central limit theorems) are given in this section for any Jacobi and Jacobi circular ensemble.

As explained in the previous chapters, $\det(\mathrm{Id} - u)$ is equal in law to a product of n independent random variables, for the Haar measure on $\mathrm{U}(n)$ or $\mathrm{USp}(2n)$, and $2n$ independent random variables, for the Haar measure on $\mathrm{SO}(2n)$. We can generalize these results to the Haar measures conditioned to the existence of a stable subspace with given dimension. We first focus on the unitary group. Consider the conditional measure on $\mathrm{U}(n+p)$ such that $\theta_{n+1} = \dots = \theta_{n+p} = 0$,

$$\prod_{1 \leqslant k < l \leqslant n} |e^{\mathrm{i}\theta_k} - e^{\mathrm{i}\theta_l}|^2 \prod_{j=1}^{n} |1 - e^{\mathrm{i}\theta_j}|^{2p} \mathrm{d}\theta_1 \dots \mathrm{d}\theta_n.$$

Then the p^{th} derivative of the characteristic polynomial at 1 is

$$Z_{\mathrm{U}}^{(p)} = p! \prod_{k=1}^{n} \left(1 - e^{\mathrm{i}\theta_k}\right).$$

Hence, the following theorem is an easy consequence of Corollary 4.18.

Theorem 5.1. *With the previous notations,*

$$\frac{Z_{\mathrm{U}}^{(p)}}{p!} \overset{\text{law}}{=} \prod_{l=1}^{n} (1 - X_\ell),$$

where the X_ℓ's are independent random variables. The distribution of X_ℓ is the $|1 - X|^{2p}$-sampling (in the sense of Definition 1.16) of a random variable $X = e^{\mathrm{i}\theta}\sqrt{B_{1,\ell-1}}$, where θ is uniform on $(-\pi, \pi)$ and independent from $B_{1,\ell-1}$, a beta variable with the indicated parameters.

The analogue on $\mathrm{SO}(2n)$ and $\mathrm{USp}(2n)$ relies on the following results by Killip and Nenciu [84]. Lemma 5.2 and Proposition 5.3 in [84] immediately imply that under the probability measure

$$\mathrm{cst} |\Delta(x_1, \dots, x_n)|^\beta \prod_{j=1}^{n} (2 - x_j)^a (2 + x_j)^b \mathrm{d}x_1 \dots \mathrm{d}x_n$$

on $(-2, 2)^n$, the following identity in law holds (the x_k's being the eigenvalues of a matrix u) :

$$(\det(2\mathrm{Id} - u), \det(2\mathrm{Id} + u)) \overset{\text{law}}{=} \left(2 \prod_{k=0}^{2n-2} (1 - \alpha_k), 2 \prod_{k=0}^{2n-2} (1 + (-1)^k \alpha_k) \right), \tag{5.3}$$

with the α_k's independent with density $f_{s(k),t(k)}$ ($f_{s,t}$ is defined below) on $(-1, 1)$ with

$$\begin{cases} s(k) = \frac{2n-k-2}{4}\beta + a + 1, & t(k) = \frac{2n-k-2}{4}\beta + b + 1 \quad \text{if } k \text{ is even} \\ s(k) = \frac{2n-k-3}{4}\beta + a + b + 2, & t(k) = \frac{2n-k-1}{4}\beta \quad \text{if } k \text{ is odd} \end{cases}.$$

Definition 5.2. *The density $f_{s,t}$ on $(-1, 1)$ is*

$$f_{s,t}(x) = \frac{2^{1-s-t}\Gamma(s+t)}{\Gamma(s)\Gamma(t)} (1-x)^{s-1} (1+x)^{t-1},$$

Moreover, for X with density $f_{s,t}$, $\mathbb{E}(X) = \frac{t-s}{t+s}$, $\mathbb{E}(X^2) = \frac{(t-s)^2+(t+s)}{(t+s)(t+s+1)}$.

The $2n$ eigenvalues of $u \in \mathrm{SO}(2n)$ or $\mathrm{USp}(2n)$ are pairwise conjugated, and noted $\left(e^{\pm \mathrm{i}\theta_1}, \dots, e^{\pm \mathrm{i}\theta_n}\right)$. The Weyl integration formula states that the eigenvalues statistics are

$$\mathrm{cst} \prod_{1 \leqslant k < \ell \leqslant n} (\cos\theta_k - \cos\theta_\ell)^2 \mathrm{d}\theta_1 \dots \mathrm{d}\theta_n$$

on $\mathrm{SO}(2n)$. On the symplectic group $\mathrm{USp}(2n)$, these statistics are

$$\mathrm{cst} \prod_{1 \leqslant k < \ell \leqslant n} (\cos\theta_k - \cos\theta_\ell)^2 \prod_{i=1}^{n} (1 - \cos\theta_i)(1 + \cos\theta_i) \mathrm{d}\theta_1 \dots \mathrm{d}\theta_n.$$

Hence, the change of variables

$$x_j = 2\cos\theta_j$$

implies the following links between $\mathrm{SO}(2n)$, $\mathrm{USp}(2n)$ and the Jacobi ensemble.

- On $\mathrm{SO}(2n + 2p^+ + 2p^-)$, endowed with its Haar measure conditioned to have $2p^+$ eigenvalues at 1 and $2p^-$ at -1, the distribution of $(x_1, \dots, x_n)$ is the Jacobi ensemble (4.4) with parameters $\beta = 2$, $a = 2p^+ - \frac{1}{2}$, $b = 2p^- - \frac{1}{2}$.
- On $\mathrm{USp}(2n + 2p^{(+)} + 2p^-)$, endowed with its Haar measure conditioned to have $2p^+$ eigenvalues at 1 and $2p^-$ at -1, the distribution of $(x_1, \dots, x_n)$ is the Jacobi ensemble (4.4) with parameters $\beta = 2$, $a = 2p^+ + \frac{1}{2}$, $b = 2p^- + \frac{1}{2}$.

Moreover, for the above groups $\mathcal{G} = \mathrm{SO}(2n + 2p^+ + 2p^-)$ or $\mathrm{USp}(2n + 2p^+ + 2p^-)$ with $2p^+$ eigenvalues at 1 and $2p^-$ at -1, $\mathrm{Z}_{\mathcal{G}}^{(2p^+)}$ denotes the $2p^{+th}$ derivative of the characteristic polynomial at point 1 and $\mathrm{Z}_{\mathcal{G}}^{(2p^-)}$ the $2p^{-th}$ derivative of the characteristic polynomial at point -1 :

$$\begin{cases} \frac{\mathrm{Z}_{\mathcal{G}}^{(2p^+)}}{(2p^+)!2^{p^-}} & = \prod_{k=1}^{n} (1 - e^{\mathrm{i}\theta_k})(1 - e^{-\mathrm{i}\theta_k}) & = \prod_{k=1}^{n} (2 - x_k) \\ \frac{\mathrm{Z}_{\mathcal{G}}^{(2p^-)}}{(2p^-)!2^{p^+}} & = \prod_{k=1}^{n} (-1 - e^{\mathrm{i}\theta_k})(-1 - e^{-\mathrm{i}\theta_k}) & = \prod_{k=1}^{n} (2 + x_k) \end{cases}.$$

Combining this with formula (5.3) leads to the following analogue of Theorem 5.1.

Theorem 5.3. *With the above notations and definition of conditional spectral Haar measures on* $\mathrm{SO}(2n + 2p^+ + 2p^-)$,

$$\left(\frac{\mathrm{Z}_{\mathrm{SO}}^{(2p^+)}}{(2p^+)!2^{p^-}}, \frac{\mathrm{Z}_{\mathrm{SO}}^{(2p^-)}}{(2p^-)!2^{p^+}}\right) \overset{\mathrm{law}}{=} \left(2\prod_{k=0}^{2n-2} (1 - \mathrm{X}_k), 2\prod_{k=0}^{2n-2} \left(1 + (-1)^k \mathrm{X}_k\right)\right)$$

where the X_k*'s are independent and* X_k *with density* $f_{s(k),t(k)}$ *on* $(-1, 1)$ *given by Definition 5.2 with parameters*

$$\begin{cases} s(k) = \frac{2n-k-1}{2} + 2p^+, & t(k) = \frac{2n-k-1}{2} + 2p^- \quad \text{if } k \text{ is even} \\ s(k) = \frac{2n-k-1}{2} + 2p^+ + 2p^-, & t(k) = \frac{2n-k-1}{2} \quad \text{if } k \text{ is odd} \end{cases}.$$

The same result holds for the joint law of $\mathrm{Z}_{\mathrm{USp}}^{(2p^+)}$ *and* $\mathrm{Z}_{\mathrm{USp}}^{(2p^-)}$*, but with the parameters*

$$\begin{cases} s(k) = \frac{2n-k+1}{2} + 2p^+, & t(k) = \frac{2n-k+1}{2} + 2p^- \quad \text{if } k \text{ is even} \\ s(k) = \frac{2n-k+3}{2} + 2p^+ + 2p^-, & t(k) = \frac{2n-k-1}{2} \quad \text{if } k \text{ is odd} \end{cases}.$$

1.1. Central limit theorems.

From (5.3), $\log\det(2\mathrm{Id} - u)$ and $\log\det(2\mathrm{Id} + u)$ (respectively abbreviated as $\log\det^{(+)}$ and $\log\det^{(-)}$) can be jointly decomposed as sums of independent random variables. Consequently, the classical central limit theorems in probability theory imply the following result. Note that, despite the dependence appearing from (5.3), $\log\det^{(+)}$ and $\log\det^{(-)}$ are independent in the limit.

Theorem 5.4. *Let u have spectral measure the Jacobi ensemble (4.4), with $\beta > 0$, $a, b \geqslant 0$. Then*

$$\left(\frac{\log\det^{(+)} + \left(\frac{1}{2} - \frac{2a+1}{\beta}\right)\log n}{\sqrt{\frac{2}{\beta}\log n}}, \frac{\log\det^{(-)} + \left(\frac{1}{2} - \frac{2b+1}{\beta}\right)\log n}{\sqrt{\frac{2}{\beta}\log n}}\right) \xrightarrow{\text{law}} (\mathcal{N}_1, \mathcal{N}_2)$$

as $n \to \infty$, with $\mathcal{N}_1$ and $\mathcal{N}_2$ independent standard normal variables.

Proof. We keep the notations from (5.3) :

$$\begin{cases} \log\det^{(+)} & = & \log 2 + \sum_{\text{odd } k}\log(1-\alpha_k) + \sum_{\text{even } k}\log(1-\alpha_k) \\ \log\det^{(-)} & = & \log 2 + \sum_{\text{odd } k}\log(1-\alpha_k) + \sum_{\text{even } k}\log(1+\alpha_k) \end{cases}$$

with $0 \leqslant k \leqslant 2n-2$. Let us first consider $\mathrm{X}_n = \sum_{\text{odd } k}\log(1-\alpha_k)$. From (5.3), $\mathrm{X}_n \stackrel{\text{law}}{=} \sum_{k=1}^{n-1}\log(1-x_k)$ with independent x_k's, x_k having density $f_{(k-1)\frac{\beta}{2}+a+b+2,k\frac{\beta}{2}}$. In particular, $\mathbb{E}(x_k) = \frac{-a-b-2+\beta/2}{\beta k} + \mathrm{O}\left(\frac{1}{k^2}\right)$ and $\mathbb{E}(x_k^2) = \frac{1}{\beta k} + \mathrm{O}\left(\frac{1}{k^2}\right)$. From the Taylor expansion of $\log(1-x)$

$$\mathrm{X}_n \stackrel{\text{law}}{=} \underbrace{\sum_{k=1}^{n-1}\left(-x_k - \frac{x_k^2}{2}\right)}_{\mathrm{X}_n^{(1)}} - \underbrace{\sum_{k=1}^{n-1}\sum_{\ell\geqslant 3}\frac{x_k^\ell}{\ell}}_{\mathrm{X}_n^{(2)}}.$$

Let $\mathrm{X} = \sum_{k=1}^{\infty}\sum_{\ell\geqslant 3}\frac{|x_k|^\ell}{l}$. A calculation implies $\mathbb{E}(\mathrm{X}) < \infty$, so $\mathrm{X} < \infty$ a.s. and consequently, $|\mathrm{X}_n^{(2)}|/\sqrt{\log n} \leqslant \mathrm{X}/\sqrt{\log n} \to 0$ a.s. as $n \to \infty$. Moreover,

$$\mathbb{E}\left(-x_k - \frac{x_k^2}{2}\right) = \frac{a+b+3/2-\beta/2}{\beta k} + \mathrm{O}\left(\frac{1}{k^2}\right),\ \mathrm{var}\left(-x_k - \frac{x_k^2}{2}\right) = \frac{1}{\beta k} + \mathrm{O}\left(\frac{1}{k^2}\right),$$

so the classical central limit theorem (see e.g. [108]) implies that

$$\frac{\mathrm{X}_n^{(1)} - \frac{a+b+3/2-\beta/2}{\beta}\log n}{\sqrt{\frac{1}{\beta}\log n}} \xrightarrow{\text{law}} \mathcal{N}_1$$

as $n \to \infty$, with $\mathcal{N}_1$ a standard normal random variable. Gathering the convergences for $\mathrm{X}_n^{(1)}$ and $\mathrm{X}_n^{(2)}$ gives

$$\frac{\mathrm{X}_n - \frac{a+b+3/2-\beta/2}{\beta}\log n}{\sqrt{\frac{1}{\beta}\log n}} \xrightarrow{\text{law}} \mathcal{N}_1. \tag{5.4}$$

We now concentrate on $\mathrm{Y}_n^{(+)} = \sum_{\text{even } k}\log(1-\alpha_k)$ and $\mathrm{Y}_n^{(-)} = \sum_{\text{even } k}\log(1+\alpha_k)$. From (5.3) $(\mathrm{Y}_n^{(+)}, \mathrm{Y}_n^{(-)}) \stackrel{\text{law}}{=} (\sum_1^n\log(1-y_k), \sum_1^n\log(1+y_k))$ with independent y_k's, y_k having density $f_{(k-1)\frac{\beta}{2}+a+1,(k-1)\frac{\beta}{2}+b+1}$. We now have

$$\mathbb{E}\left(\pm y_k - \frac{y_k^2}{2}\right) = \frac{\pm(b-a)-1/2}{\beta k} + \mathrm{O}\left(\frac{1}{k^2}\right), \mathrm{var}\left(\pm y_k - \frac{y_k^2}{2}\right) = \frac{1}{\beta k} + \mathrm{O}\left(\frac{1}{k^2}\right).$$

Consequently, as previously the two first terms in the Taylor expansions of $\log(1 \pm y_k)$ can be isolated to get the following central limit theorem for any real numbers $\lambda^{(+)}$ and $\lambda^{(-)}$:

$$\lambda^{(+)}\frac{\mathrm{Y}_n^{(+)} - \frac{a-b-1/2}{\beta}\log n}{\sqrt{\frac{1}{\beta}\log n}} + \lambda^{(-)}\frac{\mathrm{Y}_n^{(-)} - \frac{b-a-1/2}{\beta}\log n}{\sqrt{\frac{1}{\beta}\log n}} \xrightarrow{\text{law}} (\lambda^{(+)} - \lambda^{(-)})\mathcal{N}_2, \tag{5.5}$$

with $\mathcal{N}_2$ a standard normal variable, independent of $\mathcal{N}_1$, because the odd and even α_k's are independent. Gathering convergences (5.4) and (5.5) shows that

$$\left(\frac{\log\det^{(+)}+\left(\frac{1}{2}-\frac{2a+1}{\beta}\right)\log n}{\sqrt{\frac{2}{\beta}\log n}},\frac{\log\det^{(-)}+\left(\frac{1}{2}-\frac{2b+1}{\beta}\right)\log n}{\sqrt{\frac{2}{\beta}\log n}}\right)$$

converges in law to $\frac{1}{\sqrt{2}}(\mathcal{N}_1+\mathcal{N}_2,\mathcal{N}_1-\mathcal{N}_2)\overset{\text{law}}{=}(\mathcal{N}_1,\mathcal{N}_2)$. □

Remark. The absence of drift for $a=b=0$ in the above central limit theorem requires $\beta=2$: it always has specific properties amongst the β-ensembles.

An immediate corollary of the previous theorem concerns the derivatives of characteristic polynomials on $\mathrm{SO}(2n)$ and $\mathrm{USp}(2n)$.

Corollary 5.5. *With the notations of Theorem 5.3,*

$$\left(\frac{\log \mathrm{Z}_{\mathrm{SO}}^{(2p^+)}-(2p^+-\frac{1}{2})\log n}{\sqrt{\log n}},\frac{\log \mathrm{Z}_{\mathrm{SO}}^{(2p^-)}-(2p^--\frac{1}{2})\log n}{\sqrt{\log n}}\right)\xrightarrow{\text{law}}(\mathcal{N}_1,\mathcal{N}_2)$$

as $n\to\infty$, with $\mathcal{N}_1$ and $\mathcal{N}_2$ independent standard normal variables. The same result holds on the symplectic group conditioned to have $2p^+$ eigenvalues at 1 and $2p^-$ at -1, but with the parameters $2p^{(+)}$ and $2p^{(-)}$ replaced by $2p^{(+)}+1$ and $2p^{(-)}+1$ in the above formula.

These central limit theorems about the Jacobi ensemble on the segment have analogues for Jacobi ensembles on the unit circle. We only state it, the proof being similar to the previous one and relying on the decomposition as a product of independent random variables, Corollary 4.18. In the following the complex logarithm is defined continuously on $(0,1)$ as the value of $\log\det(\mathrm{Id}-xu)$ from $x=0$ to $x=1$.

Theorem 5.6. *Let $\beta>0$ and $\delta\in\mathbb{C}$, $\mathfrak{Re}(\delta)>-1/2$. If the eigenvalues of u_n are distributed as (4.5), then*

$$\frac{\log\det(\mathrm{Id}-u_n)-\frac{2\delta}{\beta}\log n}{\sqrt{\frac{2}{\beta}\log n}}\xrightarrow{\text{law}}\frac{1}{\sqrt{2}}\left(\mathcal{N}_1+\mathrm{i}\mathcal{N}_2\right)$$

as $n\to\infty$, with $\mathcal{N}_1$ and $\mathcal{N}_2$ independent standard normal variables. In particular, if $\mathrm{Z}_{\mathrm{U}}^{(p)}$ is the p^{th} derivative of the characteristic polynomial at 1, for the Haar measure on $\mathrm{U}(n)$ conditioned to have p eigenvalues equal to 1, as $n\to\infty$

$$\frac{\log \mathrm{Z}_{\mathrm{U}}^{(p)}-p\log n}{\sqrt{\log n}}\xrightarrow{\text{law}}\frac{1}{\sqrt{2}}\left(\mathcal{N}_1+\mathrm{i}\mathcal{N}_2\right).$$

Remark. The above technique does not provide the existence of a limit in law for $(\log \mathrm{Z}(1),\log \mathrm{Z}'(1))$ under the Haar measure $\mu_{\mathrm{U}(n)}$ as $n\to\infty$. This point remains open, for more on this topic, see [36].

1.2. Limit densities.

Let $(x_1,\dots,x_n)$ have the Jacobi distribution (4.4) on $(-2,2)^n$. The asymptotics of the density of

$$\det^{(+)}:=\prod_{k=1}^{n}(2-x_i)$$

near 0 can be easily evaluated from (5.3). Indeed, let f be a continuous function and h_n denote the density of $\det^{(+)}$ on $(0,\infty)$. With the notations of (5.3), as α_{2n-2} has law $f_{a+1,b+1}$,

$$\mathbb{E}\left(f(\det^{(+)})\right) = c\int_{-1}^{1}(1-x)^a(1+x)^b\,\mathbb{E}\left(f\left(2(1-x)\prod_{k=0}^{2n-3}(1-\alpha_k)\right)\right)\mathrm{d}x.$$

with $c = 2^{-1-a-b}\Gamma(a+b+2)/(\Gamma(a+1)\Gamma(b+1))$. The change of variable $\varepsilon = 2(1-x)\prod_{k=0}^{2n-3}(1-\alpha_k)$ therefore yields

$$h_n(\varepsilon) = c\,\mathbb{E}\left(\left(\frac{1}{2\prod_{k=0}^{2n-3}(1-\alpha_k)}\right)^{a+1}\left(2-\frac{\varepsilon}{2\prod_{k=0}^{2n-3}(1-\alpha_k)}\right)^b\right)\ \varepsilon^a,$$

implying immediately the following corollary of Killip and Nenciu's formula (5.3).

Corollary 5.7. *For the Jacobi distribution (4.4) on $(-2,2)^n$, the density of the characteristic polynomial* $\det^{(+)}$ *near* 0 *is, for some constant $c(n)$,*

$$h_n(\varepsilon) \underset{\varepsilon\to 0}{\sim} c(n)\ \varepsilon^a$$

Note that this constant is effective :

$$c(n) = \frac{\Gamma(a+b+2)}{2^{2(1+a)}\Gamma(a+1)\Gamma(b+1)}\prod_{k=0}^{2n-3}\mathbb{E}\left(\left(\frac{1}{1-\alpha_k}\right)^{1+a}\right).$$

As an application of Corollary 5.7, the correspondence $a = 2p - \frac{1}{2}$ shows that for the Haar measure on $\mathrm{SO}(2n+2p)$, conditioned to have $2p$ eigenvalues equal to 1, this density has order $\varepsilon^{2p-\frac{1}{2}}$, which agrees with (5.2). Of course, Corollary 5.7 gives in the same manner the asymptotic density of the characteristic polynomial for the symplectic ($a = 2p+1/2$) groups or the orthogonal groups with odd dimensions.

Moreover, the same method, based on Theorem 5.1, gives an analogous result for the unitary group.

Corollary 5.8. *Let $h_n^{(\mathrm{U})}$ be the density of $|\mathrm{Z}_{\mathrm{U}}^{(p)}|$, with the notations of Theorem 5.1. Then, for some constant $d(n)$,*

$$h_n^{(\mathrm{U})}(\varepsilon) \underset{\varepsilon\to 0}{\sim} d(n)\ \varepsilon^{2p}.$$

Remark. With a similar method (decomposition of $\det^{(+)}$ as a product of independent random variables), such asymptotics were already obtained by Yor [142] for the density of the characteristic polynomial on the group $\mathrm{SO}(n)$.

2. The trace and a remarkable martingale.

Using arguments from the theory of representations and the method of moments, Diaconis and Shahshahani [42] have shown that for $u \sim \mu_{\mathrm{U}(n)}$, as $n\to\infty$,

$$\left(\operatorname{Tr} u, \operatorname{Tr} u^2, \dots, \operatorname{Tr} u^k\right) \overset{\text{law}}{\longrightarrow} \left(\mathcal{X}_1, \sqrt{2}\mathcal{X}_2, \dots, \sqrt{k}\mathcal{X}_k\right) \tag{5.6}$$

with the $\mathcal{X}_k$'s independent complex standard normal variables : $\mathcal{X}_k \overset{\text{law}}{=} \frac{1}{\sqrt{2}}(\mathcal{N}_1 + \mathrm{i}\mathcal{N}_2)$, with $\mathcal{N}_1$ and $\mathcal{N}_2$ independent real standard normal variables. Johansson [74] has given a generalization of the Szegö theorem for Toeplitz determinants which allows to generalize the above central limit theorem in the following way. Take the eigenvalues of u distributed as the circular measure

$$|\Delta(e^{\mathrm{i}\theta_1},\dots,e^{\mathrm{i}\theta_n})|^{\beta}\mathrm{d}\theta_1\dots\mathrm{d}\theta_n. \tag{5.7}$$

Then

$$\left(\operatorname{Tr}u,\operatorname{Tr}u^2,\dots,\operatorname{Tr}u^k\right)\xrightarrow{\text{law}}\sqrt{\frac{2}{\beta}}\left(\mathcal{X}_1,\sqrt{2}\mathcal{X}_2,\dots,\sqrt{k}\mathcal{X}_k\right). \tag{5.8}$$

Moreover (5.6) can be generalized to the Haar measure on $\mathrm{U}(n)$ conditioned to have specified eigenvalues. The proof relies on the Fisher-Hartwig asymptotics for the Toeplitz determinants with singularities [139]. We will then give an interpretation of (5.8) from the point of view of the Killip and Nenciu matrix model for the circular ensembles : (5.8) can be interpreted as the almost sure convergence of a remarkable martingale, the limit being normally distributed, without any normalization.

2.1. Limit theorems for the traces

In a seminal paper [42], Diaconis and Shahshahani proved central limit theorems for the traces on the classical compact Lie groups endowed with their Haar measure, in particular (5.6) for the unitary group.

They proved it showing that the moments of the traces converge to those of the independent normal variables. Actually, for sufficiently large n, they showed that these moments even coincide, suggesting that the convergence to the normal law is very fast. This was shown by Johansson [75] : the rate of convergence is $\mathrm{O}(n^{-\delta n})$ for some $\delta>0$. He also gave an alternative proof of (5.6) relying on the Szegö asymptotics of Toeplitz determinants.

More precisely, let $\mathbb{T}$ be the unit circle and, for $a_\ell,b_\ell\in\mathbb{R}$ $(1\leqslant\ell\leqslant m)$,

$$g:\left\{\begin{array}{ccc}\mathbb{T} & \to & \mathbb{R}\\ e^{\mathrm{i}\theta} & \mapsto & \sum_{\ell=1}^m\left(a_\ell\cos(\ell\theta)+b_\ell\sin(\ell\theta)\right)\end{array}\right.$$

The above central limit theorem is strictly equivalent to the convergence of the Laplace transforms, that is to say

$$\mathbb{E}_{\mu_{\mathrm{U}(n)}}\left(e^{\sum_{k=1}^n g\left(e^{\mathrm{i}\theta_k}\right)}\right)\underset{n\to\infty}{\longrightarrow}e^{\frac{1}{2}\sum_{\ell=1}^m\ell(a_\ell^2+b_\ell^2)} \tag{5.9}$$

for any reals a_ℓ and b_ℓ. From Heine's identity (see e.g. [133]) the LHS is equal to $\mathrm{D}_n(e^g)$ with D_n the Toeplitz determinant defined by

$$\mathrm{D}_n(f)=\det(\hat{f}_{i-j})_{0\leqslant i,j\leqslant n},$$

the $(\hat{f}_k)$ being the Fourier coefficients of f. Szegö's Theorem states that, for any $g\in\mathrm{L}^1(\mathbb{T})$ with real values, if $c=\sum_1^\infty k|\hat{g}_k|^2<\infty$,

$$\mathrm{D}_n\left(e^g\right)=e^{n\hat{g}_0+c+o(1)}$$

as $n\to\infty$. Hence (5.9) is a consequence of both Heine's identity and Szegö's limit theorem. The above method cannot be directly applied to get a central limit theorem for the traces for any circular ensemble. Indeed, for the distribution (5.7) with $\beta\neq 2$, the LHS in (5.9) cannot be interpreted as a Toeplitz determinant anymore. However, the following theorem gives a direct answer.

Theorem 5.9. *(Johansson [74]) Take $(\theta_1,\dots,\theta_n)$ with distribution (4.2) and $g\in\mathrm{L}^1(\mathbb{T})$ with real values, and suppose $c=\sum_1^\infty k|\hat{g}_k|^2<\infty$. Then, as $n\to\infty$,*

$$\mathbb{E}\left(\prod_{k=1}^n e^{g(\theta_k)}\right)=e^{\frac{2}{\beta}(n\hat{g}_0+c+o(1))}.$$

Johansson's work immediately implies the following convergence of the traces for any circular ensemble.

Corollary 5.10. *Take $(\theta_1, \dots, \theta_n)$ with distribution (5.7). Then*

$$\left(\operatorname{Tr} u, \operatorname{Tr} u^2, \dots, \operatorname{Tr} u^m\right) \xrightarrow{\text{law}} \sqrt{\frac{2}{\beta}}\left(\mathcal{X}_1, \sqrt{2}\mathcal{X}_2, \dots, \sqrt{m}\mathcal{X}_m\right)$$

with the $\mathcal{X}_\ell$'s independent complex standard normal variables.

We now generalize (5.6). Consider the Haar measure on $\mathrm{U}(n+p)$ conditioned to have p eigenvalues equal to 1 : this is the Jacobi circular ensemble with $\beta = 2$ and $\delta = p$. Is there still a central limit theorem for the traces ? If so, one may hesitate between the two following statements, or an intermediate regime :

- $\operatorname{Tr} u \xrightarrow{\text{law}} p + \mathcal{X}$, with $\mathcal{X}$ a standard complex normal variable, if conditioning with p eigenvalues equal to 1 does not affect the sum of the other eigenvalues in the limit ;
- $\operatorname{Tr} u \xrightarrow{\text{law}} \mathcal{X}$, the strict analogue of the Diaconis-Shahsahani Theorem, if the additional repulsion due to the p eigenvalues is strong enough to affect the sum of the other eigenvalues in the limit.

The answer is is the second one. We give the proof in a more general context : consider the Haar measure conditioned to have p_1 eigenvalues equal to $e^{\mathrm{i}\varphi_1}$, ..., p_k eigenvalues equal to $e^{\mathrm{i}\varphi_k}$, where the φ_ℓ's are distinct. This induces the spectral density for the other eigenvalues $(\theta_1, \dots, \theta_n)$ on $\mathrm{U}(n + p_1 + \dots + p_k)$ (c is the normalization constant)

$$\mathrm{c}\ |\Delta(e^{\mathrm{i}\theta_1}, \dots, e^{\mathrm{i}\theta_n})|^2 \prod_{\ell=1}^{k} \prod_{j=1}^{n} |e^{\mathrm{i}\varphi_\ell} - e^{\mathrm{i}\theta_j}|^{2p_\ell}. \tag{5.10}$$

Theorem 5.11. *For the above spectral measure on $\mathrm{U}(n + p_1 + \dots + p_k)$, as $n \to \infty$,*

$$\left(\operatorname{Tr} u, \operatorname{Tr} u^2, \dots, \operatorname{Tr} u^m\right) \xrightarrow{\text{law}} \left(\mathcal{X}_1, \sqrt{2}\mathcal{X}_2, \dots, \sqrt{m}\mathcal{X}_m\right)$$

with the $\mathcal{X}_\ell$'s independent complex standard normal variables.

Proof. As in the previous proof, with the function g defined above, Heine's identity gives

$$\mathbb{E}\left(e^{\sum_{k=1}^{n} g\left(e^{\mathrm{i}\theta_k}\right)}\right) = \frac{\mathrm{D}_n(fe^g)}{\mathrm{D}_n(f)},$$

the expectation being with respect to (5.10) and the function f defined by

$$f\left(e^{\mathrm{i}\theta}\right) = \prod_{\ell=1}^{k} |e^{\mathrm{i}\theta} - e^{\mathrm{i}\varphi_\ell}|^{2p_\ell}.$$

The Fisher-Hartwig asymptotics (see Theorem 5.12 below) implies that

$$\frac{\mathrm{D}_n(fe^g)}{\mathrm{D}_n(f)} \underset{n\to\infty}{\sim} \mathrm{D}_n(e^g)\left(\prod_{\ell=1}^{k} e^{-p_\ell g(e^{\mathrm{i}\varphi_\ell})}\right) = e^{c - \sum_{\ell=1}^{k} p_\ell g(e^{\mathrm{i}\varphi_\ell}) + o(1)}.$$

with $c = \sum_1^\infty k|\hat{g}_k|^2$. Consequently, our choice of g implies the convergence of the Laplace transform of the traces to those of independent normal variables, with an expectation shift $-\sum_{\ell=1}^{k} p_\ell g(e^{\mathrm{i}\varphi_\ell})$. Adding the contribution of the $e^{\mathrm{i}\varphi_\ell}$'s concludes the proof. ☐

The indispensable tool for the above proof is the following theorem about the asymptotics of Toeplitz determinants with singularities. For a status report on the extensions of the result below, see [46].

Theorem 5.12. *(Fisher-Hartwig asymptotics, Widom [139]) Suppose that* $h : \mathbb{T} \to \mathbb{C}$ *is non-zero, with a winding number 0, and with a derivative satisfying a Lipschitz condition. Define* $f\left(e^{\mathrm{i}\theta}\right) = \prod_{\ell=1}^{k} |e^{\mathrm{i}\theta} - e^{\mathrm{i}\varphi_\ell}|^{2p_\ell}$ $(\mathfrak{Re}(p_\ell) > -1/2)$*. Then*

$$\begin{aligned}\mathrm{D}_n(hf) \underset{n\to\infty}{\sim} \mathrm{D}_n(h)&\left(\prod_{\ell=1}^{k} h(e^{\mathrm{i}\varphi_\ell})^{-p_\ell}\right)\mathrm{D}_n(f)\\ &\underset{n\to\infty}{\sim} e^{\sum_1^\infty k\hat{g}_k\hat{g}_{-k}}\left(\prod_{\ell=1}^{k} h(e^{\mathrm{i}\varphi_\ell})^{-p_\ell}\right)\\ &\prod_{\ell=1}^{k}\frac{\mathrm{G}(1+p_\ell)^2}{\mathrm{G}(1+2p_\ell)}\prod_{1\leqslant s<r\leqslant k}\frac{1}{|e^{\mathrm{i}\varphi_r}-e^{\mathrm{i}\varphi_s}|^{p_r p_s}}\left(e^{n\hat{g}_0}n^{\sum_1^k p_\ell^2}\right)\end{aligned}$$

where the $\hat{g}_k$*'s are the Fourier coefficients of* $\log h$ *and* G *is the Barnes function.*

Remark. An analogue of the Fisher-Hartwig asymptotics has been conjectured by Forrester and Frankel [49] for circular ensembles with $\beta > 0$. This allows us to conjecture that for the spectral measure density

$$\mathrm{c}\ |\Delta(e^{\mathrm{i}\theta_1},\dots,e^{\mathrm{i}\theta_n})|^\beta \prod_{\ell=1}^{k}\prod_{j=1}^{n}|e^{\mathrm{i}\varphi_\ell}-e^{\mathrm{i}\theta_j}|^{2p_\ell}$$

on $\mathrm{U}(n+p_1+\dots+p_k)$,

$$\left(\operatorname{Tr} u, \operatorname{Tr} u^2,\dots,\operatorname{Tr} u^m\right) \xrightarrow{\text{law}} \sqrt{\frac{2}{\beta}}\left(\mathcal{X}_1,\sqrt{2}\mathcal{X}_2,\dots,\sqrt{m}\mathcal{X}_m\right).$$

2.2. Connections with the Killip-Nenciu Theorem.

Let

$$u = \Theta^{(0)}(\alpha_0)\Theta^{(1)}(\alpha_1)\dots\Theta^{(n-1)}(\alpha_{n-1}),$$

with the notation (4.9). A calculation gives

$$\operatorname{Tr} u = \overline{\alpha}_0 - \alpha_0\overline{\alpha}_1 - \alpha_1\overline{\alpha}_2 - \dots - \alpha_{n-2}\overline{\alpha}_{n-1}.$$

If the α_k's are independent with a rotationally invariant distribution, then an easy induction gives

$$\operatorname{Tr} u \overset{\text{law}}{=} \alpha_0 + \alpha_0\alpha_1 + \alpha_1\alpha_2 + \dots + \alpha_{n-2}\alpha_{n-1}. \tag{5.11}$$

Let $(\mathrm{X}_k, k \geqslant 0)$ be independent random variables and $\mathrm{X}_k \overset{\text{law}}{=} e^{\mathrm{i}\omega}\sqrt{\mathrm{B}_{1,\frac{\beta}{2}(k-1)}}$, with ω uniform on $(-\pi,\pi)$ independent of the beta variable with the indicated parameters. Let

$$\mathrm{M}_n = \mathrm{X}_1\mathrm{X}_2 + \mathrm{X}_2\mathrm{X}_3 + \dots + \mathrm{X}_{n-1}\mathrm{X}_n.$$

Then, from the Killip-Nenciu Theorem and (5.11), for any circular distribution (5.7) with parameter $\beta > 0$,

$$\operatorname{Tr} u \overset{\text{law}}{=} \mathrm{M}_n + \mathrm{X}_n,$$

with $\mathrm{X}_n \to 0$ in probability. Moreover a little calculation shows that, if $\beta > 0$, $(\mathrm{M}_n, n \geqslant 1)$ is a L^2-bounded martingale, so it converges almost surely, and we call μ the distribution of its limit. Hence $\operatorname{Tr} u$ converges in law to μ, with no normalization, which follows from the repulsion of the eigenvalues (independent and uniform eigenangles would require a normalization by $1/\sqrt{n}$ for the convergence in law of the trace). From Corollary 5.10 we know that μ is the centered normal complex distribution with variance $2/\beta$. The above discussion can be summarized as follows.

Corollary 5.13. *The discrete martingale* $(\mathrm{M}_n, n \geqslant 1)$ *converges a.s. to a random variable, whose law is complex Gaussian with variance* $2/\beta$.

Moreover, if $\beta = 2$, the convergence of the law of $\mathrm{Tr}(u)$ to the normal distribution is extremely fast : it was shown by Johansson [75] that this is $\mathrm{O}(n^{-\delta n})$ for some $\delta > 0$. For $\beta \neq 2$, there is no evidence for such a fast convergence. Note that for classical central limit theorems this speed is generally only $\mathrm{O}(1/\sqrt{n})$ as predicted by the Berry-Esseen inequality.

In conclusion, note that there are many examples of discrete martingales with bounded increments, converging almost surely, with a Gaussian limit. For example, consider $(\mathrm{B}_t, t \geqslant 0)$ a standard Brownian motion in $\mathbb{C}$ and consider the sequence of stopping times

$$\left\{ \begin{array}{l} \mathrm{T}_0 = 0 \\ \mathrm{T}_{n+1} = \min\left(1, \inf\{t \geqslant \mathrm{T}_n \mid |\mathrm{B}_t - \mathrm{B}_{\mathrm{T}_n}| = 1\}\right) \end{array} \right. \tag{5.12}$$

Then $\mathrm{A}_n = \mathrm{B}_{\mathrm{T}_n}$ defines a martingale with bounded increments converging almost surely to B_1. The increments of $(\mathrm{A}_n, n \geqslant 0)$ are not independent : more generally, a martingale with independent increments converging to a Gaussian limit only has Gaussian increments as immediately shown by the Cramer lemma [35] : if the sum of independent random variables X and Y is Gaussian, then X and Y are Gaussian.

It would be interesting to find a sequence of stopping times (T_n), in the same manner as (5.12), such that

$$\mathrm{M}_n \stackrel{\text{law}}{=} \mathrm{B}_{\mathrm{T}_n},$$

because it would give some new insight in the surprisingly fast convergence of the trace shown by Johansson.

3. Joint convergence of characteristic polynomial and trace

The purpose of this section is to show the following joint convergence.

Theorem 5.14. *Suppose the eigenvalues of* $u_n \in \mathrm{U}(n)$ *have the distribution (5.7). Then as* $n \to \infty$,

$$\left(\frac{\log\det(\mathrm{Id} - u_n)}{\sqrt{\log n}}, \mathrm{Tr}\, u_n\right) \xrightarrow{\text{law}} \sqrt{\frac{2}{\beta}}(\mathcal{X}_1, \mathcal{X}_2)$$

with $\mathcal{X}_1, \mathcal{X}_2$ *independent complex standard normal variables.*

Proof. Take independent X_k's with distribution $\mathrm{X}_k \stackrel{\text{law}}{=} e^{\mathrm{i}\omega_k}\sqrt{\mathrm{B}_{1,\frac{\beta}{2}(k-1)}}$, ω_k uniform on $(-\pi, \pi)$ and $\mathrm{B}_{1,\frac{\beta}{2}(k-1)}$ a beta random variable with the indicated parameters. From Theorem 4.16, the eigenvalues of

$$u_n = \Xi^{(0)}(\mathrm{X}_{n-1})\Xi^{(1)}(\mathrm{X}_{n-2})\dots\Xi^{(n-1)}(\mathrm{X}_0)$$

have distribution (5.7). For this matrix model, we know that

$$\left\{ \begin{array}{ccc} \det(\mathrm{Id} - u_n) & = & (1-\mathrm{X}_0)(1-\mathrm{X}_1)\dots(1-\mathrm{X}_{n-1}) \\ \mathrm{Tr}\, u_n & = & f(\mathrm{X}_0, \dots, \mathrm{X}_{n-1}) \end{array} \right.$$

with the notation

$$f(\mathrm{X}_0, \dots, \mathrm{X}_{n-1}) = -\mathrm{X}_0\overline{\mathrm{X}_1}\frac{1-\mathrm{X}_1}{1-\overline{\mathrm{X}_1}} - \dots - \mathrm{X}_{n-2}\overline{\mathrm{X}_{n-1}}\frac{1-\mathrm{X}_{n-1}}{1-\overline{\mathrm{X}_{n-1}}} + \mathrm{X}_{n-1}.$$

Therefore, for any $0 \leqslant m \leqslant n-1$, the following equality in law holds :

$$\begin{pmatrix} \mathrm{Tr}\, u_n \\ \frac{\log\det(\mathrm{Id}-u_n)}{\sqrt{\log n}} \end{pmatrix} \overset{\text{law}}{=} \begin{pmatrix} f(\mathrm{X}_0,\dots,\mathrm{X}_m) \\ \frac{\sum_{k=m+1}^n \log(1-\mathrm{X}_k)}{\sqrt{\log n}} \end{pmatrix} + \begin{pmatrix} f(\mathrm{X}_0,\dots,\mathrm{X}_{n-1}) - f(\mathrm{X}_0,\dots,\mathrm{X}_m) \\ \frac{\sum_{k=0}^m \log(1-\mathrm{X}_k)}{\sqrt{\log n}} \end{pmatrix}$$

Suppose $m \to \infty$ and $\log m / \log n \to 0$. From Lemma 5.15, $\frac{\sum_{k=0}^m \log(1-\mathrm{X}_k)}{\sqrt{\log m}}$ converges in law, so $\frac{\sum_{k=0}^m \log(1-\mathrm{X}_k)}{\sqrt{\log n}}$ converges in law to 0. Moreover, as $n \to \infty$, $f(\mathrm{X}_0,\dots,\mathrm{X}_{n-1}) - f(\mathrm{X}_0,\dots,\mathrm{X}_m)$ also converges in law to 0 : it actually converges to 0 in L^2 as shown by Lemma 5.16. Hence, thanks to Slutsky's lemma, we only need to show that

$$\begin{pmatrix} f(\mathrm{X}_0,\dots,\mathrm{X}_m) \\ \frac{\sum_{k=m+1}^n \log(1-\mathrm{X}_k)}{\sqrt{\log n}} \end{pmatrix}$$

converges in law to a vector of independent complex Gaussians with the required variance. This is straightforward because the coordinates are now independent and each one converges to the required Gaussian : $f(\mathrm{X}_0,\dots,\mathrm{X}_m) \overset{\text{law}}{=} \mathrm{Tr}\, u_m$ with $u_m \sim \mu_{\mathrm{U}(m)}$, so as $m \to \infty$ it converges to a Gaussian thanks to Corollary 5.10, and $\frac{\sum_{k=m+1}^n \log(1-\mathrm{X}_k)}{\sqrt{\log n}}$ is the difference of a variable converging to the required Gaussian and a variable converging to 0. □

Lemma 5.15. *Suppose the eigenvalues of $u_n \in \mathrm{U}(n)$ have the distribution (5.7). Then as $n \to \infty$,*

$$\frac{\log\det(\mathrm{Id}-u_n)}{\sqrt{\log n}} \overset{\text{law}}{\longrightarrow} \sqrt{\frac{2}{\beta}}\mathcal{X}_1$$

with $\mathcal{X}_1$ a complex standard normal variable.

Proof. This is straightforward from the decomposition as a product of independent random variables

$$\det(\mathrm{Id}-u_n) = (1-\mathrm{X}_0)(1-\mathrm{X}_1)\dots(1-\mathrm{X}_{n-1}),$$

where the independent X_k's have distribution $\mathrm{X}_k \overset{\text{law}}{=} e^{\mathrm{i}\omega_k}\sqrt{\mathrm{B}_{1,\frac{\beta}{2}(k-1)}}$, ω_k uniform on $(-\pi,\pi)$ and $\mathrm{B}_{1,\frac{\beta}{2}(k-1)}$ a beta random variable with the indicated parameters : one just needs to follow the proof of Theorem 1.8. □

Lemma 5.16. *Take independent X_k's with distribution $\mathrm{X}_k \overset{\text{law}}{=} e^{\mathrm{i}\omega_k}\sqrt{\mathrm{B}_{1,\frac{\beta}{2}(k-1)}}$, ω_k uniform on $(-\pi,\pi)$ and $\mathrm{B}_{1,\frac{\beta}{2}(k-1)}$ a beta random variable with the indicated parameters. For $1 \leqslant m \leqslant n$, let*

$$\mathrm{Y} = -\mathrm{X}_{m-1}\overline{\mathrm{X}_m}\frac{1-\mathrm{X}_m}{1-\overline{\mathrm{X}_m}} - \dots - \mathrm{X}_{n-2}\overline{\mathrm{X}_{n-1}}\frac{1-\mathrm{X}_{n-1}}{1-\overline{\mathrm{X}_{n-1}}} + \mathrm{X}_{n-1}.$$

Then

$$\mathbb{E}\left(|\mathrm{Y}^2|\right) = \frac{2}{\beta}\frac{1}{1+\frac{\beta}{2}(m-1)} + \left(1-\frac{2}{\beta}\right)\frac{1}{1+\frac{\beta}{2}(n-1)}.$$

Proof. As the X_k's are independent and have a distribution invariant by rotation,

$$\mathrm{Y} \overset{\text{law}}{=} \mathrm{X}_{m-1}\mathrm{X}_m + \mathrm{X}_m\mathrm{X}_{m+1} + \dots + \mathrm{X}_{n-2}\mathrm{X}_{n-1} + \mathrm{X}_{n-1}.$$

For every k, $\mathbb{E}(\mathrm{X}_k)=0$ hence only the diagonal terms survive in the second moment :

$$\begin{aligned}
\mathbb{E}\left(|\mathrm{Y}^2|\right) &= \sum_{k=m}^{n-1}\mathbb{E}\left(|\mathrm{X}_{k-1}\mathrm{X}_k|^2\right)+\mathbb{E}\left(|\mathrm{X}_{n-1}|^2\right)\\
&= \sum_{k=m}^{n-1}\frac{1}{1+\frac{\beta}{2}(k-1)}\frac{1}{1+\frac{\beta}{2}k}+\frac{1}{1+\frac{\beta}{2}(n-1)}\\
&= \frac{2}{\beta}\sum_{k=m}^{n-1}\left(\frac{1}{1+\frac{\beta}{2}(k-1)}-\frac{1}{1+\frac{\beta}{2}k}\right)+\frac{1}{1+\frac{\beta}{2}(n-1)}\\
&= \frac{2}{\beta}\frac{1}{1+\frac{\beta}{2}(m-1)}+\left(1-\frac{2}{\beta}\right)\frac{1}{1+\frac{\beta}{2}(n-1)}.
\end{aligned}$$

□

Remark. Lemma 5.16 implies in particular ($\beta=2$ and $m=1$) that for $u_n\sim\mu_{\mathrm{U}(n)}$, $\mathbb{E}(|\operatorname{Tr}u_n|^2)$ is constantly equal to 1, which is a sign that the convergence of $\operatorname{Tr}u_n$ to a normal variable is very fast.

Suppose now that $\beta\neq 2$ and the eigenvalues of u_n have distribution (5.7). Lemma 5.16 shows that $\mathbb{E}(|\operatorname{Tr}u_n|^2)$ converges to $2/\beta$ with speed $1/n$: the convergence to a normal variable is much slower for $\beta\neq 2$.

4. Moments for $u\sim\mu_{\mathrm{SU}(n)}$

We have seen in the previous chapters that for $u\sim\mu_{\mathrm{U}(n)},\mu_{\mathrm{SO}(2n)}$ or $\mu_{\mathrm{USp}(2n)}$, the characteristic polynomial $\mathrm{Z}_u=\det(\mathrm{Id}-u)$ is equal in law to a product of independent random variables. An important example of limit monodromy group in the work by Katz and Sarnak [78] is $\mathrm{SU}(n)$, and they study more generally the groups ($m\geqslant 1$)

$$\mathrm{U}_m(n)=\{u\in\mathrm{U}(n)\mid\det(u)^m=1\}.$$

We do not know a decomposition in law of Z_u for $u\sim\mu_{\mathrm{U}_m(n)}$, however we present in this section how an important lemma by Katz and Sarnak allows to calculate the asymptotics of

$$\mathbb{E}_{\mu_{\mathrm{U}_m(n)}}\left(|\mathrm{Z}_u|^{2k}e^{2\ell\mathrm{i}\arg\mathrm{Z}_u}\right),\ k,\ell\in\mathbb{N},l\leqslant k.$$

Lemma 5.17 (Katz-Sarnak [78]). *Let f be a $\mathscr{C}^\infty$ class function. Then the following equality holds, where the series is absolutely convergent :*

$$\mathbb{E}_{\mu_{\mathrm{U}_m(n)}}\left(f(u)\right)=\sum_{j\in\mathbb{Z}}\mathbb{E}_{\mu_{\mathrm{U}(n)}}\left(f(u)(\det u)^{jm}\right).$$

An application of the above lemma yields

$$\mathbb{E}_{\mu_{\mathrm{U}_m(n)}}\left(|\mathrm{Z}_u|^{2k}e^{2\ell\mathrm{i}\arg\mathrm{Z}_u}\right)=\sum_{j\in\mathbb{Z}}\mathbb{E}_{\mu_{\mathrm{U}(n)}}\left(\det(\mathrm{Id}-u)^{k+\ell}\det(\mathrm{Id}-\overline{u})^{k-\ell}(\det u)^{jm}\right)$$

The sum is actually finite : all indexes j with $jm+\ell>k$ of $jm+\ell<-k$ give a zero contribution. It is intuitive that, as $m\to\infty$, $\mu_{\mathrm{U}_m(n)}$ *approaches* $\mu_{\mathrm{U}(n)}$. The above lines show that, in this particular case, much more is true.

Corollary 5.18. *Suppose $m>k+l$. Then*

$$\mathbb{E}_{\mu_{\mathrm{U}_m(n)}}\left(|\mathrm{Z}_u|^{2k}e^{2\ell\mathrm{i}\arg\mathrm{Z}_u}\right)=\mathbb{E}_{\mu_{\mathrm{U}(n)}}\left(|\mathrm{Z}_u|^{2k}e^{2\ell\mathrm{i}\arg\mathrm{Z}_u}\right).$$

Moreover, for any fixed m, the asymptotics of $\mathbb{E}_{\mu_{\mathrm{U}_m(n)}}\left(|\mathrm{Z}_u|^{2k}e^{2\ell\mathrm{i}\arg\mathrm{Z}_u}\right)$ can be computed explicitly. As

$$\det(u)=(-1)^n\frac{\det(\mathrm{Id}-u)}{\det(\mathrm{Id}-\overline{u})},$$

we can write

$$\begin{aligned}&\mathbb{E}_{\mu_{\mathrm{U}_m(n)}}\left(|\mathrm{Z}_u|^{2k}e^{2\ell\mathrm{i}\arg\mathrm{Z}_u}\right)\\&\qquad=\sum_{-k\leqslant jm+\ell\leqslant k}(-1)^{njm}\,\mathbb{E}_{\mu_{\mathrm{U}(n)}}\left(\det(\mathrm{Id}-u)^{k+\ell+jm}\det(\mathrm{Id}-\overline{u})^{k-\ell-jm}\right).\end{aligned}$$

For each term of this sum, we know the asymptotics (for example thanks to the Fisher-Hartwig asymptotics, see e.g. [46]), which leads to the following result, where G is the Barnes function.

Corollary 5.19. *Let $m\in\mathbb{N}^*$, $k,\ell\in\mathbb{N}$, $\ell\leqslant k$. Let k' be the minimal value of $\{|jm+\ell|,j\in\mathbb{Z}\}$, obtained at j^*. Then*

$$\mathbb{E}_{\mu_{\mathrm{U}_m(n)}}\left(|\mathrm{Z}_u|^{2k}e^{2\ell\mathrm{i}\arg\mathrm{Z}_u}\right)\underset{n\to\infty}{\sim}(-1)^{nj^*m}\frac{\mathrm{G}(1+k+k')\mathrm{G}(1+k-k')}{\mathrm{G}(1+2k)}\;n^{k^2-k'^2}.$$

In particular,

$$\mathbb{E}_{\mu_{\mathrm{U}_m(n)}}\left(|\mathrm{Z}_u|^{2k}\right)\underset{n\to\infty}{\sim}\mathbb{E}_{\mu_{\mathrm{U}(n)}}\left(|\mathrm{Z}_u|^{2k}\right).$$

Remark. Thanks to the Katz-Sarnak lemma, many other results about $\mathrm{U}(n)$ can be extended to $\mathrm{U}_m(n)$. For example, if $u_n\sim\mu_{\mathrm{U}_m(n)}$, then as $n\to\infty$

$$\left(\mathrm{Tr}\,u,\mathrm{Tr}\,u^2,\dots,\mathrm{Tr}\,u^k\right)\xrightarrow{\text{law}}\left(\mathcal{X}_1,\sqrt{2}\mathcal{X}_2,\dots,\sqrt{k}\mathcal{X}_k\right)$$

with the $\mathcal{X}_k$'s independent complex standard normal variables. This is the analogue of the Diaconis-Shahshahani theorem (5.6), and its proof is a direct application of the method of moments and the result concerning $\mathrm{U}(n)$ in [42].

5. On a theorem by Eric Rains

Let $(\theta_1,\dots,\theta_n)$ be distributed on $\mathbb{T}^n$, the torus with dimension n. For a large class of distributions on the θ_k's, one can expect that $m\theta_1,\dots,m\theta_n$ tend to independent uniform random variables as $m\to\infty$. A surprising result by E. Rains states that, if the θ_k's have distribution

$$\frac{1}{n!}|\Delta(e^{\mathrm{i}\theta_1},\dots,e^{\mathrm{i}\theta_n})|^2\frac{\mathrm{d}\theta_1}{2\pi}\dots\frac{\mathrm{d}\theta_n}{2\pi},$$

then for any $m\geqslant n$ the angles $m\theta_1,\dots,m\theta_n$ are *exactly* uniform and independent.

Theorem 5.20 (E. Rains [114]). *Let $u\sim\mu_{\mathrm{U}(n)}$. Then for any $m\geqslant n$, the eigenvalues of u^m are independent and uniform on the unit circle.*

This can be extended to any circular ensemble with Hua-Pickrell singularities, with density on $(-\pi,\pi)^n$ proportional to

$$|\Delta(e^{\mathrm{i}\theta_1},\dots,e^{\mathrm{i}\theta_n})|^{2\gamma}\prod_{j=1}^{\ell}\prod_{k=1}^{n}|e^{\mathrm{i}\theta_k}-e^{\mathrm{i}\varphi_j}|^{2a_j},\tag{5.13}$$

where γ and the a_j's are elements in $\mathbb{N}$.

Theorem 5.21. *Let $(\theta_1,\dots,\theta_n)$ have the circular distribution (5.13). Then for any integer $m>\gamma(n-1)+a_1+\dots+a_\ell$, $e^{\mathrm{i}m\theta_1},\dots,e^{\mathrm{i}m\theta_n}$ are independent and uniform on the unit circle.*

Proof. We proceed as Rains, using the method of moments : we need to show that the joint moments of $e^{\mathrm{i}m\theta_1}, \dots, e^{\mathrm{i}(\gamma(n-1)+k)\theta_n}$ coincide with those of uniform independent random variables. This is easily reduced to showing that for $m_1, m_2, \dots, m_n$ integers, not all of which are zero,

$$\int |\Delta(e^{\mathrm{i}\theta_1}, \dots, e^{\mathrm{i}\theta_n})|^{2\gamma} \prod_{j=1}^{\ell} \prod_{k=1}^{n} |e^{\mathrm{i}\theta_k} - e^{\mathrm{i}\varphi_j}|^{2a_j} e^{\mathrm{i}m_1 m\theta_1} \dots e^{\mathrm{i}m_n m\theta_n} \mathrm{d}\theta_1 \dots \mathrm{d}\theta_n = 0. \tag{5.14}$$

Suppose for example $m_p \neq 0$ for some $1 \leqslant p \leqslant n$. As $\gamma \in \mathbb{N}$, an expansion of the Vandermonde determinant shows that the above integrated expression has the form

$$e^{\mathrm{i}m_p m\theta_p} \sum_{j=-\gamma(n-1)-a_1-\dots-a_\ell}^{\gamma(n-1)+a_1+\dots+a_\ell} c_j e^{\mathrm{i}j\theta_p}$$

for some complex coefficients c_j, functions of γ, the a_k's, the φ_k's, the θ_k's distinct from θ_p. As $m > \gamma(n-1) + a_1 + \dots + a_\ell$, this is actually a polynomial in $e^{\mathrm{i}\theta_p}$ with no constant term, so the integral equals zero. $\square$

Theorem (5.21) cannot be extended to non-integer values of γ : we are not in situation to apply Carlson's theorem here. Indeed, for $n = 2$, all a_j's equal to 0, one can see that $e^{\mathrm{i}\theta_1(1+\beta)}$ and $e^{\mathrm{i}\theta_2(1+\beta)}$ are generally not independent.

Chapter 6

Mesoscopic fluctuations of the zeta zeros

This chapter corresponds to *Mesoscopic fluctuations of the zeta zeros* [18], to appear in Probability Theory and Related Fields.

This chapter gives a multidimensional extension of Selberg's central limit theorem for $\log\zeta$, in which a non-trivial dependence appears. In particular, this is related to a question by Coram and Diaconis about the mesoscopic correlations of the zeros of the Riemann zeta function.

Similar results are given in the context of random matrices from the unitary group. This indicates that the validity of the correspondence $\log t \leftrightarrow n$ holds not only between the dimension of the matrix (n) and the height on the critical line ($\log t$), but it also holds, at a local scale, for small deviations from the critical axis or the unit circle.

Remark. All results below hold for L-functions from the Selberg class, for concision we state them for ζ.

In this chapter we talk about correlations between random variables to express the idea of dependence, which is equivalent as all the involved variables are Gaussian.

The Vinogradov symbol, $a_n \ll b_n$, means $a_n = \mathrm{O}(b_n)$, and $a_n \gg b_n$ means $b_n \ll a_n$. In this chapter, we implicitly assume that, for all n and t, $\varepsilon_n \geqslant 0$, $\varepsilon_t \geqslant 0$.

1. Introduction

1.1. Main result

Selberg's central limit theorem states that, if ω is uniform on $(0,1)$, then

$$\frac{\log\zeta\left(\frac{1}{2}+\mathrm{i}\omega t\right)}{\sqrt{\log\log t}} \xrightarrow{\text{law}} \mathrm{Y}, \tag{6.1}$$

as $t\to\infty$, Y being a standard complex normal variable (see paragraph 1.4 below for precise definitions of $\log\zeta$ and complex normal variables). This result has been extended in two distinct directions, both relying on Selberg's original method.

First similar central limit theorems appear in Tsang's thesis [136] far away from the critical axis, and Joyner [77] generalized these results to a larger class of L-functions. In particular, (6.1) holds also for $\log\zeta$ evaluated close to the critical axis $(1/2+\varepsilon_t+\mathrm{i}\omega t)$ provided that $\varepsilon_t \ll 1/\log t$; for $\varepsilon_t\to 0$ and $\varepsilon_t \gg 1/\log t$, Tsang proved that a change of normalization is necessary :

$$\frac{\log\zeta\left(\frac{1}{2}+\varepsilon_t+\mathrm{i}\omega t\right)}{\sqrt{-\log\varepsilon_t}} \xrightarrow{\text{law}} \mathrm{Y}', \tag{6.2}$$

with ω uniform on $(0,1)$ and Y' a standard complex normal variable.

Second, a multidimensional extension of (6.1) was given by Hughes, Nikeghbali and Yor [72], in order to get a dynamic analogue of Selberg's central limit theorem : they showed that for any $0<\lambda_1<\dots<\lambda_\ell$

$$\frac{1}{\sqrt{\log\log t}}\left(\log\zeta\left(\frac{1}{2}+\mathrm{i}\omega e^{(\log t)^{\lambda_1}}\right),\dots,\log\zeta\left(\frac{1}{2}+\mathrm{i}\omega e^{(\log t)^{\lambda_\ell}}\right)\right) \xrightarrow{\text{law}} \left(\lambda_1\mathrm{Y}_1,\dots,\lambda_\ell\mathrm{Y}_\ell\right), \tag{6.3}$$

all the Y_k's being independent standard complex normal variables. The evaluation points $\frac{1}{2}+\mathrm{i}\omega e^{(\log t)^{\lambda_k}}$ in the above formula are very distant from each other and a natural question is whether, for closer points, a non-trivial correlation structure appears for the values of zeta. Actually, the average values of $\log\zeta$ become correlated for small shifts, and the Gaussian kernel appearing in the limit coincides with the one of Brownian motion off the diagonal. More precisely, our main result is the following.

Theorem 6.1. *Let ω be uniform on $(0,1)$, $\varepsilon_t\to 0$, $\varepsilon_t \gg 1/\log t$, and functions $0\leqslant f_t^{(1)}<\cdots<f_t^{(\ell)}<c<\infty$. Suppose that for all $i\neq j$*

$$\frac{\log|f_t^{(j)}-f_t^{(i)}|}{\log\varepsilon_t}\to c_{i,j}\in[0,\infty]. \tag{6.4}$$

Then the vector

$$\frac{1}{\sqrt{-\log\varepsilon_t}}\left(\log\zeta\left(\frac{1}{2}+\varepsilon_t+\mathrm{i}f_t^{(1)}+\mathrm{i}\omega t\right),\dots,\log\zeta\left(\frac{1}{2}+\varepsilon_t+\mathrm{i}f_t^{(\ell)}+\mathrm{i}\omega t\right)\right) \tag{6.5}$$

converges in law to a complex Gaussian vector $(\mathrm{Y}_1,\dots,\mathrm{Y}_\ell)$ with mean 0 and covariance function

$$\operatorname{cov}(\mathrm{Y}_i,\mathrm{Y}_j)=\left\{\begin{array}{ccc}1 & if & i=j\\ 1\wedge c_{i,j} & if & i\neq j\end{array}\right. . \tag{6.6}$$

Moreover, the above result remains true if $\varepsilon_t\ll 1/\log t$, replacing the normalization $-\log\varepsilon_t$ with $\log\log t$ in (6.4) and (6.5).

The covariance structure (6.6) of the limit Gaussian vector actually depends only on the $\ell-1$ parameters $c_{1,2},\dots,c_{\ell-1,\ell}$ because formula (6.4) implies, for all $i<k<j$, $c_{i,j}=c_{i,k}\wedge c_{k,j}$. We will explicitly construct Gaussian vectors with the correlation structure (6.6) in section 4.

We now illustrate Theorem 6.1. Take $\ell=2$, $\varepsilon_t\to 0$, $\varepsilon_t\gg 1/\log t$. Then for any $0\leqslant\delta\leqslant 1$ and ω uniform on $(0,1)$, choosing $f_t^{(1)}=0$ and $f_t^{(2)}=\varepsilon_t^{\delta}$,

$$\frac{1}{\sqrt{-\frac{1}{2}\log\varepsilon_t}}\left(\log\left|\zeta\left(\frac{1}{2}+\varepsilon_t+\mathrm{i}\omega t\right)\right|,\log\left|\zeta\left(\frac{1}{2}+\varepsilon_t+\mathrm{i}\omega t+\mathrm{i}\varepsilon_t^{\delta}\right)\right|\right)$$

converges in law to

$$(\mathcal{N}_1,\delta\mathcal{N}_1+\sqrt{1-\delta^2}\mathcal{N}_2), \tag{6.7}$$

where $\mathcal{N}_1$ and $\mathcal{N}_2$ are independent standard real normal variables. A similar result holds if $\varepsilon_t\ll 1/\log t$, in particular we have a central limit theorem on the critical axis $\varepsilon_t=0$:

$$\frac{1}{\sqrt{\frac{1}{2}\log\log t}}\left(\log\left|\zeta\left(\frac{1}{2}+\mathrm{i}\omega t\right)\right|,\log\left|\zeta\left(\frac{1}{2}+\mathrm{i}\omega t+\frac{\mathrm{i}}{(\log t)^{\delta}}\right)\right|\right)$$

also converges in law to (6.7). Note the change of normalization according to ε_t, i.e. the distance to the critical axis. Finally, if all shifts $f_t^{(i)}$ are constant and distinct, $c_{i,j}=0$ for all i and j, so the distinct means of ζ converge in law to independent complex normal variables, after normalization.

Remark. In this chapter we are concerned with distinct shifts along the ordinates, in particular because it implies the following Corollary 6.3 about counting the zeros of the zeta function. The same method equally applies to distinct shifts along the abscissa, not enounced here for simplicity. For example, the Gaussian variables Y and Y′ in (6.1) and (6.2) have correlation $1\wedge\sqrt{\delta}$ if $\varepsilon_t=1/(\log t)^{\delta}$ with $\delta>0$.

Theorem 6.1 can be understood in terms of Gaussian processes : it has the following immediate consequence, enounced for $\varepsilon_t=0$ for simplicity.

Corollary 6.2. *Let ω be uniform on $(0,1)$. Consider the random function*

$$\left(\frac{1}{\sqrt{\log\log t}}\log\left|\zeta\left(\frac{1}{2}+\mathrm{i}\omega t+\frac{\mathrm{i}}{(\log t)^{\delta}}\right)\right|, 0\leqslant\delta\leqslant 1\right)$$

Then its finite dimensional distribution converge, as $t\to\infty$, to those of a centered Gaussian process with kernel $\Gamma_{\gamma,\delta}=\gamma\wedge\delta$ if $\gamma\neq\delta$, 1 if $\gamma=\delta$.

There is an effective construction of a centered Gaussian process $(\mathrm{X}_\delta, 0\leqslant\delta\leqslant 1)$ with covariance function $\Gamma_{\gamma,\delta}$: let $(\mathrm{B}_\delta, 0\leqslant\delta\leqslant 1)$ be a standard Brownian motion and independently let $(\mathrm{D}_\delta, 0\leqslant\delta\leqslant 1)$ be a totally disordered process, meaning that all its coordinates are independent centered Gaussians with variance $\mathbb{E}(\mathrm{D}_\delta^2)=\delta$. Then

$$\mathrm{X}_\delta=\mathrm{B}_\delta+\mathrm{D}_{1-\delta}$$

defines a Gaussian process with the desired covariance function. Note that there is no measurable version of this process : if there were, then $(\mathrm{D}_\delta, 0\leqslant\delta\leqslant 1)$ would have a measurable version which is absurd because, by Fubini's Theorem, for all $0\leqslant a<b\leqslant 1$ $\mathbb{E}\left(\left(\int_a^b \mathrm{D}_\delta \mathrm{d}\delta\right)^2\right)=0$, so $\int_a^b \mathrm{D}_\delta\mathrm{d}\delta=0$ a.s. and $\mathrm{D}_\delta=0$ a.s. giving the contradiction.

1.2. Counting the zeros

Theorem 6.1 also has a strange consequence for the counting of zeros of ζ on intervals in the critical strip. Write $\mathrm{N}(t)$ for the number of non-trivial zeros z of ζ with $0<\mathfrak{Im} z\leqslant t$, counted with their multiplicity. Then (see e.g. Theorem 9.3 in Titchmarsh [135])

$$\mathrm{N}(t)=\frac{t}{2\pi}\log\frac{t}{2\pi e}+\frac{1}{\pi}\mathfrak{Im}\log\zeta\left(1/2+\mathrm{i}t\right)+\frac{7}{8}+\mathrm{O}\left(\frac{1}{t}\right) \tag{6.8}$$

with $\mathfrak{Im}\log\zeta\left(1/2+\mathrm{i}t\right)=\mathrm{O}(\log t)$. For $t_1<t_2$ we will write

$$\Delta(t_1,t_2)=(\mathrm{N}(t_2)-\mathrm{N}(t_1))-\left(\frac{t_2}{2\pi}\log\frac{t_2}{2\pi e}-\frac{t_1}{2\pi}\log\frac{t_1}{2\pi e}\right),$$

which represents the fluctuations of the number of zeros z $(t_1<\mathfrak{Im} z\leqslant t_2)$ minus its *expectation*. A direct consequence of Theorem 6.1, choosing $\ell=2$, $f^{(1)}(t)=0$ and $f^{(2)}(t)=\frac{1}{(\log t)^\delta}$ $(0\leqslant\delta\leqslant 1)$, is the following central limit theorem obtained by Fujii [55] :

$$\frac{\Delta\left(\omega t,\omega t+\frac{1}{(\log t)^\delta}\right)}{\frac{1}{\pi}\sqrt{\log\log t}}\xrightarrow{\text{law}}\sqrt{1-\delta}\mathcal{N}$$

as $t\to\infty$, where ω is uniform on $(0,1)$ and $\mathcal{N}$ is a standard real normal variable. A more general result actually holds, being a direct consequence of Theorem 6.1 and (6.8). This confirms numerical experiments by Coram and Diaconis [33], who after making extensive tests (based on data by Odlyzko) suggested that the correlation structure (6.9) below should appear when counting the zeros of ζ. Following [33] the phenomenon presented below can be seen as the *mesoscopic* repulsion of the zeta zeros, different from the Montgomery-Odlyzko law, describing the repulsion at a microscopic scale.

Corollary 6.3. *Let (K_t) be such that, for some $\varepsilon>0$ and all t, $\mathrm{K}_t>\varepsilon$. Suppose $\log\mathrm{K}_t/\log\log t\to\delta\in[0,1)$ as $t\to\infty$. Then the finite dimensional distributions of the process*

$$\frac{\Delta\left(\omega t+\alpha/\mathrm{K}_t,\omega t+\beta/\mathrm{K}_t\right)}{\frac{1}{\pi}\sqrt{(1-\delta)\log\log t}},\ 0\leqslant\alpha<\beta<\infty$$

converge to those of a centered Gaussian process $(\tilde{\Delta}(\alpha,\beta), 0 \leqslant \alpha < \beta < \infty)$ *with the covariance structure*

$$\mathbb{E}\left(\tilde{\Delta}(\alpha,\beta)\tilde{\Delta}(\alpha',\beta')\right) = \begin{cases} 1 & \text{if } \alpha=\alpha' \text{ and } \beta=\beta' \\ 1/2 & \text{if } \alpha=\alpha' \text{ and } \beta\neq\beta' \\ 1/2 & \text{if } \alpha\neq\alpha' \text{ and } \beta=\beta' \\ -1/2 & \text{if } \beta=\alpha' \\ 0 & \text{elsewhere} \end{cases}. \tag{6.9}$$

This correlation structure is surprising : for example $\tilde{\Delta}(\alpha,\beta)$ and $\tilde{\Delta}(\alpha',\beta')$ are independent if the segment $[\alpha,\beta]$ is strictly included in $[\alpha',\beta']$, and positively correlated if this inclusion is not strict. Note that there is again an effective construction of $\tilde{\Delta}$: if $(\tilde{\mathrm{D}}_\delta, \delta \geqslant 0)$ is a real valued *process* with all coordinates independent centered Gaussians with variance $\mathbb{E}(\tilde{\mathrm{D}}_\delta^2) = 1/2$, then

$$\tilde{\Delta}(\alpha,\beta) = \tilde{\mathrm{D}}_\beta - \tilde{\mathrm{D}}_\alpha$$

has the required correlation structure. Concerning the discovery of this exotic Gaussian correlation function in the context of unitary matrices, see the remark after Theorem 6.4.

1.3. Analogous result on random matrices

We note $\mathrm{Z}(u_n, \mathrm{X})$ the characteristic polynomial of a matrix $u_n \in \mathrm{U}(n)$, and often abbreviate it as Z. Theorem 6.1 was inspired by the following analogue (Theorem 6.4) in random matrix theory. This confirms the validity of the correspondence

$$n \leftrightarrow \log t$$

between the dimension of random matrices and the length of integration on the critical axis, but it also supports this analogy at a local scale, for the evaluation points of $\log \mathrm{Z}$ and $\log \zeta$: the necessary shifts are strictly analogue both for the abscissa\radius $(\varepsilon_n \setminus \varepsilon_t)$ and the ordinate\angle $(f^{(i)} \setminus \varphi^{(i)})$.

Theorem 6.4. *Let* $u_n \sim \mu_{\mathrm{U}(n)}$, $\varepsilon_n \to 0$, $\varepsilon_n \gg 1/n$, *and functions* $0 \leqslant \varphi_n^{(1)} < \cdots < \varphi_n^{(\ell)} < 2\pi - \delta$ *for some* $\delta > 0$. *Suppose that for all* $i \neq j$

$$\frac{\log|\varphi_n^{(j)} - \varphi_n^{(i)}|}{\log \varepsilon_n} \to c_{i,j} \in [0,\infty]. \tag{6.10}$$

Then the vector

$$\frac{1}{\sqrt{-\log \varepsilon_n}}\left(\log \mathrm{Z}(u_n, e^{\varepsilon_n + \mathrm{i}\varphi_n^{(1)}}), \dots, \log \mathrm{Z}(u_n, e^{\varepsilon_n + \mathrm{i}\varphi_n^{(\ell)}})\right) \tag{6.11}$$

converges in law to a complex Gaussian vector with mean 0 and covariance function (6.6). Moreover, the above result remains true if $\varepsilon_n \ll 1/n$, *replacing the normalization* $-\log \varepsilon_n$ *with* $\log n$ *in (6.10) and (6.11).*

Remark. Let $\mathrm{N}_n(\alpha,\beta)$ be the number of eigenvalues $e^{\mathrm{i}\theta}$ of u_n with $\alpha < \theta < \beta$, and $\delta_n(\alpha,\beta) = \mathrm{N}_n(\alpha,\beta) - \mathbb{E}_{\mu_{\mathrm{U}(n)}}(\mathrm{N}_n(\alpha,\beta))$. Then, a little calculation (see [71]) yields

$$\delta_n(\alpha,\beta) = \frac{1}{\pi}\left(\mathfrak{Im}\log \mathrm{Z}(u_n, e^{\mathrm{i}\beta}) - \mathfrak{Im}\log \mathrm{Z}(u_n, e^{\mathrm{i}\alpha})\right)$$

This and the above theorem imply that, as $n \to \infty$, the vector

$$\frac{1}{\sqrt{\log n}}\left(\delta_n(\varphi_n^{(1)}, \varphi_n^{(2)}), \delta_n(\varphi_n^{(2)}, \varphi_n^{(3)}), \dots, \delta_n(\varphi_n^{(\ell-1)}, \varphi_n^{(\ell)})\right).$$

converges in law to a Gaussian limit. Central limit theorems for the counting-number of eigenvalues in intervals were discovered by Wieand [140] in the special case when all the intervals have a fixed length independent of n (included in the case $c_{i,j} = 0$ for all i, j). Her result was extended by Diaconis and Evans to the case $\varphi_n^{(i)} = \varphi^{(i)}/\mathrm{K}_n$ for some $\mathrm{K}_n \to \infty$, $\mathrm{K}_n/n \to 0$ (i. e. $c_{i,j}$ is a constant independent of i and j) : Corollary 6.3 is a number-theoretic analogue of their Theorem 6.1 in [40].

Note that, in the general case of distinct $c_{i,i+1}$'s, a similar result holds but the correlation function of the limit vector is not as simple as the one in Corollary 6.3 : it strongly depends on the relative orders of these coefficients $c_{i,i+1}$'s.

1.4. Definitions, organization of the chapter

In this chapter, for more concision we will make use of the following standard definition of complex Gaussian random variables.

Definition 6.5. *A complex standard normal random variable* Y *is defined as* $\frac{1}{\sqrt{2}}(\mathcal{N}_1 + \mathrm{i}\mathcal{N}_2)$*,* $\mathcal{N}_1$ *and* $\mathcal{N}_2$ *being independent real standard normal variables. For any* $\lambda, \mu \in \mathbb{C}$*, we will say that* $\lambda + \mu\mathrm{Y}$ *is a complex normal variable with mean* λ *and variance* $|\mu|^2$*. The covariance of two complex Gaussian variables* Y *and* Y' *is defined as* $\mathrm{cov}(\mathrm{Y},\mathrm{Y}') = \mathbb{E}(\overline{\mathrm{Y}}\mathrm{Y}') - \mathbb{E}(\overline{\mathrm{Y}})\,\mathbb{E}(\mathrm{Y}')$*, and* $\mathrm{var}(\mathrm{Y}) = \mathrm{cov}(\mathrm{Y},\mathrm{Y})$*.*

A vector $(\mathrm{Y}_1,\dots,\mathrm{Y}_\ell)$ *is a complex Gaussian vector if any linear combination of its coordinates is a complex normal variable. For such a complex Gaussian vector and any* $\mu = (\mu_1,\dots,\mu_\ell) \in \mathbb{C}_+^\ell$*,* $\sum_{k=1}^{\ell} \mu_k \mathrm{Y}_k$ *has variance* $\overline{\mu}\mathrm{C}\,{}^{\mathrm{t}}\mu$*, where* C *is said to be the covariance matrix of* $(\mathrm{Y}_1,\dots,\mathrm{Y}_\ell)$ *:* $\mathrm{C}_{i,j} = \mathrm{cov}(\mathrm{Y}_i,\mathrm{Y}_j)$*.*

As in the real case, the mean and the covariance matrix characterize a complex Gaussian vector.

Moreover, precise definitions of $\log\zeta$ and $\log \mathrm{Z}(\mathrm{X})$ are necessary : for $\sigma \geqslant 1/2$, we use the standard definition

$$\log\zeta(\sigma + \mathrm{i}t) = -\int_\sigma^\infty \frac{\zeta'}{\zeta}(s + \mathrm{i}t)\mathrm{d}s$$

if ζ has no zero with ordinate t. Otherwise, $\log\zeta(\sigma + \mathrm{i}t) = \lim_{\varepsilon\to 0}\log\zeta(\sigma + \mathrm{i}(t+\varepsilon))$.

Similarly, let $u \sim \mu_{\mathrm{U}(n)}$ have eigenvalues $e^{\mathrm{i}\theta_1},\dots,e^{\mathrm{i}\theta_n}$. For $|\mathrm{X}| > 1$, the principal branch of the logarithm of $\mathrm{Z}(\mathrm{X}) = \det(\mathrm{Id} - \mathrm{X}^{-1}u)$ is chosen as

$$\log \mathrm{Z}(\mathrm{X}) = \sum_{k=1}^{n} \log\left(1 - \frac{e^{\mathrm{i}\theta_k}}{\mathrm{X}}\right) = -\sum_{j=1}^{\infty} \frac{1}{j}\frac{\mathrm{Tr}(u^j)}{\mathrm{X}^j}.$$

Following Diaconis and Evans [40], if $\mathrm{X}_n \to \mathrm{X}$ with $|\mathrm{X}_n| > 1$ and $|\mathrm{X}| = 1$, then $\log \mathrm{Z}(\mathrm{X}_n)$ converges in L^2 to $-\sum_{j=1}^{\infty}\frac{1}{j}\frac{\mathrm{Tr}(u^j)}{\mathrm{X}^j}$; therefore this is our definition of $\log \mathrm{Z}(\mathrm{X})$ when $|\mathrm{X}| = 1$.

We will successively prove Theorems 6.4 and 6.1 in the next two sections. They are independent, but we feel that the joint central limit theorem for ζ and its analogue for the random matrices are better understood by comparing both proofs, which are similar. In particular Proposition 6.9, which is a major step towards Theorem 6.1 is a strict number-theoretic analogue of the Diaconis-Evans theorem used in the next section to prove Theorem 6.4.

Finally, in Section 4, we show that the same correlation structure as (6.6) appears in the theory of spatial branching processes.

2. The central limit theorem for random matrices.

2.1. The Diaconis-Evans method.

Diaconis and Shahshahani [42] looked at the joint moments of $\operatorname{Tr} u, \operatorname{Tr} u^2, \dots, \operatorname{Tr} u^\ell$ for $u \sim \mu_{\mathrm{U}(n)}$, and showed that any of these moments coincides with the ones of $\mathrm{Y}_1, \sqrt{2}\mathrm{Y}_2, \dots, \sqrt{\ell}\mathrm{Y}_\ell$ for sufficient large n, the Y_k's being independent standard complex normal variables. This suggests that under general assumptions, a central limit theorem can be stated for linear combinations of these traces.

Indeed, the main tool we will use for the proof of Theorem 6.4 is the following result.

Theorem 6.6 (Diaconis, Evans [40]). *Consider an array of complex constants $\{a_{nj} \mid n \in \mathbb{N}, j \in \mathbb{N}\}$. Suppose there exists σ^2 such that*

$$\lim_{n\to\infty} \sum_{j=1}^{\infty} |a_{nj}|^2 (j \wedge n) = \sigma^2. \tag{6.12}$$

Suppose also that there exists a sequence of positive integers $\{m_n \mid n \in \mathbb{N}\}$ such that $\lim_{n\to\infty} m_n/n = 0$ and

$$\lim_{n\to\infty} \sum_{j=m_n+1}^{\infty} |a_{nj}|^2 (j \wedge n) = 0. \tag{6.13}$$

Then $\sum_{j=1}^{\infty} a_{nj} \operatorname{Tr} u_n^j$ converges in distribution to $\sigma \mathrm{Y}$, where Y is a complex standard normal random variable and $u_n \sim \mu_{\mathrm{U}(n)}$.

Thanks to the above result, to prove central limit theorems for class functions, we only need to decompose them on the basis of the traces of successive powers. This is the method employed in the next subsections, where we treat separately the cases $\varepsilon_n \gg 1/n$ and $\varepsilon_n \ll 1/n$.

2.2. Proof of Theorem 6.4 for $\varepsilon_n \gg 1/n$.

From the Cramér-Wold device[1] a sufficient condition to prove Theorem 6.4 is that, for any $(\mu_1, \dots, \mu_\ell) \in \mathbb{C}^\ell$,

$$\frac{1}{\sqrt{-\log \varepsilon_n}} \sum_{k=1}^{\ell} \mu_k \log \mathrm{Z}(e^{\varepsilon_n + \mathrm{i}\varphi_n^{(k)}}) = -\sum_{j=1}^{\infty} \frac{1}{\sqrt{-\log \varepsilon_n}} \left(\sum_{k=1}^{\ell} \frac{\mu_k}{j e^{j(\varepsilon_n + \mathrm{i}\varphi_n^{(k)})}} \right) \operatorname{Tr}(u_n^j)$$

converges in law to a complex normal variable with mean 0 and variance

$$\sigma^2 = \sum_{i=1}^{\ell} |\mu_i|^2 + \sum_{s \neq t} \overline{\mu_s} \mu_t (c_{s,t} \wedge 1). \tag{6.14}$$

We need to check conditions (6.12) and (6.13) from Theorem 6.6, with

$$a_{nj} = \frac{-1}{\sqrt{-\log \varepsilon_n}} \left(\sum_{k=1}^{\ell} \frac{\mu_k}{j e^{j(\varepsilon_n + \mathrm{i}\varphi_n^{(k)})}} \right).$$

First, to calculate the limit of

$$\sum_{j=1}^{\infty} |a_{nj}|^2 (j \wedge n) = \sum_{j=1}^{n} j |a_{nj}|^2 + n \sum_{j=n+1}^{\infty} |a_{nj}|^2,$$

1. A Borel probability measure on $\mathbb{R}^\ell$ is uniquely determined by the family of its one-dimensional projections, that is the images of μ by $(x_1, \dots, x_\ell) \mapsto \sum_{j=1}^{\ell} \lambda_j x_j$, for any vector $(\lambda_j)_{1 \leqslant j \leqslant \ell} \in \mathbb{R}^\ell$.

note that this second term tends to 0 : if $a=(\sum_{k=1}^{\ell}|\mu_k|)^2$, then

$$(-\log\varepsilon_n)\,n\sum_{j=n+1}^{\infty}|a_{nj}|^2=n\sum_{j=n+1}^{\infty}\left|\sum_{k=1}^{\ell}\frac{\mu_k}{je^{j(\varepsilon_n+\mathrm{i}\varphi_n^{(k)})}}\right|^2\leqslant a\,n\sum_{j=n+1}^{\infty}\frac{1}{j^2}\leqslant a$$

so $n\sum_{j=n+1}^{\infty}|a_{nj}|^2\to 0$. The first term can be written

$$(-\log\varepsilon_n)\sum_{j=1}^{n}j|a_{nj}|^2=\sum_{j=1}^{n}j\left|\sum_{k=1}^{\ell}\frac{\mu_k}{je^{j(\varepsilon_n+\mathrm{i}\varphi_n^{(k)})}}\right|^2=\sum_{s,t}\overline{\mu_s}\mu_t\sum_{j=1}^{n}\frac{1}{j}\left(\frac{e^{\mathrm{i}(\varphi_n^{(s)}-\varphi_n^{(t)})}}{e^{2\varepsilon_n}}\right)^j.$$

Hence the expected limit is a consequence of the following lemma.

Lemma 6.7. *Let $\varepsilon_n\gg 1/n$, $\varepsilon_n\to 0$, (Δ_n) be a strictly positive sequence, bounded by $2\pi-\delta$ for some $\delta>0$, and $\log\Delta_n/\log\varepsilon_n\to c\in[0,\infty]$. Then*

$$\frac{1}{-\log\varepsilon_n}\sum_{j=1}^{n}\frac{e^{\mathrm{i}j\Delta_n}}{je^{2j\varepsilon_n}}\underset{n\to\infty}{\longrightarrow}c\wedge 1.$$

Proof. The Taylor expansion of $\log(1-\mathrm{X})$ for $|\mathrm{X}|<1$ gives

$$\sum_{j=1}^{n}\frac{e^{\mathrm{i}j\Delta_n}}{je^{2j\varepsilon_n}}=-\underbrace{\log\left(1-e^{-2\varepsilon_n+\mathrm{i}\Delta_n}\right)}_{(1)}-\underbrace{\sum_{j=n+1}^{\infty}\frac{e^{\mathrm{i}j\Delta_n}}{je^{2j\varepsilon_n}}}_{(2)}.$$

As $\varepsilon_n>d/n$ for some constant $d>0$,

$$|(2)|\leqslant\sum_{j=n+1}^{\infty}\frac{1}{je^{2j\varepsilon_n}}\leqslant\sum_{j=n+1}^{\infty}\frac{1}{je^{d\frac{j}{n}}}\underset{n\to\infty}{\longrightarrow}\int_0^{\infty}\frac{\mathrm{d}x}{(1+x)e^{d(1+x)}},$$

so (2), divided by $\log\varepsilon_n$, tends to 0.

We now look at the main contribution, coming from (1). If $c>1$, then $\Delta_n=\mathrm{o}(\varepsilon_n)$, so (1) is equivalent to $\log\varepsilon_n$ as $n\to\infty$. If $0<c<1$, then $\varepsilon_n=\mathrm{o}(\Delta_n)$ so (1) is equivalent to $\log\Delta_n$, hence to $c\log\varepsilon_n$. If $c=1$, (1) is equivalent to $(\log\varepsilon_n)\mathbb{1}_{\varepsilon_n\geqslant\Delta_n}+(\log\Delta_n)\mathbb{1}_{\Delta_n>\varepsilon_n}$, that is to say $\log\varepsilon_n$. Finally, if $c=0$, as $(\varepsilon_n)^a\ll\Delta_n<2\pi-\delta$ for all $a>0$, $(1)=\mathrm{o}(\log\varepsilon_n)$. □

Condition (6.13) in Theorem 6.6 remains to be shown. Since $n\sum_{j=n+1}^{\infty}|a_{nj}|^2\to 0$, we just need to find a sequence (m_n) with $m_n/n\to 0$ and $\sum_{j=m_n+1}^{n}j|a_{nj}|^2\to 0$. Writing as previously $a=(\sum_{k=1}^{\ell}|\mu_k|)^2$, then

$$\sum_{j=m_n+1}^{n}j|a_{nj}|^2\leqslant\frac{a}{-\log\varepsilon_n}\sum_{j=m_n+1}^{n}\frac{1}{j}.$$

Hence any sequence (m_n) with $m_n=\mathrm{o}(n)$, $(\log n-\log(m_n))/\log\varepsilon_n\to 0$ is convenient, for example $m_n=\lfloor n/(-\log\varepsilon_n)\rfloor$.

2.3. Proof of Theorem 6.4 for $\varepsilon_n\ll 1/n$.

We now need to check conditions (6.12) and (6.13) with

$$a_{nj}=\frac{-1}{\sqrt{\log n}}\left(\sum_{k=1}^{\ell}\frac{\mu_k}{je^{j(\varepsilon_n+\mathrm{i}\varphi_n^{(k)})}}\right)$$

and σ^2 as in (6.14). In the same way as the previous paragraph, $n\sum_{j=n+1}^{\infty}|a_{nj}|^2\to 0$, and (6.13) holds with $m_n=\lfloor n/\log n\rfloor$. So the last thing to prove is

$$\sum_{j=1}^{n} j|a_{nj}|^2=\sum_{s,t}\overline{\mu}_s\mu_t\frac{1}{\log n}\sum_{j=1}^{n}\frac{1}{j}\left(\frac{e^{\mathrm{i}(\varphi_n^{(s)}-\varphi_n^{(t)})}}{e^{2\varepsilon_n}}\right)^j \underset{n\to\infty}{\longrightarrow}\sigma^2,$$

that is to say, writing $x_n=e^{-2\varepsilon_n+\mathrm{i}(\varphi_n^{(s)}-\varphi_n^{(t)})}$,

$$\frac{1}{\log n}\sum_{j=1}^{n}\frac{x_n^j}{j}\underset{n\to\infty}{\longrightarrow} c_{s,t}\wedge 1.$$

First note that with no restriction we can suppose $\varepsilon_n=0$. Indeed, if we write $y_n=e^{\mathrm{i}(\varphi_n^{(s)}-\varphi_n^{(t)})}$, and $\varepsilon_n\leqslant b/n$ for some $b>0$ (since $\varepsilon_n\ll 1/n$),

$$\left|\sum_{j=1}^{n}\frac{x_n^j}{j}-\sum_{j=1}^{n}\frac{y_n^j}{j}\right|\leqslant\sum_{j=1}^{n}\frac{1}{j}\left|e^{-b\frac{j}{n}}-1\right|\leqslant b$$

because $|e^{-x}-1|\leqslant x$ for $x\geqslant 0$. The asymptotics of $\sum_{j=1}^{n}\frac{y_n^j}{j}$ are given in the next lemma, which concludes the proof.

Lemma 6.8. *Let (Δ_n) be a strictly positive sequence, bounded by $2\pi-\delta$ for some $\delta>0$, such that $-\log\Delta_n/\log n\to c\in[0,\infty]$. Then*

$$\frac{1}{\log n}\sum_{j=1}^{n}\frac{e^{\mathrm{i}j\Delta_n}}{j}\underset{n\to\infty}{\longrightarrow} c\wedge 1.$$

Proof. We successively treat the cases $c>0$ and $c=0$. Suppose first that $c>0$. By comparison between the Riemann sum and the corresponding integral,

$$\begin{aligned}\left|\sum_{j=1}^{n}\frac{e^{\mathrm{i}j\Delta_n}}{j}-\int_{\Delta_n}^{(n+1)\Delta_n}\frac{e^{\mathrm{i}t}}{t}\mathrm{d}t\right|&\leqslant\sum_{j=1}^{n}\int_{j\Delta_n}^{(j+1)\Delta_n}\left(\frac{\left|e^{\mathrm{i}j\Delta_n}-e^{\mathrm{i}t}\right|}{j\Delta_n}+\left|\frac{e^{\mathrm{i}t}}{j\Delta_n}-\frac{e^{\mathrm{i}t}}{t}\right|\right)\mathrm{d}t\\&\leqslant\sum_{j=1}^{n}\frac{\Delta_n}{j}+\sum_{j=1}^{n}\left(\frac{1}{j}-\frac{1}{j+1}\right)\\&\leqslant\Delta_n(\log n+1)+1.\end{aligned}$$

As $c>0$, $\Delta_n\to 0$ so $\frac{1}{\log n}\sum_{j=1}^{n}\frac{e^{\mathrm{i}j\Delta_n}}{j}$ has the same limit as $\frac{1}{\log n}\int_{\Delta_n}^{(n+1)\Delta_n}\frac{e^{\mathrm{i}t}}{t}\mathrm{d}t$ as $n\to\infty$. If $c>1$, $n\Delta_n\to 0$ so we easily get

$$\frac{1}{\log n}\int_{\Delta_n}^{(n+1)\Delta_n}\frac{e^{\mathrm{i}t}}{t}\mathrm{d}t\underset{n\to\infty}{\sim}\frac{1}{\log n}\int_{\Delta_n}^{(n+1)\Delta_n}\frac{\mathrm{d}t}{t}=\frac{\log(n+1)}{\log n}\underset{n\to\infty}{\longrightarrow}1.$$

If $0<c<1$, $n\Delta_n\to\infty$. As $\sup_{x>1}\left|\int_1^x\frac{e^{\mathrm{i}t}}{t}\mathrm{d}t\right|<\infty$,

$$\frac{1}{\log n}\int_{\Delta_n}^{(n+1)\Delta_n}\frac{e^{\mathrm{i}t}}{t}\mathrm{d}t\underset{n\to\infty}{\sim}\frac{1}{\log n}\int_{\Delta_n}^{1}\frac{e^{\mathrm{i}t}}{t}\mathrm{d}t\underset{n\to\infty}{\sim}\frac{1}{\log n}\int_{\Delta_n}^{1}\frac{\mathrm{d}t}{t}\underset{n\to\infty}{\longrightarrow}c.$$

If $c=1$, a distinction between the cases $n\Delta_n\leqslant 1$, $n\Delta_n>1$ and the above reasoning gives 1 in the limit.

If $c = 0$, Δ_n does not necessarily converge to 0 anymore so another method is required. An elementary summation gives

$$\sum_{j=1}^{n} \frac{e^{\mathrm{i}j\Delta_n}}{j} = \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1}\right) \sum_{j=1}^{k} e^{\mathrm{i}j\Delta_n} + \frac{1}{n+1} \sum_{j=1}^{n} e^{\mathrm{i}j\Delta_n}.$$

We will choose a sequence (a_n) $(1 \leqslant a_n \leqslant n)$ and bound $\sum_{j=1}^{k} e^{\mathrm{i}j\Delta_n}$ by k if $k < a_n$, by $|(e^{\mathrm{i}k\Delta_n} - 1)/(e^{\mathrm{i}\Delta_n} - 1)| \leqslant 2/|e^{\mathrm{i}\Delta_n} - 1|$ if $a_n \leqslant k \leqslant n$. This yields

$$\sum_{j=1}^{n} \frac{e^{\mathrm{i}j\Delta_n}}{j} \leqslant \sum_{k=1}^{a_n-1} \frac{1}{k+1} + \frac{2}{|e^{\mathrm{i}\Delta_n} - 1|} \sum_{k=a_n}^{n} \left(\frac{1}{k} - \frac{1}{k+1}\right) + 1 \leqslant \log a_n + \frac{2}{a_n|e^{\mathrm{i}\Delta_n} - 1|} + 1.$$

As $\Delta_n < 2\pi - \delta$, there is a constant $\lambda > 0$ with $|e^{\mathrm{i}\Delta_n} - 1| > \lambda\Delta_n$. So the result follows if we can find a sequence (a_n) such that $\frac{\log a_n}{\log n} \to 0$ and $a_n\Delta_n \log n \to \infty$, which is true for $a_n = \lfloor 2\pi/\Delta_n \rfloor$. □

3. The central limit theorem for ζ

3.1. Selberg's method.

Suppose the Euler product of ζ holds for $1/2 \leqslant \mathfrak{Re}(s) \leqslant 1$ (this is a conjecture) : then $\log \zeta(s) = -\sum_{p\in\mathcal{P}} \log(1 - p^{-s})$ can be approximated by $\sum_{p\in\mathcal{P}} p^{-s}$. Let $s = 1/2 + \varepsilon_t + \mathrm{i}\omega t$ with ω uniform on $(0, 1)$. As the $\log p$'s are linearly independent over $\mathbb{Q}$, the terms $\{p^{-\mathrm{i}\omega t} \mid p \in \mathcal{P}\}$ can be viewed as independent uniform random variables on the unit circle as $t \to \infty$, hence it was a natural thought that a central limit theorem might hold for $\log \zeta(s)$, which was indeed shown by Selberg [119].

The crucial point to get such arithmetical central limit theorems is the approximation by sufficiently short Dirichlet series. Selberg's ideas to approximate $\log \zeta$ appear in Goldston [59], Joyner [77], Tsang [136] or Selberg's original paper [119]. More precisely, the explicit formula for ζ'/ζ, by Landau, gives such an approximation ($x > 1$, s distinct from 1, the zeros ρ and $-2n$, $n \in \mathbb{N}$) :

$$\frac{\zeta'}{\zeta}(s) = -\sum_{n\leqslant x} \frac{\Lambda(n)}{n^s} + \frac{x^{1-s}}{1-s} - \sum_{\rho} \frac{x^{\rho-s}}{\rho - s} + \sum_{n=1}^{\infty} \frac{x^{-2n-s}}{2n+s},$$

from which we get an approximate formula for $\log \zeta(s)$ by integration. However, the sum over the zeros is not absolutely convergent, hence this formula is not sufficient. Selberg found a slight change in the above formula, that makes a great difference because all infinite sums are now absolutely convergent : under the above hypotheses, if

$$\Lambda_x(n) = \begin{cases} \Lambda(n) & \text{for } 1 \leqslant n \leqslant x, \\ \Lambda(n)\frac{\log \frac{x^2}{n}}{\log n} & \text{for } x \leqslant n \leqslant x^2, \end{cases}$$

then

$$\frac{\zeta'}{\zeta}(s) = -\sum_{n\leqslant x^2} \frac{\Lambda_x(n)}{n^s} + \frac{x^{2(1-s)} - x^{1-s}}{(1-s)^2 \log x} + \frac{1}{\log x} \sum_{\rho} \frac{x^{\rho-s} - x^{2(\rho-s)}}{(\rho-s)^2} + \frac{1}{\log x} \sum_{n=1}^{\infty} \frac{x^{-2n-s} - x^{-2(2n+s)}}{(2n+s)^2}.$$

Assuming the Riemann hypothesis, the above formulas give a simple expression for $(\zeta'/\zeta)(s)$ for $\mathfrak{Re}(s) \geqslant 1/2$: for $x \to \infty$, all terms in the infinite sums converge to 0 because $\mathfrak{Re}(\rho - s) < 0$. By subtle arguments, Selberg showed that, although RH is necessary for the *almost sure* coincidence between ζ'/ζ and its Dirichlet series, it is not required in order to get *a good* L^k *approximation.* In particular, Selberg [119] (see also Joyner [77] for similar results for more general L-functions) proved that for any $k \in \mathbb{N}^*$, $0 < a < 1$, there is a constant $c_{k,a}$ such that for any $1/2 \leqslant \sigma \leqslant 1$, $t^{a/k} \leqslant x \leqslant t^{1/k}$,

$$\frac{1}{t}\int_1^t \left|\log\zeta(\sigma+\mathrm{i}s) - \sum_{p\leqslant x}\frac{p^{-\mathrm{i}s}}{p^{\sigma}}\right|^{2k}\mathrm{d}s \leqslant c_{k,a}.$$

In the following, we only need the case $k=1$ in the above formula : with the notations of Theorem 6.1 (ω uniform on $(0,1)$),

$$\log\zeta\left(\frac{1}{2}+\varepsilon_t+\mathrm{i}f_t^{(j)}+\mathrm{i}\omega t\right) - \sum_{p\leqslant t}\frac{p^{-\mathrm{i}\omega t}}{p^{\frac{1}{2}+\varepsilon_t+\mathrm{i}f_t^{(j)}}}$$

is bounded in L^2, and after normalization by $\frac{1}{-\log\varepsilon_t}$ or $\frac{1}{\log\log t}$, it converges in probability to 0. Hence, Slutsky's lemma and the Cramér-Wold device allow us to reformulate Theorem 6.1 in the following way.

Equivalent of Theorem 6.1. *Let ω be uniform on $(0,1)$, $\varepsilon_t \to 0$, $\varepsilon_t \gg 1/\log t$, and functions $0 \leqslant f_t^{(1)} < \cdots < f_t^{(\ell)} < c < \infty$. Suppose (6.4). Then for any finite set of complex numbers $\mu_1,\dots,\mu_\ell$,*

$$\frac{1}{\sqrt{-\log\varepsilon_t}}\sum_{j=1}^{\ell}\mu_j\sum_{p\leqslant t}\frac{p^{-\mathrm{i}\omega t}}{p^{\frac{1}{2}+\varepsilon_t+\mathrm{i}f_t^{(j)}}} \tag{6.15}$$

converges in law to a complex Gaussian variable with mean 0 and variance

$$\sigma^2 = \sum_{j=1}^{\ell}|\mu_j|^2 + \sum_{j\neq k}\overline{\mu_j}\mu_k(1\wedge c_{j,k}).$$

If $\varepsilon_t \ll 1/\log t$, then the same result holds with normalization $1/\sqrt{\log\log t}$ instead of $1/\sqrt{-\log\varepsilon_t}$ in (6.15) and (6.4).

To prove this convergence in law, we need a number-theoretic analogue of Theorem 6.6, stated in the next paragraph.

3.2. An analogue of the Diaconis-Evans theorem.

Heuristically, the following proposition stems from the linear independence of the $\log p$'s over $\mathbb{Q}$, and the main tool to prove it is the Montgomery-Vaughan theorem.

Note that, generally, convergence to normal variables in a number-theoretic context is proved thanks to the convergence of all moments (see e.g. [72]). The result below is a tool showing that testing the L^2-convergence is sufficient.

Proposition 6.9. *Let a_{pt} ($p \in \mathcal{P}, t \in \mathbb{R}^+$) be complex numbers with $\sup_p |a_{pt}| \to 0$ and $\sum_p |a_{pt}|^2 \to \sigma^2$ as $t \to \infty$. Suppose also the existence of (m_t) with $\log m_t/\log t \to 0$ and*

$$\sum_{p>m_t}|a_{pt}|^2\left(1+\frac{p}{t}\right)\underset{t\to\infty}{\longrightarrow} 0. \tag{6.16}$$

Then, if ω is a uniform random variable on $(0,1)$,

$$\sum_{p\in\mathcal{P}} a_{pt}p^{-\mathrm{i}\omega t} \overset{\text{law}}{\longrightarrow} \sigma\mathrm{Y}$$

as $t \to \infty$, Y being a standard complex normal variable.

Remark. The condition $m_n = \mathrm{o}(n)$ in Theorem 6.6 is replaced here by $\log m_t = \mathrm{o}(\log t)$. A systematic substitution $n \leftrightarrow \log t$ would give the stronger condition $m_t / \log m_t = \mathrm{o}(\log t)$: the above proposition gives a better result than the one expected from the analogy between random matrices and number theory.

Proof. Condition (6.16) first allows to restrict the infinite sum over the set of primes $\mathcal{P}$ to the finite sum over $\mathcal{P} \cap [2, m_t]$. More precisely, following [102], let (a_r) be complex numbers, (λ_r) distinct real numbers and

$$\delta_r = \min_{s \neq r} |\lambda_r - \lambda_s|.$$

The Montgomery-Vaughan theorem states that

$$\frac{1}{t} \int_0^t \left| \sum_r a_r e^{\mathrm{i}\lambda_r s} \right|^2 \mathrm{d}s = \sum_r |a_r|^2 \left(1 + \frac{3\pi\theta}{t\delta_r} \right)$$

for some θ with $|\theta| \leqslant 1$. We substitute above a_r by a_{pt} and λ_r by $\log p$, and restrict the sum to the p's greater than m_t : there is a constant $c > 0$ independent of p with $\min_{p' \neq p} |\log p - \log p'| > \frac{c}{p}$, so

$$\frac{1}{t} \int_0^t \left| \sum_{p > m_t} a_{pt} p^{-\mathrm{i}s} \right|^2 \mathrm{d}s \leqslant \sum_p |a_{pt}|^2 \left(1 + c' \frac{p}{t} \right)$$

with c' bounded by $3\pi c$. Hence the hypothesis (6.16) implies that $\sum_{p > m_t} a_{pt} p^{-\mathrm{i}\omega t}$ converges to 0 in L^2, so by Slutsky's lemma it is sufficient to show that

$$\sum_{p \leqslant m_t} a_{pt} p^{-\mathrm{i}\omega t} \xrightarrow{\text{law}} \sigma \mathrm{Y}. \tag{6.17}$$

As $\sum_{p \leqslant m_t} |a_{pt}|^2 \to \sigma^2$ and $\sup_{p \leqslant m_t} |a_{pt}| \to 0$, Theorem 4.1 in Petrov [108] gives the following central limit theorem :

$$\sum_{p \leqslant m_t} a_{pt} e^{\mathrm{i}\omega_p} \xrightarrow{\text{law}} \sigma \mathrm{Y}, \tag{6.18}$$

where the ω_p's are independent uniform random variables on $(0, 2\pi)$. The $\log p$'s being linearly independent over $\mathbb{Q}$, it is well known that as $t \to \infty$ any given finite number of the $p^{\mathrm{i}\omega t}$'s are asymptotically independent and uniform on the unit circle. The problem here is that the number of these random variables increases as they become independent. If this number increases sufficiently slowly ($\log m_t / \log t \to 0$), one can expect that (6.18) implies (6.17).

The method of moments tells us that , in order to prove the central limit theorem (6.17), it is sufficient to show for all positive integers a and b that

$$\mathbb{E}\left(f_{a,b} \left(\sum_{p \leqslant m_t} a_{pt} p^{-\mathrm{i}\omega t} \right) \right) \underset{t \to \infty}{\longrightarrow} \mathbb{E}\left(f_{a,b}(\sigma \mathrm{Y}) \right),$$

with $f_{a,b}(x) = x^a \overline{x}^b$. From (6.18) we know that

$$\mathbb{E}\left(f_{a,b} \left(\sum_{p \leqslant m_t} a_{pt} e^{\mathrm{i}\omega_p} \right) \right) \underset{n \to \infty}{\longrightarrow} \mathbb{E}\left(f_{a,b}(\sigma \mathrm{Y}) \right).$$

Hence it is sufficient for us to show that, for every a and b,

$$\left| \mathbb{E}\left(f_{a,b} \left(\sum_{p \leqslant m_t} a_{pt} p^{-\mathrm{i}\omega t} \right) \right) - \mathbb{E}\left(f_{a,b} \left(\sum_{p \leqslant m_t} a_{pt} e^{\mathrm{i}\omega_p} \right) \right) \right| \underset{n \to \infty}{\longrightarrow} 0. \tag{6.19}$$

Let $n_t = |\mathcal{P}\cap[2,m_t]|$ and, for $z=(z_1,\dots,z_{n_t})\in\mathbb{R}^{n_t}$, write $f_{a,b}^{(t)}(z) = f_{a,b}\left(\sum_{p\leqslant m_t} a_{pt}e^{\mathrm{i}z_p}\right)$, which is $\mathscr{C}^\infty$ and $(2\pi\mathbb{Z})^{n_t}$-periodic. Let its Fourier decomposition be $f_{a,b}^{(t)}(z)=\sum_{k\in\mathbb{Z}^{n_t}} u_{a,b}^{(t)}(k)e^{\mathrm{i}k\cdot z}$. If we write T^s for the translation on $\mathbb{R}^{n_t}$ with vector $s\,p^{(t)} = s(\log p_1,\dots,\log p_{n_t})$, inspired by the proof of Theorem 2.1 we can write the LHS of the above equation as ($\mu^{(t)}$ is the uniform distribution on the Torus with dimension n_t)

$$\left|\frac{1}{t}\int_0^t \mathrm{d}s f_{a,b}^{(t)}(\mathrm{T}^s 0) - \int \mu^{(t)}(\mathrm{d}z) f_{a,b}^{(t)}(z)\right| = \frac{1}{t}\left|\sum_{k\in\mathbb{Z}^{n_t},k\neq 0} u_{a,b}^{(t)}(k)\frac{e^{\mathrm{i}tk\cdot p^{(t)}}-1}{k\cdot p^{(t)}}\right|.$$

Our theorem will be proven if the above difference between a mean in time and a mean in space converges to 0, which can be seen as an ergodic result. The above RHS is clearly bounded by

$$\frac{2}{t}\left(\sum_{k\in\mathbb{Z}^{n_t}}|u_{a,b}^{(t)}(k)|\right)\frac{1}{\inf_{k\in\mathcal{H}_{a,b}^{(t)}}|k\cdot p^{(t)}|},$$

where $\mathcal{H}_{a,b}^{(t)}$ is the set of the non-zero k's in $\mathbb{Z}^{n_t}$ for which $u_{a,b}^{(t)}(k)\neq 0$: such a k can be written $k^{(1)}-k^{(2)}$, with $k^{(1)}\in[\![1,a]\!]^{n_t}$, $k^{(2)}\in[\![1,b]\!]^{n_t}$, $k_1^{(1)}+\dots+k_{n_t}^{(1)}=a$, $k_1^{(2)}+\dots+k_{n_t}^{(2)}=b$.

First note that, as $\sum_{k\in\mathbb{Z}^{n_t}} u_{a,b}^{(t)}(k)e^{\mathrm{i}k\cdot z} = \left(\sum_{p\leqslant m_t} a_{pt}e^{\mathrm{i}z_p}\right)^a\left(\sum_{p\leqslant m_t}\overline{a_{pt}}e^{-\mathrm{i}z_p}\right)^b$,

$$\sum_{k\in\mathbb{Z}^{n_t}}|u_{a,b}^{(t)}(k)|\leqslant\left(\sum_{p\leqslant m_t}|a_{pt}|\right)^{a+b}\leqslant m_t^{\frac{a+b}{2}}\left(\sum_{p\leqslant m_t}|a_{pt}|^2\right)^{\frac{a+b}{2}}$$

hence for sufficiently large t

$$\left|\frac{1}{t}\int_0^t \mathrm{d}s f_{a,b}^{(t)}(\mathrm{T}^s 0) - \int \mu^{(t)}(\mathrm{d}z) f_{a,b}^{(t)}(z)\right|\leqslant\frac{2(2\sigma)^{\frac{a+b}{2}}}{t}\frac{m_t^{\frac{a+b}{2}}}{\inf_{k\in\mathcal{H}_{a,b}^{(t)}}|k\cdot p^{(t)}|}.$$

Lemma 6.10 below and the condition $\log m_t/\log t\to 0$ show that the above term tends to 0, concluding the proof. □

Lemma 6.10. *For $n\geqslant 1$ and all $k\in\mathcal{H}_{a,b}^t$,*

$$|k\cdot p^{(t)}|\geqslant\frac{1}{{n_t}^{2\max(a,b)}}.$$

Proof. For $k\in\mathbb{Z}^{n_t}$, $k\neq 0$, let $\mathcal{E}_1$ (resp $\mathcal{E}_2$) be the set of indexes $i\in[\![1,n_t]\!]$ with k_i strictly positive (resp strictly negative.) Write $u_1=\prod_{i\in\mathcal{E}_1}p_i^{|k_i|}$ and $u_2=\prod_{i\in\mathcal{E}_2}p_i^{|k_i|}$. Suppose $u_1\geqslant u_2$. Thanks to the uniqueness of decomposition as product of primes, $u_1\geqslant u_2+1$. Hence,

$$\begin{aligned}|k\cdot p^{(t)}| = (u_1-u_2)\frac{\log u_1-\log u_2}{u_1-u_2}&\geqslant(\log' u_1)(u_1-u_2)\\ &\geqslant\frac{1}{u_1}=e^{-\sum_{i\in\mathcal{E}_1}k_i\log p_i}\geqslant e^{-\log p_{n_t}\sum_{i\in\mathcal{E}_1}k_i}.\end{aligned}$$

For all $n_t\geqslant 0$, $\log p_{n_t}\leqslant 2\log n_t$. Moreover, from the decomposition $k=k^{(1)}-k^{(2)}$ in the previous section, we know that $\sum_{i\in\mathcal{E}_1}k_i\leqslant a$, so

$$|k\cdot p^{(t)}|\geqslant e^{-2a\log n_t}.$$

The case $u_1<u_2$ leads to $|k\cdot p^{(t)}|\geqslant e^{-2b\log n_t}$, which completes the proof. □

In the above proof, we showed that the remainder terms ($p > m_t$) converge to 0 in the L^2-norm to simplify a problem of convergence of a sum over primes : this method seems to appear for the first time in Soundararajan [132].

3.3. Proof of Theorem 6.1 for $\varepsilon_t \gg 1/\log t$.

To prove our equivalent of Theorem 6.1, we apply the above Proposition 6.9 to the random variable (6.15), that is to say

$$a_{pt} = \frac{1}{\sqrt{-\log \varepsilon_t}} \sum_{j=1}^{\ell} \frac{\mu_j}{p^{\frac{1}{2}+\varepsilon_t+\mathrm{i} f_t^{(j)}}}$$

if $p \leqslant t$, 0 if $p > t$. Then clearly $\sup_p |a_{pt}| \to 0$ as $t \to \infty$. For any sequence $0 < m_t < t$, writing $a = (\sum_{k=1}^{\ell} |\mu_k|)^2$,

$$\sum_{m_t<p<t} |a_{pt}|^2 \left(1+\frac{p}{t}\right) \leqslant \frac{a}{-\log \varepsilon_t} \sum_{m_t<p<t} \frac{1}{p} + \frac{a}{-\log \varepsilon_t}.$$

As $\sum_{p\leqslant t} \frac{1}{p} \sim \log\log t$, condition (6.16) is satisfied if we can find $m_t = \exp(\log t/b_t)$ with $b_t \to \infty$ and $\frac{\log b_t}{-\log \varepsilon_t} \to 0$: $b_t = -\log \varepsilon_t$ for example.

We now only need to show that $\sum_{p\leqslant t} |a_{pt}|^2 \to \sum_{j=1}^{\ell} |\mu_j|^2 + \sum_{s\neq t} \overline{\mu_s}\mu_t (1 \wedge c_{s,t})$, which is a consequence of the following lemma.

Lemma 6.11. *Let (Δ_t) be bounded and positive. If $\varepsilon_t \to 0$, $\varepsilon_t \gg 1/\log t$ and $\log \Delta_t/\log \varepsilon_t \to c \in [0,\infty]$, then*

$$\frac{1}{-\log \varepsilon_t} \sum_{p\leqslant t} \frac{p^{\mathrm{i}\Delta_t}}{p^{1+2\varepsilon_t}} \underset{t\to\infty}{\longrightarrow} c \wedge 1.$$

Proof. The first step consists in showing that $\frac{1}{-\log \varepsilon_t} \sum_{p\leqslant t} \frac{p^{\mathrm{i}\Delta_t}}{p^{1+2\varepsilon_t}}$ has the same limit as the infinite sum $\frac{1}{-\log \varepsilon_t} \sum_{p\in\mathcal{P}} \frac{p^{\mathrm{i}\Delta_t}}{p^{1+2\varepsilon_t}}$. In fact, a stronger result holds : as ε_t is sufficiently large ($\varepsilon_t > d/\log t$ for some $d > 0$), $\sum_{p>t} \frac{p^{\mathrm{i}\Delta_t}}{p^{1+2\varepsilon_t}}$ is uniformly bounded :

$$\begin{aligned}
\sum_{p>t} \frac{1}{p^{1+2\varepsilon_t}} &= \sum_{n>t} \frac{\pi(n)-\pi(n-1)}{n^{1+2\varepsilon_t}} \\
&= \sum_{n>t} \pi(n) \left(\frac{1}{n^{1+2\varepsilon_t}} - \frac{1}{(n+1)^{1+2\varepsilon_t}}\right) + \mathrm{o}(1) \\
&= (1+2\varepsilon_t) \int_t^{\infty} \frac{\pi(x)}{x^{2+2\varepsilon_t}} \mathrm{d}x + \mathrm{o}(1),
\end{aligned}$$

and this last term is bounded, for sufficiently large t (remember that $\pi(x) \sim x/\log x$ from the prime number theorem), by

$$2 \int_t^{\infty} \frac{\mathrm{d}x}{x^{1+\frac{d}{\log t}} \log x} = -2 \int_0^{e^{-d}} \frac{\mathrm{d}y}{\log y} < \infty,$$

as shown by the change of variables $y = x^{-d/\log t}$. Therefore the lemma is equivalent to

$$\frac{1}{-\log \varepsilon_t} \sum_{p\in\mathcal{P}} \frac{p^{\mathrm{i}\Delta_t}}{p^{1+2\varepsilon_t}} \underset{t\to\infty}{\longrightarrow} c \wedge 1.$$

The above term has the same limit as

$$\frac{1}{\log \varepsilon_t} \sum_{p\in\mathcal{P}} \log\left(1 - \frac{p^{\mathrm{i}\Delta_t}}{p^{1+2\varepsilon_t}}\right) = \frac{1}{-\log \varepsilon_t} \log \zeta(1+2\varepsilon_t - \mathrm{i}\Delta_t)$$

because $\log(1-x) = -x + \mathrm{O}(|x|^2)$ as $x \to 0$, and $\sum_p 1/p^2 < \infty$. The equivalent $\zeta(1+x) \sim 1/x$ ($x \to 0$) and the condition $\log \Delta_t / \log \varepsilon_t \to c$ yield the conclusion, exactly as in the end of the proof of Lemma 6.7. $\square$

3.4. Proof of Theorem 6.1 for $\varepsilon_t \ll 1/\log t$.

The equivalent of Theorem 6.1 now needs to be proven with

$$a_{pt} = \frac{1}{\sqrt{\log\log t}} \sum_{j=1}^{\ell} \frac{\mu_j}{p^{\frac{1}{2}+\varepsilon_t+\mathrm{i}f_t^{(j)}}}$$

if $p \leqslant t$, 0 if $p > t$. Reasoning as in the previous paragraph, a suitable choice for (m_t) is $m_t = \exp(\log t/\log\log t)$. Therefore, the only remaining condition to check is that, for (Δ_t) bounded and strictly positive such that $-\log \Delta_t/\log\log t \to c$ and $\varepsilon_t \ll 1/\log t$,

$$\frac{1}{\log\log t} \sum_{p \leqslant t} \frac{p^{\mathrm{i}\Delta_t}}{p^{1+\varepsilon_t}} \underset{t\to\infty}{\longrightarrow} c \wedge 1.$$

First note that we can suppose $\varepsilon_t = 0$, because (using $\varepsilon_t < d/\log t$ for some $d > 0$ and once again $|1 - e^{-x}| < x$ for $x > 0$)

$$\left| \sum_{p\leqslant t} \frac{p^{\mathrm{i}\Delta_t}}{p^{1+\varepsilon_t}} - \sum_{p\leqslant t} \frac{p^{\mathrm{i}\Delta_t}}{p^1} \right| \leqslant \sum_{p\leqslant t} \frac{\varepsilon_t \log p}{p} \leqslant \frac{d}{\log t} \sum_{p<t} \frac{\log p}{p} \to d$$

where the last limit makes use of the prime number theorem. The result therefore follows from the lemma below, a strict analogue of Lemma 6.8 used in the context of random matrices.

Lemma 6.12. *Let (Δ_t) be bounded and positive, such that $-\log \Delta_t/\log\log t \to c \in [0,\infty]$. Then*

$$\frac{1}{\log\log t} \sum_{p\leqslant t} \frac{p^{\mathrm{i}\Delta_t}}{p} \underset{t\to\infty}{\longrightarrow} c \wedge 1.$$

Proof. As calculated in the proof of Lemma 6.11,

$$\sum_{p\leqslant t} \frac{p^{\mathrm{i}\Delta_t}}{p} = \sum_{n\leqslant t} \frac{n^{\mathrm{i}\Delta_t}}{n} (\pi(n) - \pi(n-1)) = (1 - \mathrm{i}\Delta_t) \int_e^t \frac{\pi(x) x^{\mathrm{i}\Delta_t}}{x^2} \mathrm{d}x + \mathrm{o}(1).$$

The prime number theorem ($\pi(x) \sim x/\log x$) thus implies

$$\begin{aligned}\sum_{p\leqslant t} \frac{p^{\mathrm{i}\Delta_t}}{p} &= (1-\mathrm{i}\Delta_t) \int_e^t \frac{x^{\mathrm{i}\Delta_t}\mathrm{d}x}{x\log x} + (1-\mathrm{i}\Delta_t)\,\mathrm{o}\left(\int_e^t \frac{\mathrm{d}x}{x\log x}\right) + \mathrm{o}(1) \\ &= (1-\mathrm{i}\Delta_t) \int_{\Delta_t}^{\Delta_t \log t} \frac{e^{\mathrm{i}y}\mathrm{d}y}{y} + (1-\mathrm{i}\Delta_t)\,\mathrm{o}(\log\log t) + \mathrm{o}(1).\end{aligned}$$

If $c > 1$, $\Delta_t \log t \to 0$, so the above term is equivalent to $\int_{\Delta_t}^{\Delta_t \log t} \mathrm{d}y/y = \log\log t$. If $c < 1$, $\Delta_t \log t \to \infty$ so, as $\sup_{x>1} \left|\int_1^x \frac{e^{\mathrm{i}y}}{y}\mathrm{d}y\right| < \infty$, $\frac{1}{\log\log t}\sum_{p\leqslant t} \frac{p^{\mathrm{i}\Delta_t}}{p}$ tends to the same limit as $\int_{\Delta_t}^1 \mathrm{d}y/y = \log \Delta_t/\log\log t \to c$. Finally, if $c = 1$, the distinction between the cases $\Delta_t \log t > 1$ and $\Delta_t \log t < 1$ and the above reasoning give 1 in the limit. $\square$

4. Connection with spatial branching processes.

There is no easy a priori reason why the matrix (6.6) is a covariance matrix. More precisely, given positive numbers $c_1, \dots, c_{\ell-1}$, is there a reason why the symmetric matrix

$$\mathrm{C}_{i,j} = \mathbb{E}(\overline{\mathrm{Y}_i}\mathrm{Y}_j) = \begin{cases} 1 & \text{if} \quad i = j \\ 1 \wedge \inf_{[\![i,j-1]\!]} c_k & \text{if} \quad i < j \end{cases}$$

is positive semi-definite ? This is a by-product of Theorem 6.1, and a possible construction for the Gaussian vector $(\mathrm{Y}_1, \dots, \mathrm{Y}_\ell)$ is as follows. Define the angles $\varphi_n^{(k)}$, $1 \leqslant k \leqslant \ell$, by $\varphi_n^{(1)} = 0$ and

$$\varphi_n^{(k)} = \varphi_n^{(k-1)} + \frac{1}{n^{c_{k-1,k}}},\ 2 \leqslant k \leqslant \ell. \tag{6.20}$$

Let $(\mathcal{X}_r)_{r\geqslant 1}$ be independent standard complex Gaussian variables. For $1 \leqslant k \leqslant \ell$, let

$$\mathrm{Y}_k^{(n)} = \frac{1}{\sqrt{\log n}} \sum_{r=1}^{n} e^{\mathrm{i}r\varphi_n^{(k)}} \frac{\mathcal{X}_r}{\sqrt{r}}.$$

Then $(\mathrm{Y}_1^{(n)}, \dots, \mathrm{Y}_\ell^{(n)})$ is a complex Gaussian vector, and Lemma 6.8 implies that its covariance matrix converges to (6.20).

Instead of finding a Gaussian vector with covariance structure (6.20), we consider this problem : given $c_1, \dots, c_\ell$ positive real numbers, can we find a centered (real or complex) Gaussian vector $(\mathrm{X}_1, \dots, \mathrm{X}_\ell)$ with

$$\mathbb{E}(\mathrm{X}_i\mathrm{X}_j) = \inf_{i\leqslant k\leqslant j} c_k \tag{6.21}$$

for all $i \leqslant j$? A matrix C of type (6.20) can always be obtained as a $\lambda\mathrm{C}' + \mathrm{D}$ with $\lambda > 0$, C' of type (6.21) and D diagonal with positive entries, so the above problem is more general than the original one.

Equation (6.21) is the discrete analogue of the following problem, considered in the context of spatial branching processes by Le Gall (see e.g. [89]). Strictly following his work, we note $e : [0,\sigma] \to \mathbb{R}^+$ a continuous function such that $e(0) = e(\sigma) = 0$. Le Gall associates to such a function e a continuous tree by the following construction : each $s \in [0,\sigma]$ corresponds to a vertex of the tree after identification of s and t ($s \sim t$) if

$$e(s) = e(t) = \inf_{[s,t]} e(r).$$

This set $[0,\sigma]/\sim$ of vertices is endowed with the partial order $s \prec t$ (s is an ancestor of t) if

$$e(s) = \inf_{[s,t]} e(r).$$

Independent Brownian motions can diffuse on the distinct branches of the tree : this defines a Gaussian process B_u with $u \in [0,\sigma]/\sim$ (see [89] for the construction of this diffusion). For $s \in [0,\sigma]$ writing $\mathrm{X}_s = \mathrm{B}_{\overline{s}}$ (where $\overline{s}$ is the equivalence class of s for $\sim$), we get a continuous centered Gaussian process on $[0,\sigma]$ with correlation structure

$$\mathbb{E}(\overline{\mathrm{X}_s}\mathrm{X}_t) = \inf_{[s,t]} e(u), \tag{6.22}$$

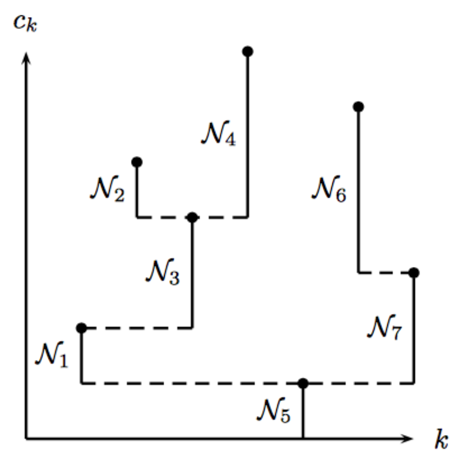
which is the continuous analogue of (6.21). This construction by Le Gall yields a solution of our discrete problem (6.21). More precisely, suppose for simplicity that all the c_i's are distinct (this is not a restrictive hypothesis by a continuity argument), and consider the graph $i \mapsto c_i$. We say that i is an ancestor of j if

$$c_i = \inf_{k\in[\![i,j]\!]} c_k.$$

The father of i is its nearest ancestor, for the distance $d(i,j) = |c_i - c_j|$. It is noted $p(i)$. We can write $c_{\sigma(1)} < \dots < c_{\sigma(\ell)}$ for some permutation σ, and $(\mathcal{N}_1, \dots, \mathcal{N}_\ell)$ a vector of independent centered complex Gaussian variables, $\mathcal{N}_k$ with variance $c_k - c_{p(k)}$ (by convention $c_{p(\sigma(1))} = 0$). Then the Gaussian vector $(\mathrm{X}_1, \dots, \mathrm{X}_\ell)$ iteratively defined by

$$\left\{ \begin{array}{lcl} \mathrm{X}_{\sigma(1)} & = & \mathcal{N}_{\sigma(1)} \\ \mathrm{X}_{\sigma(i+1)} & = & \mathrm{X}_{p(\sigma(i+1))} + \mathcal{N}_{\sigma(i+1)} \end{array} \right.$$

satisfies (6.21), by construction.



Appendix

1. Beta random variables

We recall here some well known facts about the beta variables which are often used in this thesis. A beta random variable $\mathrm{B}_{a,b}$ with strictly positive coefficients a and b has density on $(0,1)$ given by

$$\mathbb{P}\left(\mathrm{B}_{a,b} \in \mathrm{d}t\right) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} t^{a-1}(1-t)^{b-1}\,\mathrm{d}t.$$

Its Mellin transform is $(s>0)$

$$\mathbb{E}\left(\mathrm{B}_{a,b}^{s}\right) = \frac{\Gamma(a+s)}{\Gamma(a)} \frac{\Gamma(a+b)}{\Gamma(a+b+s)}. \tag{7.1}$$

For the uniform measure on the real sphere

$$\mathscr{S}_{\mathbb{R}}^{n} = \{(r_1,\dots,r_n) \in \mathbb{R}^n \ : \ r_1^2+\dots+r_n^2 = 1\},$$

the sum of the squares of the first k coordinates is equal in law to $\mathrm{B}_{\frac{k}{2},\frac{n-k}{2}}$. Consequently, under the uniform measure on the complex sphere $\mathscr{S}_{\mathbb{C}}^{n} = \{(c_1,\dots,c_n) \in \mathbb{C}^n \ : \ |c_1|^2+\dots+|c_n|^2 = 1\}$,

$$c_1 \overset{\text{law}}{=} e^{\mathrm{i}\omega}\sqrt{\mathrm{B}_{1,n-1}},$$

with ω uniform on $(-\pi,\pi)$ and independent from $\mathrm{B}_{1,n-1}$.

2. A remarkable identity in law.

Using Mellin-Fourier transforms, we will prove the following equality in law.

Theorem 7.1. *Let $\mathfrak{Re}(\delta) > -1/2$, $\lambda > 1$. Take independently :*

- *ω uniform on $(-\pi,\pi)$ and $\mathrm{B}_{1,\lambda}$ a beta variable with the indicated parameters ; Y distributed as the $(1-x)^{\overline{\delta}}(1-\overline{x})^{\delta}$-sampling of $x = e^{\mathrm{i}\omega}\sqrt{\mathrm{B}_{1,\lambda}}$;*
- *$\mathrm{B}_{1,\lambda-1}$ a beta variable with the indicated parameters.*
- *Z distributed as the $(1-x)^{\overline{\delta}+1}(1-\overline{x})^{\delta+1}$-sampling of $x = e^{\mathrm{i}\omega}\sqrt{\mathrm{B}_{1,\lambda-1}}$;*

Then

$$\mathrm{Y} - \frac{(1-|\mathrm{Y}|^2)\,\mathrm{B}_{1,\lambda-1}}{1-\overline{\mathrm{Y}}} \overset{\text{law}}{=} \mathrm{Z}.$$

Proof. Actually, we will show that

$$\mathrm{X} = 1 - \left(\mathrm{Y} - \frac{(1-|\mathrm{Y}|^2)\,\mathrm{B}_{1,\lambda-1}}{1-\overline{\mathrm{Y}}}\right) \overset{\text{law}}{=} 1-\mathrm{Z}.$$

First note that, by Lemma 7.4,

$$1-\mathrm{Y} \overset{\text{law}}{=} 2\cos\varphi\, e^{\mathrm{i}\varphi}\,\mathrm{B}_{\lambda+\delta+\overline{\delta}+1,\lambda},$$

where φ has probability density $c\ (1+e^{2\mathrm{i}\varphi})^{\lambda+\delta}(1+e^{-2\mathrm{i}\varphi})^{\lambda+\overline{\delta}}\mathbb{1}_{(-\pi/2,\pi/2)}$ (c is the normalization constant). Consequently, by a straightforward calculation,

$$\mathrm{X} \overset{\text{law}}{=} 2\cos\varphi\ e^{\mathrm{i}\varphi}\left(\mathrm{B}_{\lambda+\delta+\overline{\delta}+1,\lambda} + (1-\mathrm{B}_{\lambda+\delta+\overline{\delta}+1,\lambda})\,\mathrm{B}_{1,\lambda-1}\right).$$

We first suppose that $2(2\lambda+\delta+\overline{\delta}+1)-1\in\mathbb{N}$. Consider the uniform distribution on $\mathscr{S}_{\mathbb{R}}^{2(2\lambda+\delta+\overline{\delta}+1)-1}$. Then the sum of the squares of the first $2(\lambda+\delta+\overline{\delta}+2)$ coordinates is equal in law to $\mathrm{B}_{\lambda+\delta+\overline{\delta}+2,\lambda-1}$, but also to $\mathrm{B}_{\lambda+\delta+\overline{\delta}+1,\lambda} + (1-\mathrm{B}_{\lambda+\delta+\overline{\delta}+1,\lambda})\,\mathrm{B}_{1,\lambda-1}$ by counting the first $2(\lambda+\delta+\overline{\delta}+1)$ coordinates first and then the next two. Hence

$$\mathrm{X} \overset{\text{law}}{=} 2\cos\varphi\ e^{\mathrm{i}\varphi}\,\mathrm{B}_{\lambda+\delta+\overline{\delta}+2,\lambda-1}\,.$$

The result remains true if $2(2\lambda+\delta+\overline{\delta}+1)-1\notin\mathbb{N}$ by analytical continuation. Consequently Lemma 7.3 implies the following Mellin Fourier transform formula

$$\mathbb{E}\left(|\mathrm{X}|^t e^{\mathrm{i}s\arg\mathrm{X}}\right) = \frac{\Gamma(\lambda+\delta+1)\Gamma(\lambda+\overline{\delta}+1)}{\Gamma(\lambda+\delta+\overline{\delta}+2)}\frac{\Gamma(\lambda+\delta+\overline{\delta}+2+t)}{\Gamma(\lambda+\delta+\frac{t+s}{2}+1)\Gamma(\lambda+\overline{\delta}+\frac{t-s}{2}+1)}.$$

Using the following Lemma 7.2, the Mellin Fourier transform of $1-\mathrm{Z}$, coincides with the above expression, completing the proof. □

Lemma 7.2. *Let $\lambda>0$ and $\mathrm{X}=1+e^{\mathrm{i}\omega}\sqrt{\mathrm{B}_{1,\lambda}}$, where ω, uniformly distributed on $(-\pi,\pi)$, is assumed independent from $\mathrm{B}_{1,\lambda}$. Then, for all t and s with $\mathfrak{Re}(t\pm s)>-1$*

$$\mathbb{E}\left(|\mathrm{X}|^t e^{\mathrm{i}s\arg\mathrm{X}}\right) = \frac{\Gamma\left(\lambda+1\right)\Gamma\left(\lambda+1+t\right)}{\Gamma\left(\lambda+1+\frac{t+s}{2}\right)\Gamma\left(\lambda+1+\frac{t-s}{2}\right)}.$$

Proof. First, note that

$$\mathbb{E}\left(|\mathrm{X}|^t e^{\mathrm{i}s\arg\mathrm{X}}\right) = \mathbb{E}\left(\left(1+e^{\mathrm{i}\omega}\sqrt{\mathrm{B}_{1,\lambda}}\right)^a\left(1+e^{-\mathrm{i}\omega}\sqrt{\mathrm{B}_{1,\lambda}}\right)^b\right),$$

with $a=(t+s)/2$ and $b=(t-s)/2$. Recall that if $|x|<1$ and $u\in\mathbb{R}$ then

$$(1+x)^u = \sum_{k=0}^{\infty}\frac{(-1)^k(-u)_k}{k!}x^k,$$

where $(y)_k = y(y+1)\dots(y+k-1)$ is the Pochhammer symbol. As $|e^{\mathrm{i}\omega}\sqrt{\mathrm{B}_{1,\lambda}}|<1$ a.s., we get

$$\mathbb{E}\left(|\mathrm{X}|^t e^{\mathrm{i}s\arg\mathrm{X}}\right) = \mathbb{E}\left(\left(\sum_{k=0}^{\infty}\frac{(-1)^k(-a)_k}{k!}\,\mathrm{B}_{1,\lambda}^{\frac{k}{2}}\,e^{\mathrm{i}k\omega}\right)\left(\sum_{\ell=0}^{\infty}\frac{(-1)^\ell(-b)_\ell}{\ell!}\overline{\mathrm{B}}_{1,\lambda}^{\frac{\ell}{2}}e^{-\mathrm{i}\ell\omega}\right)\right).$$

After expanding this double sum (it is absolutely convergent because $\mathfrak{Re}(t\pm s)>-1$), all terms with $k\neq\ell$ will give an expectation equal to 0. Hence, since $\mathbb{E}(\mathrm{B}_{1,\lambda}^k) = \frac{\Gamma(1+k)\Gamma(\lambda+1)}{\Gamma(1)\Gamma(\lambda+1+k)} = \frac{k!}{(\lambda+1)_k}$, we obtain

$$\mathbb{E}\left(|\mathrm{X}|^t e^{\mathrm{i}s\arg\mathrm{X}}\right) = \sum_{k=0}^{\infty}\frac{(-a)_k(-b)_k}{k!(\lambda+1)_k}.$$

Note that this series is equal to the value at $z=1$ of the hypergeometric function ${}_2\mathrm{F}_1(-a,-b,\lambda+1;z)$. Hence Gauss formula (3.7) yields :

$$\mathbb{E}\left(|\mathrm{X}|^t e^{\mathrm{i}s\arg\mathrm{X}}\right) = \frac{\Gamma(\lambda+1)\Gamma(\lambda+1+a+b)}{\Gamma(\lambda+1+a)\Gamma(\lambda+1+b)}.$$

This is the desired result. □

Lemma 7.3. *Take φ with probability density $c\ (1+e^{2\mathrm{i}\varphi})^{\overline{z}}(1+e^{-2\mathrm{i}\varphi})^{z}\mathbb{1}_{(-\pi/2,\pi/2)}$, where c is the normalization constant, $\mathfrak{Re}(z)>-1/2$. Let $\mathrm{X}=2\cos\varphi\ e^{\mathrm{i}\varphi}$. Then*

$$\mathbb{E}\left(|\mathrm{X}|^t e^{\mathrm{i}s\arg \mathrm{X}}\right)=\frac{\Gamma\left(z+1\right)\Gamma\left(\overline{z}+1\right)}{\Gamma\left(z+\overline{z}+1\right)}\frac{\Gamma(z+\overline{z}+t+1)}{\Gamma(\overline{z}+\frac{t+s}{2}+1)\Gamma(z+\frac{t-s}{2}+1)}.$$

Proof. From the definition of X,

$$\mathbb{E}\left(|\mathrm{X}|^t e^{\mathrm{i}s\arg \mathrm{X}}\right)=c\int_{-\pi/2}^{\pi/2}\left(1+e^{2\mathrm{i}x}\right)^{\overline{z}+\frac{t+s}{2}}\left(1+e^{-2\mathrm{i}x}\right)^{z+\frac{t-s}{2}}\mathrm{d}x.$$

Both terms on the RHS can be expanded as a series in $e^{2\mathrm{i}x}$ or $e^{-2\mathrm{i}x}$ for all $x\neq 0$. Integrating over x between $-\pi/2$ and $\pi/2$, only the diagonal terms remain, hence

$$\mathbb{E}\left(|\mathrm{X}|^t e^{\mathrm{i}s\arg \mathrm{X}}\right)=c\sum_{k=0}^{\infty}\frac{\left(-\overline{z}-\frac{+t+s}{2}\right)_k\left(-z-\frac{t-s}{2}\right)_k}{(k!)^2}.$$

The value of a ${}_2\mathrm{F}_1$ hypergeometric function at $z=1$ is given by the Gauss formula (3.7), hence

$$\mathbb{E}\left(|\mathrm{X}|^t e^{\mathrm{i}s\arg \mathrm{X}}\right)=c\,\frac{\Gamma(z+\overline{z}+t+1)}{\Gamma(\overline{z}+\frac{t+s}{2}+1)\Gamma(z+\frac{t-s}{2}+1)}.$$

As this is 1 when $s=t=0$, $c=\frac{\Gamma(z+1)\Gamma(\overline{z}+1)}{\Gamma(z+\overline{z}+1)}$, which completes the proof. $\square$

Lemma 7.4. *Let $\lambda>2$, $\mathfrak{Re}(\delta)>-1/2$, ω uniform on $(-\pi,\pi)$ and $\mathrm{B}_{1,\lambda-1}$ a beta variable with the indicated parameters. Let Y be distributed as the $(1-\overline{x})^{\delta}(1-x)^{\overline{\delta}}$-sampling of $x=e^{\mathrm{i}\omega}\sqrt{\mathrm{B}_{1,\lambda-1}}$. Then*

$$1-\mathrm{Y}\overset{\text{law}}{=}2\cos\varphi\ e^{\mathrm{i}\varphi}\,\mathrm{B}_{\lambda+\delta+\overline{\delta},\lambda-1},$$

with $\mathrm{B}_{\lambda+\delta+\overline{\delta},\lambda-1}$ a beta variable with the indicated parameters and, independently, φ having probability density $c\ (1+e^{2\mathrm{i}\varphi})^{\lambda+\delta-1}(1+e^{-2\mathrm{i}\varphi})^{\lambda+\overline{\delta}-1}\mathbb{1}_{(-\pi/2,\pi/2)}$ (c is the normalization constant).

Proof. The Mellin Fourier transform of $\mathrm{X}=1-\mathrm{Y}$ can be evaluated using Lemma 7.2, and equals

$$\mathbb{E}\left(|\mathrm{X}|^t e^{\mathrm{i}s\arg \mathrm{X}}\right)=\frac{\Gamma\left(\lambda+\delta\right)\Gamma\left(\lambda+\overline{\delta}\right)\Gamma\left(\lambda+t+\delta+\overline{\delta}\right)}{\Gamma\left(\lambda+\frac{t+s}{2}+\delta\right)\Gamma\left(\lambda+\frac{t-s}{2}+\overline{\delta}\right)\Gamma\left(\lambda+\delta+\overline{\delta}\right)}.$$

On the other hand, using Lemma 7.3 and (7.1), the Mellin Fourier transform of $2\cos\varphi\ e^{\mathrm{i}\varphi}\,\mathrm{B}_{n+\delta+\overline{\delta},\lambda}$ coincides with the above result. $\square$

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