The Paley–Wiener excess and a local law for the Sine_β process

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Abstract: The main result of this paper is the proof that the Paley–Wiener excess of the Sine_β process almost surely equals $\lfloor\sqrt{2/\beta}\rfloor$ for all $\beta > 0$. We also derive the loop equations for the resolvent of the Sine_β point process and use them to obtain accurate bounds on the moments of the associated Stieltjes transform.

1 Introduction and statement of the main results

Let PW denote the Paley–Wiener space, that is the space of functions on $\mathbb{R}$ which are the Fourier transform of $L^2(\mathbb{R})$ functions supported in $[-\pi, \pi]$, where the convention for the Fourier transform in this paper is

$$\widehat{f}(x) = \frac{1}{2\pi}\int_{\mathbb{R}} f(t)e^{-\mathrm{i}tx}\,\mathrm{d}t$$

so that

$$f(t) = \int_{\mathbb{R}} \hat{f}(x)e^{\mathrm{i}tx}\,\mathrm{d}x.$$

We say that an arbitrary countable collection $\mathcal{X}$ of distinct points in $\mathbb{R}$ is *a uniqueness set* for PW (in short: uniqueness set) if there does not exist a non-zero $f \in \mathsf{PW}$ so that $f(\lambda) = 0$ for all $\lambda \in \mathcal{X}$. Equivalently, the system of complex exponentials $\{e^{\mathrm{i}t\lambda},\, t \in [-\pi, \pi]\}_{\lambda\in\mathcal{X}}$ is complete in $L^2[-\pi, \pi]$.

For $k \in \mathbb{Z}$, we say that $\mathcal{X}$ has Paley–Wiener excess k (in short, excess k) if removing any k points from $\mathcal{X}$ gives a uniqueness set, while removing any $k + 1$ points creates a set which is not a uniqueness set[1,2]. Thus, $\mathcal{X}$ has excess 0 if the system of complex exponentials $\{e^{\mathrm{i}t\lambda},\, t \in [-\pi, \pi]\}_{\lambda\in\mathcal{X}}$ is complete and minimal in $L^2[-\pi, \pi]$. In general, the excess of a set may be infinite, but in Proposition 1.3 below we show that for (deterministic) regular sets satisfying appropriate growth conditions, the excess is well defined and finite.

[1] In the Paley–Wiener setting, if this happens for one choice of $k + 1$ removed points, then it happens for every such choice.

[2] By removing a negative number of points we mean adding that many distinct points to the set $\mathcal{X}$.

We study in this paper the random infinite point process Sine_β on $\mathbb{R}$, introduced by Killip and Stoiciu [21] and by Valkó and Virág [30], which is defined as the limit (in distribution) of the process of eigenvalues of the β-ensemble of random matrices, rescaled to have mean spacing equal to 1, see Section 3. Throughout the paper we denote this process by $\mathcal{X}_\beta$. Our main result gives the excess of $\mathcal{X}_\beta$, as follows.

Theorem 1.1. *For $\beta > 0$, the process $\mathcal{X}_\beta$ almost surely has excess $\lfloor \sqrt{2/\beta} \rfloor$.*

We note that for $\beta = 2$, Alexander Bufetov [9] proved Theorem 1.1, using the determinantal structure of $\mathcal{X}_2$. This work is inspired by his results[3].

Similarly to [9], our proof of Theorem 1.1 is based on a general deterministic criterion, which reduces the Paley–Wiener excess problem to growth- and L^2-estimates for the canonical product with zeros at the point configuration $\mathcal{X}$. This criterion requires a certain regularity of $\mathcal{X}$.

Assumption 1.2. *We say a set $\mathcal{X}$ is regular if the following hold.*

$$\lim_{R\to\infty} \frac{|\mathcal{X} \cap [0,R]|}{R} = 1, \tag{1.1}$$

$$\lim_{R\to\infty} \frac{|\mathcal{X} \cap [-R,0]|}{R} = 1, \tag{1.2}$$

$$\ell_{\mathcal{X}} := \lim_{R\to\infty} \ell_{\mathcal{X}}(R) := \lim_{R\to\infty} \sum_{\substack{|\lambda|\leqslant R,\\ \lambda\in\mathcal{X}}} \frac{1}{\lambda} \quad \text{exists.} \tag{1.3}$$

For regular $\mathcal{X}$, introduce the function

$$G_{\mathcal{X}}(z) = \lim_{R\to\infty} \prod_{\substack{|\lambda|\leqslant R,\\ \lambda\in\mathcal{X}}} (1 - \frac{z}{\lambda}) \tag{1.4}$$

which exists due to (1.3). The criterion we use is the following.[4]

Proposition 1.3. *Suppose that $\mathcal{X}$ is a regular set.*

(a) *If*

$$G_{\mathcal{X}}(t+\mathrm{i})(1+|t|)^{-(k_*+1)} \in L^2(\mathbb{R}), \tag{1.5}$$

then the excess of $\mathcal{X}$ is at most k_*.

(b) *If for some $A \geqslant 0$,*

$$\liminf_{\eta\to+\infty} |G_{\mathcal{X}}(\mathrm{i}\eta)|(1+\eta)^A e^{-\pi\eta} > 0 \tag{1.6}$$

[3] After this work was completed, we learnt about an alternative argument by S. Gorbunov [17] for non-integer $\sqrt{2/\beta} > 1$.

[4] After this paper was completed, we learnt of A. Bufetov's paper [10], where a version of Proposition 1.3 appears as Proposition 1.

and further

$$G_{\mathcal{X}}(t+\mathrm{i})(1+|t|)^{-k_*} \notin L^2(\mathbb{R}), \tag{1.7}$$

then the excess of $\mathcal{X}$ is at least k_*.

(c) *Furthermore, set $\Lambda = \alpha\mathbb{Z}$ with $0 < \alpha < 1$. Then (1.7) and (1.5) can be replaced by*

$$G_{\mathcal{X}}(t+\mathrm{i})(1+|t|)^{-k_*} \notin \ell^2(\Lambda), \tag{1.8}$$

$$G_{\mathcal{X}}(t+\mathrm{i})(1+|t|)^{-(k_*+1)} \in \ell^2(\Lambda), \tag{1.9}$$

with the same conclusion.

This proposition will suffice for our purpose except at the discontinuity points $2/\beta = \ell^2$, $\ell = 1, 2, \dots$, of the excess. To tackle these points, we will also use another deterministic result.

Proposition 1.4. *Suppose that $\mathcal{X}$ is a regular set, and that $G_{\mathcal{X}}$ satisfies the lower bound (1.6) on the imaginary axis. Let $\eta\colon \mathbb{R} \to [1, \infty)$ be a locally bounded measurable function satisfying*

$$\eta(t) = o(|t|), \qquad t \to \pm\infty.$$

Suppose that, for some $k \in \mathbb{Z}$,

$$\int_{\mathbb{R}} e^{-2\pi\eta(t)}(1+|t|)^{-2k}|G_{\mathcal{X}}(t+\mathrm{i}\eta(t))|^2 \,\mathrm{d}t = \infty. \tag{1.10}$$

Then the set $\mathcal{X}$ with any k points removed is a uniqueness set for the Paley–Wiener space.

In other words, under the assumptions of Proposition 1.4, the excess of the set $\mathcal{X}$ is at least k (which does not exclude the possibility of infinite excess).

The technical heart of the proof of Theorem 1.1 is an evaluation of exponential moments of the logarithmic potential associated with $\mathcal{X}_\beta$, which in turn is based on a local law for the Stieltjes transform of the Sine$_\beta$ process. We state and prove these results in Section 3.[5] The derivation is based on a similar one for the β-ensembles presented in [8]. Once these are in place, in Section 4 we employ methods from the theory of logarithmically correlated fields to obtain the proof of Theorem 1.1. In Section 5, we prove a version of an optimal local law for the Sine$_\beta$ process, with accurate dependence on β. This version will not be used in the proof of Theorem 1.1.

[5] There is some overlap between Sections 3.1, 3.2 and 5 in this paper and Sections 3.2-3.4 and Appendix B in [6]. We provide here proofs with full details. See also [14] for local laws for finite N and explicit dependence on β.

Previous work. Shannon's sampling theorem [28] and its predecessors [27, 31] deal with sampling a function in PW and thus can be translated to the statement that the regular lattice of spacing 1 has non-negative excess (of course, we know that its excess is indeed 0 by just looking at the Fourier transform of $\sin(\pi x)/x$); we refer to [18] for a review of the rich (mostly applied) literature on extensions of this basic result to other deterministic and random sampling models. On the mathematical complex-analytic side, such questions are discussed in [26, Lecture 18], see in particular Theorem 4. Our treatment in Section 2 follows the latter reference.

Our results in Section 3 build on the techniques of [8]. We note that for the case $\beta = 2$, similar results (including convergence toward a GMC), based on determinantal techniques, were derived in [23].

As mentioned before, our work is motivated by [9], who considered the case $\beta = 2$ by determinantal methods. We also note [12], where a Gaussian Multiplicative Chaos convergence (sharper than the arguments we use) is announced in the determinantal case $\beta = 2$; see also [11], where relevant computations are carried out by determinantal techniques, and [10] where a general condition for the computation of the excess of a point process $\mathcal{X}$ is given in terms of the two and four-point correlation functions of $G_{\mathcal{X}}$.

An earlier example of a random set having excess 0 is the model of a randomly perturbed lattice considered by Chistyakov and Lyubarskii in [13, Theorem 5]. We are not aware of an explicit construction in the literature of a stationary random process of arbitrary Paley–Wiener excess.

In a work in progress [3], Borichev and Sodin studied a similar question in the Bargmann–Fock Hilbert space B of entire functions endowed with the scalar product

$$\langle F, G\rangle_{\mathsf{B}} = \int_{\mathbb{C}} F(z)\overline{G(z)}e^{-\pi|z|^2}\, \mathrm{d}m(z),$$

where m is the planar Lebesgue measure, for the random point process obtained by perturbation of the lattice $\mathbb{Z}^2$ by independent complex Gaussians with variance b^2. They show that the Bargmann–Fock excess of this process almost surely equals $\lfloor d(b)\rfloor$, where

$$d(b) = \begin{cases} 1 + \pi b^2, & b \leqslant 1/\sqrt{\pi}; \\ 2\sqrt{\pi}b, & b \geqslant 1/\sqrt{\pi}. \end{cases}$$

The proofs in [3] are also based on the theory of log-correlated fields. In their case, the log-correlated fields are planar and Gaussian.

Notation.

- Throughout the paper, c and C denote various positive constants that may depend on β, unless explicitly stated otherwise. In Section 5 they will denote numerical constants. Dependence on any other parameters will be indicated explicitly. We tacitly assume that $c < 1 < C$, and that decreasing the value of c or increasing the value of C does not cause any damage. The values of these constants may vary from line to line.

- For A, B positive, we write $A \lesssim B$ if there exists a constant C so that $A \leqslant CB$. We also write $A = O(B)$ in this case. Then $A \gtrsim B$ means that $B \lesssim A$ and $A \simeq B$ means that $A \lesssim B$, and $A \gtrsim B$.
- We use the notation $A = o_j(1)$ if $A = A(j)$ is such that $A(j) \to 0$ as $j \to \infty$.
- The following notation is introduced in Section 3. $\mu_N = \mu_{N,\beta}$ denotes the distribution law of the β-ensemble, $\mathbb{E}_N = \mathbb{E}_{N,\beta}$ denotes the expectation with respect to μ_N, and s_N denotes the corresponding random empirical Stieltjes transform

$$s_N(z) = \frac{1}{N} \sum_{k=1}^{N} \frac{1}{\lambda_k - z} \,.$$

- μ_{sc} denotes the semicircle law, ρ_{sc} denotes its density, and m_{sc} denotes its Stieltjes transform.
- $\mathcal{X}_\beta$ denotes the limiting Sine_β process, $\mathbb{P}_\beta$ denotes its law, $\mathbb{E}_\beta$ denotes the expectation with respect to $\mathbb{P}_\beta$, and s denotes the corresponding random Stieltjes transform defined by

$$s(z) = \lim_{K \to \infty} \sum_{\lambda \in \mathcal{X}_\beta, |\lambda| \leqslant K} \frac{1}{\lambda - z} \,.$$

- We often write $\sum_\lambda$ as shorthand for $\sum_{\lambda \in \mathcal{X}_\beta}$, understood in the principal value sense.
- $\mathbb{H}$ denotes the upper half-plane in $\mathbb{C}$.

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2 Entire functions preliminaries: Proof of Propositions 1.3 and 1.4

We begin by collecting several facts about the Paley–Wiener space. Our standard references are [25] and [26]. Recall that $F \in \mathsf{PW}$ means

$$F(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-\mathrm{i}xt} f(t)\, \mathrm{d}t, \quad \text{for some } f \in L^2[-\pi, \pi].$$

Such F has an entire extension of exponential type, as defined below.

Definition 2.1. *An entire function F has finite exponential type if there exist constants A_F and C_F such that $|F(z)| \leqslant C_F e^{A_F|z|}$, for all $z \in \mathbb{C}$. The infimum of A_F for which this property holds, denoted σ_F, is called the exponential type of F.*

We have the following.

(R_1) $F \in \mathsf{PW}$ implies that F has an analytic continuation to $\mathbb{C}$, and $\sigma_F \leqslant \pi$.

(R_2) By Cauchy–Schwarz, $F \in \mathsf{PW}$ implies that $|F(z)| \leqslant c_F e^{\pi|\eta|}/\sqrt{1+|\eta|}$, with $z \in \mathbb{C}$ and $\eta = \operatorname{Im} z$.

(R_3) $F \in \mathsf{PW}$ is equivalent to F entire, $\sigma_F \leqslant \pi$ and $F \in L^2(\mathbb{R})$ (this is the Paley–Wiener theorem). It follows that if $\sigma_F \leqslant \pi$ and $F \in L^2(\mathbb{R}+\mathrm{i}\eta)$ for some $\eta \in \mathbb{R}$, then $F \in \mathsf{PW}$.

Recall the entire function $G_{\mathcal{X}}$ defined in (1.4). We have the following.

Lemma 2.2. *If $\mathcal{X}$ is regular, then $\sigma_{G_{\mathcal{X}}} = \pi$ and, for any $\delta > 0$,*

$$\log|G_{\mathcal{X}}(re^{\mathrm{i}\theta})| = \pi|\sin(\theta)|r + o(r),$$

uniformly in $|\theta| \in [\delta, \pi-\delta]$.

Proof. Write

$$f_R(z) = \prod_{\substack{\lambda\in\mathcal{X},\\ |\lambda|\leqslant R}} \left(1-\frac{z}{\lambda}\right) \cdot \prod_{\substack{\lambda\in\mathcal{X},\\ |\lambda|>R}} \left(1-\frac{z}{\lambda}\right) e^{z/\lambda}.$$

Then,

$$G_{\mathcal{X}}(z) = e^{(\ell_{\mathcal{X}}(R)-\ell_{\mathcal{X}})z} f_R(z),$$

which gives

$$\frac{\log|G_{\mathcal{X}}(Re^{\mathrm{i}\theta})|}{R} = (\ell_{\mathcal{X}}(R)-\ell_{\mathcal{X}})\cos(\theta) + \frac{\log|f_R(Re^{\mathrm{i}\theta})|}{R}.$$

By [25, Lemma 9 in Ch.I, §17] (with $\rho = \Delta = 1, \rho(r) = r$), we get that

$$\lim_{R\to\infty} \frac{\log|f_R(Re^{\mathrm{i}\theta})|}{R} = -\operatorname{Re}\left[(1-\mathrm{i}(\theta-\pi))e^{\mathrm{i}\theta} + (1-\mathrm{i}\theta)e^{\mathrm{i}(\theta+\pi)}\right] = \pi\sin\theta,$$

uniformly in $\theta \in [\delta, \pi-\delta]$. The argument for $\theta \in [-\pi+\delta, -\delta]$ is very similar. □

We are now ready to give the proof of Propositions 1.3 and 1.4.

Proof of Proposition 1.3. We assume without loss of generality that $0 \notin \mathcal{X}$. For $k \in \mathbb{Z}$, set $\mathcal{X}_k$ as the set obtained by removing (arbitrary) k points from $\mathcal{X}$ if $k \geqslant 0$, and by adding (arbitrary) $|k|$ points if $k < 0$. Clearly,

$$|G_{\mathcal{X}_k}(z)| \lesssim (1+|z|)^{-k}|G_{\mathcal{X}}(z)|, \quad \text{as } |z| \to \infty. \tag{2.1}$$

Note that by (1.5), we have that $G_{\mathcal{X}_k}(t+\mathrm{i}) \in L^2(\mathbb{R})$, if $k \geqslant k_* + 1$. This implies by (R_3) above that for such k, $G_{\mathcal{X}_k} \in \mathsf{PW}$, which in turn implies that the excess of $\mathcal{X}$ is at most k_*.

To see that the excess is at least k_*, we use (1.7). We need to prove that $\mathcal{X}_{k_*}$ is a uniqueness set. Assuming $F \in \mathsf{PW}$ vanishes on $\mathcal{X}_{k_*}$, we will show that F vanishes everywhere. By [26, Theorem 1, Lecture 2], we have that the function $H := F/G_{\mathcal{X}_{k_*}}$ is entire of finite exponential type. We claim its exponential type is 0. Indeed, fix $\varepsilon, \delta > 0$ small, then with $|\theta| \in [\delta, \pi - \delta]$ and $r \geqslant r(\varepsilon, \delta)$ we have, using (R_2) and Lemma 2.2, that

$$\log|H(re^{\mathrm{i}\theta})| = \log|F(re^{\mathrm{i}\theta})| - \log|G_{\mathcal{X}_{k_*}}(re^{\mathrm{i}\theta})| \leqslant \pi r|\sin(\theta)| - \pi r|\sin(\theta)| + \varepsilon r \leqslant \varepsilon r.$$

In particular the Phragmén–Lindelöf indicator of growth of H satisfies

$$k_H(\theta) := \limsup_{r\to\infty} \frac{\log|H(re^{\mathrm{i}\theta})|}{r} \leqslant \varepsilon|\sin(\theta)|. \tag{2.2}$$

Since k_H is trigonometrically convex (see [26, Section 8, Theorem 1]), for $|\theta| \leqslant \delta$ we have $k_H(\theta) \leqslant \varepsilon \sin(\delta)\cos(\theta)/\cos(\delta)$, with a similar estimate for $|\theta - \pi| \leqslant \delta$. Together with (2.2), this shows that $\sigma_H = 0$. We next claim that H is actually a polynomial. Indeed, from (R_2) together with (1.6) we have that for $\eta \in \mathbb{R}_+$,

$$|H(\mathrm{i}\eta)| = \frac{|F(\mathrm{i}\eta)|}{|G_{\mathcal{X}_{k_*}}(\mathrm{i}\eta)|} \lesssim \frac{(1+\eta)^{-1/2+k_*}e^{\pi\eta}}{(1+\eta)^{-A}e^{\pi\eta}} = (1+\eta)^{C(k_*,A)}, \tag{2.3}$$

for some constant $C(k_*, A)$. Since H has exponential type 0, the Hadamard factorization theorem implies that, if it is not a polynomial, then it has infinitely many zeroes. Consider a polynomial P of degree at least $C(k_*, A)$ that divides H, and consider the (entire) function $\widetilde{H} = H/P$, which also has exponential type 0. Further, it stays bounded on the imaginary axis (because P divides H). It follows from the Phragmén–Lindelöf theorem, see e.g. [26, Section 6.1, Theorem 3] or [25, Corollary to Theorem 22, Ch. I] that $\widetilde{H}$ is bounded in the whole complex plane, and thus constant. It follows that H is a polynomial.

On the other hand, because $F = HG_{\mathcal{X}_{k_*}} \in \mathsf{PW}$, we have that $HG_{\mathcal{X}_{k_*}} \in L^2(\mathbb{R}+\mathrm{i})$ (see [26, Theorem 4 in Lecture 7]), which implies that, as a function of t,

$$G_{\mathcal{X}}(t+\mathrm{i})(1+|t|)^{-k_*+\deg(H)} \in L^2(\mathbb{R}),$$

which contradicts (1.7) unless $H = 0$.

The proof of the claim concerning (1.8) and (1.9) proceeds along similar lines. Indeed, by the Plancherel–Pólya theorem ([26, Theorem 5 in Lecture 21] which requires $\alpha < 1$ in our setting), we have that if F is an entire function of exponential type π then

$$F \in \ell^2(\Lambda) \quad \text{implies that} \quad F \in \mathsf{PW} \quad \text{and} \quad \|F\|_{L^2(\mathbb{R})} \asymp \|F\|_{\ell^2(\Lambda)}. \tag{2.4}$$

So, if (1.9) holds then we deduce that (1.5) holds and therefore the excess is at most k_*. On the other hand, following the proof in the case of (1.7) yields as before (under the assumption that $F \in \mathsf{PW}$ vanishes on $\mathcal{X}_{k_*}$) that H is a polynomial, which in turn yields that

$$G_{\mathcal{X}_{k_*}}(t+\mathrm{i}\eta)(1+|t|)^{\deg(H)} \in L^2(\mathbb{R}), \text{ uniformly in } |\eta| \leqslant 2$$

(here we used again [26, Theorem 4 in Lecture 7]). Recalling that for any analytic function f we have

$$|f(z)|^2 \leqslant \pi^{-1} \iint\limits_{\{w\colon |w-z|\leqslant 1\}} |f(w)|^2 \,\mathrm{d}^2 w,$$

we obtain that

$$G_{\mathcal{X}}(t+\mathrm{i})(1+|t|)^{-k_*+\deg(H)} \in \ell^2(\Lambda),$$

which contradicts (1.8) unless $H = 0$. □

Proof of Proposition 1.4. To simplify writing, we assume that $k \geqslant 0$ (the same proof works for $k < 0$ when we add to $\mathcal{X}$ any new $|k|$ points). Let $\mathcal{X}_k$ be $\mathcal{X}$ with any k points removed, and suppose that $\mathcal{X}_k$ is not a uniqueness set, that is, there is $F \in \mathsf{PW} \setminus \{0\}$ that vanishes on $\mathcal{X}_k$.

First, as in the proof of Proposition 1.3, we claim that $F = HG_{\mathcal{X}_k}$, where H is a polynomial. This step only used the regularity of $\mathcal{X}$ and the lower bound (1.6), so the argument applies without change.

Next we use several basic facts about the Hardy space H^2_+ in the upper half-plane, see for instance [26, Lecture 19]. Put

$$\widetilde{F}(z) = e^{\mathrm{i}\pi z} F(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(s) e^{-\mathrm{i}z(s-\pi)} \,\mathrm{d}s = \frac{1}{2\pi} \int_{-2\pi}^{0} f(s+\pi) e^{-\mathrm{i}zs} \,\mathrm{d}s\,.$$

Then, by the half-plane version of the Paley–Wiener theorem [26, Section 19.4], $\widetilde{F} \in H^2_+$.

The map $t \mapsto t + \mathrm{i}\eta(t)$ maps the Lebesgue measure on $\mathbb{R}$ to a measure μ on the graph $\Gamma_\eta = \{t + \mathrm{i}\eta(t)\colon t \in \mathbb{R}\}$ (in our application in Section 4, the function η will be piecewise constant, $\eta(t) = \eta_j$, $t \in \mathfrak{T}_j = [e^j, e^{j+1}]$, so the measure μ will be the Lebesgue measure on the intervals $\mathfrak{T}_j + \mathrm{i}\eta_j$). Note that, for every square of the form $Q_I = I \times (0, |I|\,] \subset \mathbb{H}$ with I an interval, we have

$$\mu(Q_I) = |\{t \in I\colon 0 < \eta(t) \leqslant |I|\}| \leqslant |I|,$$

i.e., the measure μ has the Carleson norm at most 1. Hence, by the Carleson embedding theorem, see [22, Theorem VIII-F] or [15, Chapter 1, Theorem 5.6], applied to $\widetilde{F}^2$,

$$\|\widetilde{F}\|_{L^2(\mu)} \leqslant C \|\widetilde{F}\|_{H^2_+} < \infty.$$

That is,

$$\int_{\mathbb{R}} e^{-2\pi\eta(t)} |F(t+\mathrm{i}\eta(t))|^2 \,\mathrm{d}t = \int_{\mathbb{R}} |\widetilde{F}(t+\mathrm{i}\eta(t))|^2 \,\mathrm{d}t < \infty.$$

Now, let $d = \deg H$. Then on the graph Γ_η we have

$$|F(z)|^2 = |H(z) G_{\mathcal{X}_k}(z)|^2 \simeq |z|^{2(d-k)} |G_{\mathcal{X}}(z)|^2, \qquad z \to \infty,\ z \in \Gamma_\eta.$$

Hence,

$$\int_{\mathbb{R}} e^{-2\pi\eta(t)} |G_{\mathcal{X}}(t+\mathrm{i}\eta(t))|^2 (1+|t+\mathrm{i}\eta(t)|)^{2(d-k)} \,\mathrm{d}t < \infty.$$

Since $d \geqslant 0$ and $|t + \mathrm{i}\eta(t)| \simeq |t|$, this contradicts the divergence assumption (1.10). □

3 Rigidity and fluctuations for the Sine$_\beta$ processes

In this section, we derive quantitative bounds for the counting measure of the Sine$_\beta$ process (Lemma 3.6) and for its Stieltjes transform (Corollary 3.9). Then we describe in Theorem 3.10 the logarithmic asymptotics of the Laplace transform of the Sine$_\beta$ process, which yield logarithmic correlations and asymptotic normality of the process.

3.1 β-ensembles and convergence to the* Sine$_\beta$ *process. The β-ensembles are both a generalization of the Gaussian orthogonal, unitary and symplectic ensembles, and a natural statistical physics model, the 1D log-gas. Their distribution, on N points $\lambda_1 \leqslant \ldots \leqslant \lambda_N$, is given by the probability measure $\mu_N = \mu_{\beta,N}$ defined as

$$\mathrm{d}\mu_N(\lambda_1, \ldots, \lambda_N) \coloneqq \frac{1}{Z_N} \prod_{1 \leqslant k < l \leqslant N} |\lambda_k - \lambda_l|^\beta e^{-\frac{\beta N}{2} \sum_{k=1}^N V(\lambda_k)} \, \mathrm{d}\lambda_1 \ldots \mathrm{d}\lambda_N. \tag{3.1}$$

In this paper, $\beta > 0$ is fixed and our assumptions on V are the following.

(A1) V is analytic on $\mathbb{R}$.

(A2) There exist positive constants M_0, $C = C_V$, and $c = c_V$ such that

$$V'(x) \geqslant c \quad \text{and} \quad \sup_{y \in [M_0, x]} \frac{V'(y)}{y} \leqslant C V(x) \quad \text{for all} \quad x \geqslant M_0,$$

and similar estimates apply for $x \leqslant -M_0$, i.e. the above holds for $\widetilde{V}(x) := V(-x)$.

(A3) Under the previous assumptions, it is known that the sequence of probability measures $\mathbb{E}[N^{-1} \sum_{i=1}^N \delta_{\lambda_i}]$ converges weakly to a probability measure μ_V, with density ϱ_V. We assume ϱ_V is positive and supported on a single interval $[A, B]$ (one-cut hypothesis), with square root singularities at A and B ("generic behaviour").

(A4) We also assume that the function

$$x \mapsto V(x)/2 - \int \log|x - t| \, \mathrm{d}\mu_V(t)$$

achieves its minimum value only on the interval $[A, B]$.

Under these assumptions it is well-known that in the bulk of the spectrum the point process converges to a universal point process. The version we need is at fixed energy, as proved in Valkó–Virág [30] for quadratic V (which is all we need) or in Shcherbina [29, Theorem 2], Landon–Sosoe–Yau [24], Bourgade–Erdős–Yau [5] and Bekerman-Figalli-Guionnet [1] for more general V.

Theorem 3.1. *Let $E \in (A, B)$. The sequence of point processes $\sum_{i=1}^N \delta_{N\varrho_V(E)(\lambda_i - E)}$ converges in distribution as $N \to \infty$ to a translation-invariant point process $\mathcal{X}_\beta$ with intensity 1, which does not depend on E and V satisfying the above constraints.*

The limiting process $\mathcal{X}_\beta$ is called the Sine$_\beta$ process, and its law is denoted $\mathbb{P}_\beta$.

3.2 Rigidity. Given an interval $I \subset \mathbb{R}$ of length $|I|$, we aim to bound the probability that the deviation $|\,\#\mathcal{X}_\beta(I) - |I|\,|$ becomes large.

From now on, we choose $V(x) = x^2/2$ and, for convenience, $E = 0$, and set

$$\rho_{\mathrm{sc}}(x) = \frac{1}{2\pi}\sqrt{4-x^2}\,\mathbf{1}_{[-2,2]}(x), \quad \mathrm{d}\mu_{\mathrm{sc}}(x) = \rho_{\mathrm{sc}}(x)\,\mathrm{d}x.$$

With $\mathbb{E}_N = \mathbb{E}_{N,\beta}$ denoting expectation with respect to μ_N and $\mathbb{E}_\beta$ expectation with respect to $\mathbb{P}_\beta$, Theorem 3.1 (as stated in [29], [24, Theorem 2.4]) implies that, for any $p \geqslant 1$ and $f_1, \dots f_p$, smooth compactly supported,

$$\mathbb{E}_N\Big[\prod_{n=1}^{p}\sum_i f_n\big(\frac{N\lambda_i}{\pi}\big)\Big] \to \mathbb{E}_\beta\Big[\prod_{n=1}^{p}\sum_{\lambda\in\mathcal{X}_\beta} f_n(\lambda)\Big], \quad N\to\infty. \tag{3.2}$$

This convergence will be applied to functions f_n of the form $g\mathbb{1}_I$, where g is smooth and I an interval, using a standard approximation of $\mathbb{1}_I$ by its upper and lower smooth envelopes. The approximation errors are controlled by the fact that $\mathbb{P}(\#(\mathcal{X}_\beta \cap [a, a+\varepsilon]) \geqslant 1) \to 0$ as $\varepsilon \to 0$ (which follows from stationarity of Sine_β).

We introduce the following notation. For z with $\operatorname{Im} z \neq 0$, we let

$$s_N(z) = \frac{1}{N}\sum_{k=1}^{N}\frac{1}{\lambda_k - z}, \quad m_{\mathrm{sc}}(z) = \int_{\mathbb{R}}\frac{\mathrm{d}\mu_{\mathrm{sc}}(x)}{x-z} = \frac{-z+\sqrt{z^2-4}}{2},$$

where the sign of the square root is chosen so that $\operatorname{Im} m_{\mathrm{sc}}(z) > 0$ in the upper half-plane, and note that $m_{\mathrm{sc}}(z)$ solves the quadratic equation $m^2 + zm + 1 = 0$.

The following result [8, Theorem 1.1 and Remark 1.3] provides us with "a local law" for the β-ensemble, i.e., high-probability estimates of "the resolvent function" s_N near the spectrum:

Theorem 3.2. *Given $\beta > 0$, there exist η_0 and $C \geqslant 1$ such that for any $q \geqslant 1$, $N \geqslant 1$ and $z = E + \mathrm{i}\eta$ with $0 < \eta \leqslant \eta_0$ and $-2-\eta \leqslant E \leqslant 2+\eta$, we have*

$$\mathbb{E}_N[|s_N(z) - m_{\mathrm{sc}}(z)|^q] \leqslant \frac{(Cq)^{q/2}}{(N\eta)^q}. \tag{3.3}$$

Note that the above is equivalent to

$$\mathbb{P}_N\big(|s_N(z) - m_{\mathrm{sc}}(z)| \geqslant \frac{u}{N\eta}\big) \leqslant Ce^{-cu^2}, \tag{3.4}$$

uniformly in N, $u \geqslant 1$, and $0 < \eta \leqslant \eta_0$. In Theorem 5.2 we prove a sharp version of (3.3) with an explicit dependence of the constant C in β.

Using a mesh argument, we make the above estimate uniform:

Lemma 3.3. *Let $\mathscr{D}_\gamma = \{z = E + \mathrm{i}\eta \colon 0 < \eta \leqslant \frac{\gamma}{N}, |E| \leqslant \frac{\gamma}{N}\}$. Then, uniformly in $u \geqslant 1$ and $1 \leqslant \gamma \leqslant \delta N$ with $\delta > 0$ depending only on β, we have*

$$\mathbb{P}_N\big(\max_{z=E+\mathrm{i}\eta\in\mathscr{D}_\gamma} \eta|s_N(z) - m_{\mathrm{sc}}(z)| \geqslant \frac{u}{N}\big) \leqslant C\gamma^2 e^{-cu^2}. \tag{3.5}$$

Proof. Put

$$\widetilde{\mathscr{D}}_\gamma = \{z = E+\mathrm{i}\eta \colon \tfrac{1}{N} \leqslant \eta \leqslant \tfrac{\gamma}{N},\ |E| \leqslant \tfrac{\gamma}{N}\}, \quad \mathscr{D}'_\gamma = \{E+\mathrm{i}\eta \in \tfrac{1}{N}\,\mathbb{Z}^2 \colon |E| \leqslant \tfrac{\gamma+1}{N}, \tfrac{1}{N} \leqslant \eta \leqslant \tfrac{\gamma+1}{N}\}.$$

Throughout the proof we assume that $\gamma \leqslant \delta N$, so that $\mathscr{D}_\gamma$ lies in a fixed neighbourhood of the origin, and the one-point estimate (3.4) applies to all points of the mesh $\mathscr{D}'_\gamma$.

Let

$$M_\gamma = \sup_{z\in\mathscr{D}_\gamma} \eta|s_N(z) - m_{\mathrm{sc}}(z)|,$$
$$\widetilde{M}_\gamma = \sup_{z\in\widetilde{\mathscr{D}}_\gamma} \eta|s_N(z) - m_{\mathrm{sc}}(z)|,$$

and

$$M'_\gamma = \max_{z\in\mathscr{D}'_\gamma} \eta|s_N(z) - m_{\mathrm{sc}}(z)|.$$

First, we prove deterministic estimates:

$$\widetilde{M}_\gamma \lesssim M'_\gamma + \frac{1}{N} \tag{3.6}$$

and

$$M_\gamma \lesssim \widetilde{M}_\gamma + \frac{1}{N}, \tag{3.7}$$

with implicit constants independent of γ, u and N. Their proofs are based on elementary bounds:

$$|s'_N(z)| \leqslant \frac{1}{\eta}\,\operatorname{Im} s_N(z), \tag{3.8}$$

$$\Big|\frac{\mathrm{d}}{\mathrm{d}E}\log\operatorname{Im} s_N(E+\mathrm{i}\eta)\Big| + \Big|\frac{\mathrm{d}}{\mathrm{d}\eta}\log\operatorname{Im} s_N(E+\mathrm{i}\eta)\Big| \leqslant \frac{2}{\eta}, \tag{3.9}$$

and

$$\operatorname{Im} s_N(E+\mathrm{i}\eta) \leqslant \frac{\widetilde{\eta}}{\eta}\,\operatorname{Im} s_N(E+\mathrm{i}\widetilde{\eta}), \quad 0 < \eta \leqslant \widetilde{\eta}. \tag{3.10}$$

These estimates follow directly from the definition of s_N.

Proof of (3.6): Take $z = E + \mathrm{i}\eta \in \widetilde{\mathscr{D}}_\gamma$, and choose $z' = E' + \mathrm{i}\eta' \in \mathscr{D}'_\gamma$ so that $|E - E'| \leqslant 1/N$ and $|\eta - \eta'| \leqslant 1/N$. Since $\eta, \eta' \geqslant 1/N$, we have $\eta \simeq \eta'$. At the mesh point z', by definition of M'_γ, we have $|s_N(z') - m_{\mathrm{sc}}(z')| \leqslant M'_\gamma/\eta'$. Since m_{sc} is bounded in the upper half-plane and $\eta' \geqslant 1/N$, this gives us

$$|s_N(z')| \leqslant C + \frac{M'_\gamma}{\eta'} \lesssim 1 + NM'_\gamma,$$

and hence the same bound holds for $\operatorname{Im} s_N(z')$.

Consider the polygonal path Γ joining z' with z. Integrating (3.9) along Γ we get

$$\operatorname{Im} s_N(w) \lesssim \operatorname{Im} s_N(z') \lesssim 1 + NM'_\gamma, \quad w \in \Gamma,$$

and, by (3.8), $|s'_N(w)| \lesssim \eta^{-1}(1 + NM'_\gamma)$ (recall that $\operatorname{Im} w \simeq \eta$ on Γ). Thus,

$$\eta |s_N(z) - s_N(z')| \lesssim M'_\gamma + \frac{1}{N}.$$

Since $\eta \simeq \eta'$, we have $\eta |s_N(z') - m_{\mathrm{sc}}(z')| \lesssim M'_\gamma$. Finally, m'_{sc} is bounded in a fixed neighbourhood of the origin. Since $\eta \leqslant \gamma/N \leqslant \delta$, we get $\eta |m_{\mathrm{sc}}(z) - m_{\mathrm{sc}}(z')| \lesssim N^{-1}$. Combining these estimates, we obtain

$$\eta |s_N(z) - m_{\mathrm{sc}}(z)| \lesssim M'_\gamma + \frac{1}{N}, \quad z \in \widetilde{\mathscr{D}}_\gamma,$$

proving (3.6).

Proof of (3.7): we need to consider the lower part $0 < \eta \leqslant 1/N$ of $\mathscr{D}_\gamma$. We claim that

$$|s_N(E + \mathrm{i}\eta)| \lesssim \frac{1}{\eta N}(N\widetilde{M}_\gamma + 1) \quad 0 < \eta \leqslant \frac{1}{N}. \tag{3.11}$$

First, we estimate $\operatorname{Im} s_N$. Using (3.10) with $\widetilde{\eta} = 1/N$, we get

$$\begin{aligned} \operatorname{Im} s_N(E + \mathrm{i}\eta) &\leqslant \frac{1}{\eta N} \operatorname{Im} s_N(E + \tfrac{\mathrm{i}}{N}) \leqslant \frac{1}{\eta N} |s_N(E + \tfrac{\mathrm{i}}{N})| \\ &\leqslant \frac{1}{\eta N}(|s_N(E + \tfrac{\mathrm{i}}{N}) - m_{\mathrm{sc}}(E + \tfrac{\mathrm{i}}{N})| + O(1)) \lesssim \frac{1}{\eta N}(1 + N\widetilde{M}_\gamma). \end{aligned}$$

For the real part, we use both (3.8) and (3.10) (again with $\widetilde{\eta} = 1/N$). We get

$$\Big|\frac{\mathrm{d}}{\mathrm{d}y} \operatorname{Re} s_N(E + \mathrm{i}y)\Big| \leqslant \frac{1}{y} \operatorname{Im} s_N(E + \mathrm{i}y) \leqslant \frac{1}{y^2 N} \operatorname{Im} s_N(E + \tfrac{\mathrm{i}}{N}) \leqslant \frac{1}{y^2 N}\,(1 + N\widetilde{M}_\gamma).$$

Integrating from $y = \eta$ to $y = 1/N$, we obtain

$$|\operatorname{Re} s_N(E + \mathrm{i}\eta)| \lesssim (1 + N\widetilde{M}_\gamma) + \frac{1}{\eta N}\,(1 + N\widetilde{M}_\gamma) \overset{\eta N \leqslant 1}{\lesssim} \frac{1}{\eta N}\,(1 + N\widetilde{M}_\gamma),$$

proving (3.11).

Thus,

$$\eta |s_N(z) - m_{\mathrm{sc}}(z)| \leqslant \eta |s_N(z)| + \eta |m_{\mathrm{sc}}(z)| \overset{(3.11)}{\lesssim} \widetilde{M}_\gamma + \frac{1}{N}, \quad z \in \mathscr{D}_\gamma,\ 0 < \eta \leqslant \frac{1}{N},$$

proving (3.7).

The rest is straightforward. Combining (3.6) and (3.7), we get $M_\gamma \lesssim M'_\gamma + N^{-1}$. The mesh $\mathscr{D}'_\gamma$ contains at most $C\gamma^2$ points. Hence, by the one-point estimate (3.4),

$$\mathbb{P}_N\Big(M'_\gamma > \frac{u}{N}\Big) \leqslant C\gamma^2 e^{-cu^2}.$$

On the complementary event, $M'_\gamma \leqslant u/N$, and therefore, $M_\gamma \leqslant Cu/N$. Therefore

$$\mathbb{P}_N\Big(M_\gamma > \frac{Cu}{N}\Big) \leqslant C\gamma^2 e^{-cu^2}.$$

Renaming u and changing the constants, we prove the lemma. □

Next, we will use a version of the Helffer–Sjöstrand-type estimate [8, Lemma 3.7] that transfers the bound for the Stieltjes transform of a signed measure to the bound for the distribution of the measure.

Lemma 3.4. *Let ν be a finite signed measure on $\mathbb{R}$, and let*

$$\widehat{\nu}(z) = \int_{\mathbb{R}} \frac{\mathrm{d}\nu(t)}{t - z}.$$

Let f be a C^2-function compactly supported in $[A, B]$. Suppose that

$$|\widehat{\nu}(E + \mathrm{i}\eta)| \leqslant \frac{M}{\eta}, \qquad E \in [A, B],\ 0 < \eta \leqslant \gamma. \tag{3.12}$$

Then for every $0 < \eta \leqslant \gamma$ we have

$$\Big|\int_{\mathbb{R}} f(t)\,\mathrm{d}\nu(t)\Big| \lesssim M\Big(\frac{\|f\|_1}{\gamma} + (1 + \log\frac{\gamma}{\eta})\|f'\|_1 + \eta\|f''\|_1\Big).$$

For the reader's convenience, we supply the proof.

Proof. It suffices to prove the lemma for real-valued f. The complex-valued case follows by applying the result to $\operatorname{Re} f$ and $\operatorname{Im} f$.

Define the first-order Taylor extension of f to the upper half-plane by $F(x + \mathrm{i}y) = f(x) + \mathrm{i}yf'(x)$. Then $\bar\partial F = \frac{1}{2}(\partial_x + \mathrm{i}\partial_y)(f + \mathrm{i}yf') = \frac{\mathrm{i}}{2}yf''(x)$. Since $\widehat{\nu}$ is analytic in the upper half-plane, $\bar\partial(F\widehat{\nu}) = (\bar\partial F)\widehat{\nu}$. Now, applying the Cauchy–Green formula to the function $F\widehat{\nu}$ in the strip $\{\varepsilon < y < \eta\}$ (since F is compactly supported in x, the vertical boundary terms vanish), we get

$$\begin{aligned}\int_{\mathbb{R}} F(x + \mathrm{i}\eta)\widehat{\nu}(x + \mathrm{i}\eta)\,\mathrm{d}x - \int_{\mathbb{R}} F(x + \mathrm{i}\varepsilon)\widehat{\nu}(x + \mathrm{i}\varepsilon)\,\mathrm{d}x &= -2\mathrm{i}\int_\varepsilon^\eta \int_{\mathbb{R}} (\bar\partial F)\widehat{\nu}\,\mathrm{d}x\mathrm{d}y \\ &= \int_\varepsilon^\eta \int_{\mathbb{R}} yf''(x)\widehat{\nu}(x + \mathrm{i}y)\,\mathrm{d}x\mathrm{d}y.\end{aligned}$$

The next step is to let $\varepsilon \downarrow 0$. This will require a little care. We will be using a version of the Stieltjes inversion formula.

Claim 3.5. *Let ν be a finite signed measure on $\mathbb{R}$, and let*

$$\widehat{\nu}(z) = \int_{\mathbb{R}} \frac{\mathrm{d}\nu(t)}{t-z}\,.$$

Then, for every compactly supported real-valued C^1-function f,

$$\lim_{y\downarrow 0} \operatorname{Im} \int_{\mathbb{R}} (f(x) + \mathrm{i}yf'(x))\,\widehat{\nu}(x+\mathrm{i}y)\,\mathrm{d}x = \pi \int_{\mathbb{R}} f(t)\,\mathrm{d}\nu(t).$$

We proceed with the proof of Lemma 3.4, and then return to the proof of Claim 3.5.

Letting $\varepsilon \to 0$ in the Cauchy–Green formula, we get

$$\Big|\pi \int_{\mathbb{R}} f\,\mathrm{d}\nu - \operatorname{Im} \int_{\mathbb{R}} F(x+\mathrm{i}\eta)\widehat{\nu}(x+\mathrm{i}\eta)\,\mathrm{d}x\Big| \leqslant \int_0^\eta \int_{\mathbb{R}} y|f''(x)||\widehat{\nu}(x+\mathrm{i}y)|\,\mathrm{d}x\mathrm{d}y \overset{(3.12)}{\leqslant} M\eta\|f''\|_1.$$

Hence,

$$\Big|\pi \int_{\mathbb{R}} f\,\mathrm{d}\nu\Big| \leqslant \Big|\int_{\mathbb{R}} f(x)\widehat{\nu}(x+\mathrm{i}\eta)\,\mathrm{d}x\Big| + \eta\Big|\int_{\mathbb{R}} f'(x)\widehat{\nu}(x+\mathrm{i}\eta)\,\mathrm{d}x\Big| + M\eta\|f''\|_1.$$

By (3.12), the f'-term on the RHS is bounded by $M\|f'\|_1$. So, we have

$$\Big|\pi \int_{\mathbb{R}} f\,\mathrm{d}\nu\Big| \leqslant \Big|\int_{\mathbb{R}} f(x)\widehat{\nu}(x+\mathrm{i}\eta)\,\mathrm{d}x\Big| + M\|f'\|_1 + M\eta\|f''\|_1.$$

It remains to estimate the first term on the RHS; the trivial bound by $M\|f\|_1/\eta$ is not good for us.

To get a better bound, we move up the integration contour to $\{y = \gamma\}$, applying the Cauchy–Green formula to the function $f(x)\widehat{\nu}(x+\mathrm{i}y)$ in the strip $\{\eta < y < \gamma\}$. Using that $\bar{\partial} f(x) = f'(x)/2$, we get

$$\int_{\mathbb{R}} f(x)\widehat{\nu}(x+\mathrm{i}\eta)\,\mathrm{d}x - \int_{\mathbb{R}} f(x)\widehat{\nu}(x+\mathrm{i}\gamma)\,\mathrm{d}x = 2\mathrm{i}\int_\eta^\gamma \int_{\mathbb{R}} \tfrac{1}{2}f'(x)\widehat{\nu}(x+\mathrm{i}y)\,\mathrm{d}x\mathrm{d}y,$$

whence,

$$\begin{aligned}\Big|\int_{\mathbb{R}} f(x)\widehat{\nu}(x+\mathrm{i}\eta)\,\mathrm{d}x\Big| &\leqslant \int_{\mathbb{R}} |f(x)|\,|\widehat{\nu}(x+\mathrm{i}\gamma)|\,\mathrm{d}x + \int_\eta^\gamma \int_{\mathbb{R}} |f'(x)|\,|\widehat{\nu}(x+\mathrm{i}y)|\,\mathrm{d}x\mathrm{d}y \\ &\overset{(3.12)}{\leqslant} \frac{M}{\gamma}\,\|f\|_1 + M\log(\gamma/\eta)\,\|f'\|_1.\end{aligned}$$

Combining the estimates, we get what we wanted.

It remains to prove the claim. Denote by

$$P_y(t) = \frac{y}{\pi}\,\frac{1}{t^2+y^2}$$

the Poisson kernel. Then $\operatorname{Im}\widehat{\nu}(x+\mathrm{i}y) = \pi(P_y * \nu)(x)$, whence,

$$\int_{\mathbb{R}} f(x)\operatorname{Im}\widehat{\nu}(x+\mathrm{i}y)\,\mathrm{d}x = \pi\int_{\mathbb{R}}(P_y * f)(t)\,\mathrm{d}\nu(t).$$

Since f is continuous and compactly supported, $P_y * f \to f$ uniformly. Therefore,

$$\int_{\mathbb{R}} f(x)\operatorname{Im}\widehat{\nu}(x+\mathrm{i}y)\,\mathrm{d}x \to \pi\int_{\mathbb{R}} f\,\mathrm{d}\nu, \qquad \text{as } y\to 0.$$

Now, consider the second term

$$\begin{aligned}\operatorname{Im}\Big[\mathrm{i}y\int_{\mathbb{R}} f'(x)\widehat{\nu}(x+\mathrm{i}y)\,\mathrm{d}x\Big] &= y\int_{\mathbb{R}} f'(x)\operatorname{Re}\widehat{\nu}(x+\mathrm{i}y)\,\mathrm{d}x\\ &= \int_{\mathbb{R}} H_y^f(t)\,\mathrm{d}\nu(t),\end{aligned}$$

where

$$H_y^f(t) = y\int_{\mathbb{R}} f'(x)\frac{t-x}{(t-x)^2+y^2}\,\mathrm{d}x.$$

We claim that $H_y^f(t)\to 0$ pointwise in t, as $y\downarrow 0$. To see this, we split the integration into $|t-x|\leqslant\delta$ and $|t-x|>\delta$. The first integral is small when δ is small due to the bound

$$\frac{y|t-x|}{(t-x)^2+y^2}\leqslant\frac{1}{2},$$

and to the absolute continuity of the integral of $|f'|$. In the second integral, the kernel is bounded by y/δ, and hence tends to zero as $y\to 0$. Since $|H_y^f(t)|\leqslant\frac{1}{2}\|f'\|_1$, dominated convergence gives

$$\int_{\mathbb{R}} H_y^f(t)\,\mathrm{d}\nu(t)\to 0,$$

as $y\downarrow 0$, completing the proof of the claim, and hence, of the lemma. □

Now, we are ready to prove rigidity estimates for the counting measures of the β-ensemble and of the Sine_β process. For $\gamma>0$, let

$$n(\gamma) = \#(\mathcal{X}_\beta\cap(0,\gamma]), \quad n_N(\gamma) = \#\{j\colon 0<\lambda_j\leqslant\pi\gamma/N\},$$

where (λ_j) are points of the β-ensemble with $V(x)=x^2/2$. By $a_N(\gamma) = N\mu_{\mathrm{sc}}([0,\pi\gamma/N])$ we denote the scaled semi-circle measure.

Lemma 3.6. *For any fixed $\beta>0$, the following holds.*
I. *There exist $0<c<C$ and $\delta>0$ such that for any $e\leqslant\gamma\leqslant\delta N$ and $u\geqslant 1$, we have*

$$\mathbb{P}_N\big(|n_N(\gamma)-a_N(\gamma)|\geqslant u\log\gamma\big)\leqslant C\gamma^2 e^{-cu^2}, \tag{3.13}$$

and, for $p \geqslant 1$,

$$\mathbb{E}_N[|n_N(\gamma) - a_N(\gamma)|^p] \leqslant \big(C(\sqrt{p} + \sqrt{\log \gamma}) \log \gamma\big)^p. \tag{3.14}$$

II. *There exist* $0 < c < C$ *such that, for any* $\gamma \geqslant e$ *and* $u \geqslant 1$, *we have:*

$$\mathbb{P}_\beta\big(|n(\gamma) - \gamma| \geqslant u(\log \gamma)^{3/2}\big) \leqslant Ce^{-cu^2}, \tag{3.15}$$

$$\mathbb{E}_\beta[|n(\gamma) - \gamma|^p] \leqslant \big(C(\sqrt{p} + \sqrt{\log \gamma}) \log \gamma\big)^p, \quad p \geqslant 1. \tag{3.16}$$

Proof. First, combining Lemmata 3.3 and 3.4, we prove (3.13). Other estimates are more or less straightforward consequences of this one.

Consider the empirical measure $\chi_N = \dfrac{1}{N} \sum_{k=1}^{N} \delta_{\lambda_k}$, and denote by $\nu_N = \chi_N - \mu_{\mathrm{sc}}$ the centred empirical measure. Its Stieltjes transform is $\widehat{\nu}_N = s_N - m_{\mathrm{sc}}$. Then, choose smooth envelopes $f_\pm$ so that

$$0 \leqslant f_- \leqslant \mathbb{1}_{[0,\pi\gamma/N]} \leqslant f_+ \leqslant 1,$$

and

$$\operatorname{spt} f_+ \subset [-1/N, \pi(\gamma+1)/N], \quad f_- = 1 \text{ on } [1/N, \pi(\gamma-1)/N].$$

They can be chosen so that $\|f_\pm^{(k)}\|_\infty \leqslant C_k N^k$, hence,

$$\|f_\pm\|_1 \lesssim \gamma/N, \ \|f_\pm'\|_1 \lesssim 1, \ \|f_\pm''\|_1 \lesssim N.$$

Lemma 3.3 tells us that, outside an event of probability at most $C\gamma^2 e^{-cu^2}$, we have

$$|\widehat{\nu}_N(E + \mathrm{i}\eta)| \leqslant \frac{u}{\eta N}, \quad |E| \leqslant \frac{C_1\gamma}{N}, \ 0 < \eta \lesssim \frac{C_1\gamma}{N}.$$

We can now apply Lemma 3.4 to $f_\pm$ with $C_1\gamma/N$ instead of γ, $\eta = 1/N$, and $M = u/N$. The Helffer–Sjöstrand bound contains three terms. The terms containing f and f'' are bounded, while the term containing f' is $\lesssim \log \gamma$. Hence, the error term provided by this bound is $\lesssim uN^{-1} \log \gamma$.

Put $I_N = [0, \pi\gamma/N]$. Then, on a good event,

$$\int f_- \, \mathrm{d}\mu_{\mathrm{sc}} - O(\frac{u}{N} \log \gamma) \leqslant \int f_- \, \mathrm{d}\chi_N \leqslant \chi_N(I_N) \leqslant \int f_+ \, \mathrm{d}\chi_N \leqslant \int f_+ \, \mathrm{d}\mu_{\mathrm{sc}} + O(\frac{u}{N} \log \gamma),$$

i.e.,

$$N \int f_- \, \mathrm{d}\mu_{\mathrm{sc}} - O(u \log \gamma) \leqslant N\chi_N(I_N) \leqslant N \int f_+ \, \mathrm{d}\mu_{\mathrm{sc}} + O(u \log \gamma).$$

Since the functions $f_\pm$ differ from $\mathbb{1}_{I_N}$ only on two intervals of total length $\lesssim 1/N$ and the semi-circle density ρ_{sc} is bounded near the origin, we have

$$N\Big|\int f_\pm \, \mathrm{d}\mu_{\mathrm{sc}} - \mu_{\mathrm{sc}}(I_N)\Big| = O(1),$$

which is $O(u\log\gamma)$ since $u \geqslant 1$ and $\gamma \geqslant e$. Absorbing the additive constant into $O(u\log\gamma)$ and renaming u, we arrive at (3.13).

To derive the L^p estimate, we write

$$\mathbb{E}_N[|n_N(\gamma) - a_N(\gamma)|^p] = p\int_0^\infty t^{p-1}\mathbb{P}_N(|n_N(\gamma) - a_N(\gamma)| > t)\,\mathrm{d}t\,.$$

Splitting the integral at $t = A(\log\gamma)^{3/2}$ with sufficiently large A, we get

$$\mathbb{E}_N[|n_N(\gamma) - a_N(\gamma)|^p] \leqslant (A(\log\gamma)^{3/2})^p + Cp\gamma^2\int_{A(\log\gamma)^{3/2}}^\infty t^{p-1}e^{-c(t/\log\gamma)^2}\,\mathrm{d}t.$$

For $t \geqslant A(\log\gamma)^{3/2}$, choosing A sufficiently large, we have $\gamma^2 e^{-c(t/\log\gamma)^2} \leqslant e^{-\frac{1}{2}c(t/\log\gamma)^2}$. Hence,

$$\mathbb{E}_N[|n_N(\gamma) - a_N(\gamma)|^p] \leqslant (A(\log\gamma)^{3/2})^p + Cp\int_{A(\log\gamma)^{3/2}}^\infty t^{p-1}e^{-c'(t/\log\gamma)^2}\,\mathrm{d}t.$$

The second term on the RHS does not exceed

$$Cp(\log\gamma)^p\int_0^\infty t^{p-1}e^{-c't^2}\,dt \leqslant (C'\sqrt{p}\log\gamma)^p,$$

proving (3.14).

For the rest of the proof, we fix $\gamma \geqslant e$, and put $X_N = n_N(\gamma) - a_N(\gamma)$ and $X = n(\gamma) - \gamma$. Then, X_N converges in distribution to X. The limiting L^p-bound (3.16) is an immediate consequence of (3.14). Letting $f_M(x) = \min(|x|^p, M)$, we get

$$\mathbb{E}_\beta[f_M(X)] = \lim_{N\to\infty}\mathbb{E}_N[f_M(X_N)] \leqslant \liminf_{N\to\infty}\mathbb{E}_N[|X_N|^p],$$

while, by the monotone convergence theorem, $\mathbb{E}_\beta[|X|^p] = \lim\limits_{M\to\infty}\mathbb{E}_\beta[f_M(X)]$, proving (3.16).

It remains to prove (3.15). By (3.13), for every $v \geqslant 1$ and $N \geqslant \gamma/\delta$,

$$\mathbb{P}_N(|X_N| \geqslant v\log\gamma) \leqslant C\gamma^2 e^{-cv^2}.$$

First, we pass to the limit in this estimate. Fix $v > 1$, and choose $0 < \varepsilon < (v-1)\log\gamma$. The set $G_\varepsilon = \{|x| > v\log\gamma - \varepsilon\}$ is open. Hence, $\mathbb{P}_\beta(X \in G_\varepsilon) \leqslant \liminf_{N\to\infty}\mathbb{P}_N(X_N \in G_\varepsilon)$. Put $v_\varepsilon = v - \varepsilon/\log\gamma$. Applying (3.13), we get $\mathbb{P}_\beta(|X| > v\log\gamma - \varepsilon) \leqslant C\gamma^2 e^{-cv_\varepsilon^2}$. As $\varepsilon \downarrow 0$, the events on the LHS decrease to $\{|X| \geqslant v\log\gamma\}$. Therefore, $\mathbb{P}_\beta(|X| \geqslant v\log\gamma) \leqslant C\gamma^2 e^{-cv^2}$. This holds for $v > 1$, but after increasing C, we get the same for $v \geqslant 1$.

Now, put $v = u\sqrt{\log\gamma}$. We get

$$\mathbb{P}_\beta(|X| \geqslant u(\log\gamma)^{3/2}) \leqslant C\gamma^2 e^{-cu^2\log\gamma} = Ce^{-(cu^2-2)\log\gamma} \overset{\gamma\geqslant e}{\leqslant} Ce^{-(cu^2-2)}$$

If $u \geqslant u_0$ with sufficiently large numerical u_0, the RHS is $\leqslant Ce^{-\frac{1}{2}cu^2}$. For the remaining bounded interval $1 \leqslant u \leqslant u_0$, the same holds after increasing C. This completes the proof of the lemma. □

3.3 The Stieltjes transform of the Sine_β ***process.*** We define the Stieltjes transform $s(z)$ of $\mathcal{X}_\beta$ using the principal value summation at infinity, and then show that it coincides with the limit of the rescaled functions

$$\widetilde{s}_N(z) = \pi s_N(\pi z/N) = \sum_{1\leqslant j\leqslant N} \frac{1}{\lambda_{j,N} - z}, \quad \lambda_{j,N} = N\lambda_j/\pi.$$

Given $z \in \mathbb{C}\setminus\mathbb{R}$, define

$$\widetilde{s}_N^{(K)}(z) = \sum_{|\lambda_{j,N}|\leqslant K} \frac{1}{\lambda_{j,N} - z}, \quad s^{(K)}(z) = \sum_{\lambda\in\mathcal{X}_\beta\colon |\lambda|\leqslant K} \frac{1}{\lambda - z}.$$

Thanks to rigidity estimates from Lemma 3.6, we easily have the following.

Lemma 3.7.

I. *For every compact* $Q \subset \mathbb{C}\setminus\mathbb{R}$ *and every* $p \geqslant 1$,

$$\limsup_{N\to\infty} \big(\mathbb{E}_N[\sup_{z\in Q} |\widetilde{s}_N(z) - \widetilde{s}_N^{(K)}(z)|^p]\big)^{1/p} = O_{\beta,p,Q}(1)\, \frac{(\log K)^{3/2}}{K}\,.$$

II. *There exists a random holomorphic function* s *on* $\mathbb{C}\setminus\mathbb{R}$ *such that* $s^{(K)} \to s$ *as* $K\to\infty$, *almost surely and locally uniformly on* $\mathbb{C}\setminus\mathbb{R}$. *Moreover, for every compact* $Q \subset \mathbb{C}\setminus\mathbb{R}$ *and every* $p \geqslant 1$,

$$\big(\mathbb{E}_\beta[\sup_{z\in Q} |s(z) - s^{(K)}(z)|^p]\big)^{1/p} = O_{\beta,p,Q}(1)\, \frac{(\log K)^{3/2}}{K}\,.$$

In particular, for any $z_i \in \mathbb{C}\setminus\mathbb{R}$, $i = 1,\dots,m$, *the random vectors* $\{s^{(K)}(z_i)\}_{i=1}^m$ *converge to* $\{s(z_i)\}_{i=1}^m$ *almost surely and in* L^p *as* $K\to\infty$.

Proof. It will be convenient to slightly change our notation. For $\gamma > 0$, we set

$$n_{N,+}(\gamma) = \#\{j\colon 0 < \lambda_{j,N} \leqslant \gamma\}, \quad n_{N,-}(\gamma) = \#\{j\colon -\gamma \leqslant \lambda_{j,N} < 0\},$$

$$n_+(\gamma) = \#(\mathcal{X}_\beta \cap (0,\gamma]), \quad n_-(\gamma) = \#(\mathcal{X}_\beta \cap [-\gamma, 0)),$$

and define, with $a_N(\gamma) = N\mu_{\mathrm{sc}}([0,\pi\gamma/N])$ as before,

$$D_{N,\pm}(\gamma) = n_{N,\pm}(\gamma) - a_N(\gamma), \quad D_\pm(\gamma) = n_\pm(\gamma) - \gamma.$$

By reflection symmetry, the estimates of Lemma 3.6 hold with $n_{N,+}$, n_+ replaced by $n_{N,-}$, n_-. So, applying Lemma 3.6, we have

(a) A.s., $|D_\pm(\gamma)| \leqslant C(\omega)(\log\gamma)^2$, for all sufficiently large γ (with random $C(\omega)$). To see this, we combine (3.15) with Borel-Cantelli, which gives us the estimate for sufficiently large integer γ, and then, using monotonicity of $n_\pm$, extend it to all sufficiently large γ.

(b) For every $p \geqslant 1$ and $e^p \leqslant \gamma \leqslant \delta N$, $\|D_{N,\pm}(\gamma)\|_{L^p} \leqslant C(\log\gamma)^{3/2}$ (here, δ is a constant from Lemma 3.6, Part I, depending only on β).

Proof of I. Fix a compact set $Q \subset \mathbb{C}\setminus\mathbb{R}$ and assume that $K \geqslant 2\sup\{|z|\colon z \in Q\}+1$. We split

$$\begin{aligned}(\widetilde{s}_N - \widetilde{s}_N^{(K)})(z) &= \Big(\sum_{K<|\lambda_{j,N}|\leqslant\delta N} + \sum_{|\lambda_{j,N}|>\delta N}\Big)\frac{1}{\lambda_{j,N}-z}\\ &= A_{N,K}(z) + B_N(z).\end{aligned}$$

If $K \geqslant \delta N$, the first sum is empty. Since we eventually take $N\to\infty$ with fixed K, we may assume that $K < \delta N$.

Note that

$$\begin{aligned}A_{N,K}(z) &= \int_K^{\delta N}\frac{\mathrm{d}n_{N,+}(\gamma)}{\gamma-z} - \int_K^{\delta N}\frac{\mathrm{d}n_{N,-}(\gamma)}{\gamma+z}\\ &= \int_K^{\delta N}\Big(\frac{1}{\gamma-z}-\frac{1}{\gamma+z}\Big)\,\mathrm{d}a_N(\gamma) + \int_K^{\delta N}\frac{\mathrm{d}D_{N,+}(\gamma)}{\gamma-z} - \int_K^{\delta N}\frac{\mathrm{d}D_{N,-}(\gamma)}{\gamma+z}\end{aligned}$$

(in the Stieltjes integrals we are using, endpoint conventions are immaterial almost surely).

Since $\mathrm{d}a_N(\gamma) = \pi\rho_{\mathrm{sc}}(\pi\gamma/N)\mathrm{d}\gamma \leqslant C\mathrm{d}\gamma$, the deterministic term is bounded by

$$C(Q)\int_K^\infty\frac{\mathrm{d}\gamma}{\gamma^2} = \frac{C(Q)}{K}\,.$$

Integrating by parts the discrepancy terms, we get

$$\int_K^{\delta N}\frac{\mathrm{d}D_{N,+}(\gamma)}{\gamma-z} = \frac{D_{N,+}(\delta N)}{\delta N - z} - \frac{D_{N,+}(K)}{K-z} + \int_K^{\delta N}\frac{D_{N,+}(\gamma)}{(\gamma-z)^2}\,\mathrm{d}\gamma.$$

Hence

$$\sup_{z\in Q}\Big|\int_K^{\delta N}\frac{\mathrm{d}D_{N,+}(\gamma)}{\gamma-z}\Big| \leqslant C(Q)\Big(\frac{|D_{N,+}(\delta N)|}{\delta N} + \frac{|D_{N,+}(K)|}{K} + \int_K^{\delta N}\frac{|D_{N,+}(\gamma)|}{\gamma^2}\,\mathrm{d}\gamma\Big).$$

Taking the L^p norm and applying Minkowski's inequality, we get

$$\begin{aligned}\Big\|\sup_{z\in Q}\Big|\int_K^{\delta N}\frac{\mathrm{d}D_{N,+}(\gamma)}{\gamma-z}\Big|\Big\|_p &\leqslant C(Q)\Big(\frac{\|D_{N,+}(\delta N)\|_p}{\delta N} + \frac{\|D_{N,+}(K)\|_p}{K} + \int_K^{\delta N}\frac{\|D_{N,+}(\gamma)\|_p}{\gamma^2}\,\mathrm{d}\gamma\Big)\\ &\leqslant C(\beta,p,Q)\Big(\frac{(\log(\delta N))^{3/2}}{\delta N} + \frac{(\log K)^{3/2}}{K} + \int_K^{\delta N}\frac{(\log\gamma)^{3/2}}{\gamma^2}\,\mathrm{d}\gamma\Big)\\ &\leqslant C(\beta,p,Q)\frac{(\log K)^{3/2}}{K}\,.\end{aligned}$$

The same estimate holds for the term with $D_{N,-}$. Thus,

$$\big\|\sup_Q|A_{N,K}|\big\|_p \leqslant C(\beta,p,Q)\frac{(\log K)^{3/2}}{K}\,.$$

Next, estimating the macroscopic term B_N, we will show that $\sup_Q |B_N| \to 0$ in $L^p(\mathbb{P}_N)$ as $N \to \infty$. Returning to the original variables (λ_j), we write

$$B_N(z) = \sum_{|\lambda_j|>\pi\delta} \frac{1}{N\lambda_j/\pi - z} = \pi \int_{|\lambda|>\pi\delta} \frac{\mathrm{d}\chi_N(\lambda)}{\lambda - \pi z/N}\,.$$

Let $h_\delta(\lambda) = \lambda^{-1}\mathbb{1}_{\{|\lambda|>\pi\delta\}}$. Then

$$\int_{|\lambda|>\pi\delta} \frac{\mathrm{d}\chi_N(\lambda)}{\lambda - \pi z/N} = \int_{\mathbb{R}} h_\delta \,\mathrm{d}\chi_N + O_{\delta,Q}(1)N^{-1}.$$

Recalling that χ_N converges weakly in probability to μ_{sc}, and noting that $\int_{\mathbb{R}} h_\delta \,\mathrm{d}\mu_{\mathrm{sc}} = 0$ (and that the function h_δ is bounded), we see that $\lim_{N\to\infty} \big\|\sup_Q |B_N|\,\big\|_p = 0$, thus completing the proof of part I.

Proof of II. For $T > K$, set $S_{K,T} = s^{(T)} - s^{(K)}$. Applying the same Stieltjes integration by parts argument and using the a.s. estimate (a) to estimate the random terms, we get

$$\sup_Q |S_{K,T}| \leqslant \frac{C(Q)}{K} + C(\omega, Q)\frac{(\log K)^2}{K}\,.$$

Hence,

$$\sup_{T>K} \sup_Q |S_{K,T}| \leqslant C(\omega, Q)\frac{(\log K)^2}{K} \to 0,$$

as $K \to \infty$. Thus, $s^{(K)}$ almost surely converges and the convergence is uniform on Q. Exhausting $\mathbb{C}\setminus\mathbb{R}$ by countably many compact sets, we get an exceptional event independent of Q.

To get the L^p convergence we argue as follows. The same estimates as above, applied to the functions $S_{K,T}$ (with a little simplification due to the absence of the macroscopic term), give the needed L^p-bound:

$$\big\|\sup_Q |S_{K,T}|\big\|_p \leqslant C(\beta, p, Q)\,\frac{(\log K)^{3/2}}{K}\,.$$

Then, by Cauchy's criterion, $s^{(K)}$ converges in the norm $\big(\mathbb{E}_\beta[\sup_Q |\cdot|^p]\big)^{1/p}$, and the limit coincides with the a.s. limit. Letting $T \to \infty$ in the preceding displayed bound, we get the asserted L^p estimate for $s - s^{(K)}$. □

Now, using a truncation argument, we will obtain convergence of $\widetilde{s}_N$ to s in distribution. Let $\mathcal{H} = \mathrm{Hol}(\mathbb{C}\setminus\mathbb{R})$ be a space of holomorphic functions equipped with the topology of locally uniform convergence. We use the following metric on $\mathcal{H}$:

$$d_{\mathcal{H}}(f, g) = \sum_{m\geqslant 1} 2^{-m}(1 \wedge \|f - g\|_{C(Q_m)}),$$

where (Q_m) is a fixed increasing compact exhaustion of $\mathbb{C}\setminus\mathbb{R}$, $Q_1 \subset Q_2 \subset \ldots$, $\bigcup_m Q_m = \mathbb{C}\setminus\mathbb{R}$.

Corollary 3.8. *As $N \to \infty$, $\widetilde{s}_N$ converges in distribution to s in $\mathcal{H}$.*

Proof. We first claim that, for every fixed K,

$$\widetilde{s}_N^{(K)} \xrightarrow{d} s^{(K)} \quad \text{in } \mathcal{H}. \tag{3.17}$$

Indeed, given K, the map

$$\nu \mapsto \int_{[-K,K]} \frac{\mathrm{d}\nu(\gamma)}{\gamma - \cdot}$$

from locally finite point measures on $\mathbb{R}$ with the vague topology to $\mathcal{H}$ is continuous at every ν having no points at $\pm K$. Since the Sine_β -process almost surely has no points at fixed locations, the continuous mapping theorem [2, Section 2] yields (3.17).

Next, we show that the truncation errors are negligible in the metric $d_{\mathcal{H}}$. We claim that

$$\lim_{K\to\infty} \limsup_{N\to\infty} \mathbb{E}_N\big[d_{\mathcal{H}}(\widetilde{s}_N, \widetilde{s}_N^{(K)})\big] = 0, \tag{3.18}$$

and

$$\lim_{K\to\infty} \mathbb{E}_\beta\big[d_{\mathcal{H}}(s, s^{(K)})\big] = 0. \tag{3.19}$$

Part I of Lemma 3.7 with $p = 1$ gives, for every $m \geqslant 1$,

$$\limsup_{N\to\infty} \mathbb{E}_N\big[\|\widetilde{s}_N - \widetilde{s}_N^{(K)}\|_{C(Q_m)}\big] \to 0, \qquad K \to \infty. \tag{3.20}$$

Fix $M \geqslant 1$. Then

$$\mathbb{E}_N\big[d_{\mathcal{H}}(\widetilde{s}_N, \widetilde{s}_N^{(K)})\big] \leqslant \sum_{1\leqslant m\leqslant M} 2^{-m}\mathbb{E}_N\big[\|\widetilde{s}_N - \widetilde{s}_N^{(K)}\|_{C(Q_m)}\big] + \sum_{m>M} 2^{-m}.$$

Taking $N \to \infty$, combining the resulting estimate with (3.20), and then letting $K \to \infty$, and finally $M \to \infty$, we obtain (3.18). The same argument gives us (3.19).

It remains to combine (3.17), (3.18), and (3.19). Let $\Phi\colon \mathcal{H} \to \mathbb{R}$ be a bounded function, Lipschitz with respect to the metric $d_{\mathcal{H}}$. Then

$$\begin{aligned}\big|\mathbb{E}_N[\Phi(\widetilde{s}_N)] - \mathbb{E}_\beta[\Phi(s)]\big| \leqslant{}& \big|\mathbb{E}_N[\Phi(\widetilde{s}_N) - \Phi(\widetilde{s}_N^{(K)})]\big| \\ &+ \big|\mathbb{E}_N[\Phi(\widetilde{s}_N^{(K)})] - \mathbb{E}_\beta[\Phi(s^{(K)})]\big| + \big|\mathbb{E}_\beta[\Phi(s^{(K)}) - \Phi(s)]\big|,\end{aligned}$$

where

$$\big|\mathbb{E}_N[\Phi(\widetilde{s}_N) - \Phi(\widetilde{s}_N^{(K)})]\big| \leqslant \mathsf{Lip}(\Phi)\mathbb{E}_N[d_{\mathcal{H}}(\widetilde{s}_N, \widetilde{s}_N^{(K)})],$$

and

$$\big|\mathbb{E}_\beta[\Phi(s^{(K)}) - \Phi(s)]\big| \leqslant \mathsf{Lip}(\Phi)\mathbb{E}_\beta[d_{\mathcal{H}}(s, s^{(K)})],$$

while, by (3.17), for every fixed K, the middle term tends to 0 as $N \to \infty$. Thus, as $N \to \infty$,

$$\big|\mathbb{E}_N[\Phi(\widetilde{s}_N)] - \mathbb{E}_\beta[\Phi(s)]\big| \leqslant \mathsf{Lip}(\Phi)\big(\mathbb{E}_N[d_{\mathcal{H}}(\widetilde{s}_N, \widetilde{s}_N^{(K)})] + \mathbb{E}_\beta[d_{\mathcal{H}}(s, s^{(K)})]\big) + o(1),$$

with the $o(1)$-term on the RHS depending on K. Letting $N \to \infty$ and then $K \to \infty$, and using (3.18) and (3.19), we conclude that

$$\mathbb{E}_N[\Phi(\widetilde{s}_N)] \to \mathbb{E}_\beta[\Phi(s)]$$

for every bounded Lipschitz function on $\mathcal{H}$.

It remains to use that for every closed set $X \subset \mathcal{H}$, the bounded Lipschitz functions $\Phi_k(h) = [1 - k d_{\mathcal{H}}(h, X)]_+$ decrease pointwise to $\mathbb{1}_X(h)$, $h \in \mathcal{H}$, as $k \to \infty$. Hence, convergence tested on bounded Lipschitz functions implies that $\limsup_{N\to\infty} \mathbb{P}_N(\widetilde{s}_N \in X) \leqslant \mathbb{P}_\beta(s \in X)$. By the so-called Portmanteau theorem [2, Theorem 2.1], this yields $\widetilde{s}_N \xrightarrow{d} s$ in $\mathcal{H}$. □

Putting together Theorem 3.2 with Corollary 3.8, we arrive at the following local law for the Sine_β process.

Corollary 3.9. *For every $z \in \mathbb{C} \setminus \mathbb{R}$ and every $q \geqslant 1$,*

$$\|s(z) - \pi \mathrm{i} \operatorname{sgn}(\operatorname{Im} z)\|_{L^q(\mathbb{P}_\beta)} \leqslant \frac{C\sqrt{q}}{|\operatorname{Im} z|}.$$

This estimate will suffice for the proof of Theorem 1.1. In Corollary 5.4 of Section 5 we present a sharper version of this law which is uniform in β.

Proof. Indeed, for $z \in \mathbb{C} \setminus \mathbb{R}$, Corollary 3.8 gives $\widetilde{s}_N(z) = \pi s_N(\pi z/N) \xrightarrow{d} s(z)$, while $\pi m_{\mathrm{sc}}(\pi z/N) \to \pi \mathrm{i} \operatorname{sgn}(\operatorname{Im} z)$. Hence,

$$\widetilde{s}_N(z) - \pi m_{\mathrm{sc}}(\pi z/N) \xrightarrow{d} s(z) - \pi \mathrm{i} \operatorname{sgn}(\operatorname{Im} z).$$

Applying this convergence to $f_M(x) = |x|^q \wedge M$, letting $M \to \infty$, and using Theorem 3.2, we complete the job. □

3.4 The logarithmic asymptotics of the Laplace transform. In this section we obtain (Gaussian) fluctuation bounds for our random field, exploiting the first loop equation for biased measures in infinite volume, an idea introduced by Johansson [19] for finite N. Our result is an infinite volume analogue of [7, Theorem B.1].

To state the main result of this section, we need some notation. Let $m \geqslant 2$. Given points $\boldsymbol{z} = (z_i)_{i=1}^m$ in $\mathbb{H}$ and a vector $\boldsymbol{\gamma} \in \mathbb{R}^m$ satisfying

$$\boldsymbol{\gamma} = (\gamma_i)_{i=1}^m, \quad \sum_{i=1}^m \gamma_i = 0, \tag{3.21}$$

we set

$$h(\lambda, \boldsymbol{z}) = \sum_{i=1}^m \gamma_i \log |z_i - \lambda|.$$

To simplify notation, we usually suppress $\boldsymbol{z}$, writing $h(\lambda) = h(\lambda, \boldsymbol{z})$. Then we set

$$H(\boldsymbol{z}, \boldsymbol{\gamma}) = \sum_{\lambda} h(\lambda),$$

where the sum is understood in the principal value sense, and note that $\mathbb{E}_\beta[H(\boldsymbol{z}, \boldsymbol{\gamma})] = \pi \sum_{i=1}^m \gamma_i \operatorname{Im} z_i$. This follows, for instance, by integrating the equation $\mathbb{E}_\beta[s(z)] = \mathrm{i}\pi$, $z \in \mathbb{H}$.

Next, we define the following quadratic form in $\boldsymbol{\gamma}$

$$\sigma(\boldsymbol{z}, \boldsymbol{\gamma}) = \frac{1}{2\beta} \sum_{1 \leqslant i,j \leqslant m} \gamma_i \gamma_j \log \frac{1}{|z_i - \bar{z}_j|}.$$

We also set

$$\eta_* = \min_{1 \leqslant i \leqslant m} \operatorname{Im} z_i, \quad \Delta = \max_{1 \leqslant i \leqslant j \leqslant m} |z_i - z_j|,$$

and put $\ell(\Delta) = 1 \vee \log \Delta$.

Theorem 3.10. *Let $\boldsymbol{\gamma}$ satisfy (3.21) and in addition $\|\boldsymbol{\gamma}\|_\infty \leqslant M$ for some $M \geqslant 1$ and $\eta_* \geqslant 1$. Then, uniformly in $\boldsymbol{\gamma}$ and $\boldsymbol{z}$, we have*

$$\log \mathbb{E}_\beta \Big[e^{H(\boldsymbol{z}, \boldsymbol{\gamma})} \Big] = \pi \sum_{i=1}^m \gamma_i \operatorname{Im} z_i + \sigma(\boldsymbol{z}, \boldsymbol{\gamma}) + O_{\beta,m}(1) M^3 \frac{\ell^2(\Delta)}{\eta_*}.$$

Thus, in particular, fluctuations of the random variable $H(\boldsymbol{z}, \boldsymbol{\gamma})$ are asymptotically Gaussian, and 2σ is essentially its asymptotic variance.

The proof of Theorem 3.10 will require two ingredients: an a priori bound for the exponential moment of $H(\boldsymbol{z}, \boldsymbol{\gamma})$ and a biased loop equation.

3.4.1 A priori bound. Denote by $\mathrm{d}_{\mathbb{H}}(z, w)$ the hyperbolic distance in $\mathbb{H}$. Recall that

$$\mathrm{d}_{\mathbb{H}} = 2 \log(\delta/2 + \sqrt{1 + \delta^2/4}),$$

where

$$\delta(z, w) = \frac{|z - w|}{\sqrt{\operatorname{Im} z \operatorname{Im} w}},$$

and note that under our assumptions, $\delta(z_i, z_j) \leqslant \Delta/\eta_*$. Hence, $\mathrm{d}_{\mathbb{H}}(z_i, z_j) \lesssim \varphi(\Delta/\eta_*)$, where $\varphi(x) \stackrel{\text{def}}{=} x \wedge (1 + \log_+ x)$.

Corollary 3.11. *We have*

$$\log \mathbb{E}_\beta \Big[e^{H(\boldsymbol{z}, \boldsymbol{\gamma}) - \pi \sum_{i=1}^m \gamma_i \operatorname{Im} z_i} \Big] \leqslant O_{\beta,m}(1)(M^2 \varphi(\Delta/\eta_*)^2 + 1). \tag{3.22}$$

Note that, for proving Theorem 3.10, we will be using a less accurate form of this bound: *for $\eta_* \geqslant 1$ and $M \geqslant 1$,*

$$\log \mathbb{E}_\beta \Big[e^{H(\boldsymbol{z}, \boldsymbol{\gamma}) - \pi \sum_{i=1}^m \gamma_i \operatorname{Im} z_i} \Big] \leqslant O_{\beta,m}(1) M^2 \ell^2(\Delta). \tag{3.23}$$

Proof. For any z, w in the upper half plane, and any curve from w to z included in $\mathbb{H}$, we write

$$\sum_\lambda (\log|z-\lambda| - \log|w-\lambda|) = \operatorname{Re}\int_z^w s(u)\,\mathrm{d}u = \operatorname{Re}\,\mathrm{i}\pi(w-z) + \operatorname{Re}\int_z^w (s(u)-\mathrm{i}\pi)\,\mathrm{d}u,$$

so that

$$\begin{aligned}\mathbb{E}_\beta[e^{\gamma(\sum_\lambda(\log|z-\lambda|-\log|w-\lambda|)-\pi\mathrm{Im}(z-w))}] &\leqslant \sum_{\ell\geqslant 0}\frac{|\gamma|^\ell}{\ell!}\mathbb{E}_\beta\left[\left(\int_z^w |s(u)-\mathrm{i}\pi|\cdot|\,\mathrm{d}u|\right)^\ell\right]\\ &\leqslant \sum_{\ell\geqslant 0}\frac{|\gamma|^\ell}{\ell!}\left(\int_z^w \left(\mathbb{E}_\beta|s(u)-\mathrm{i}\pi|^\ell\right)^{1/\ell}\cdot|\,\mathrm{d}u|\right)^\ell\\ &\leqslant C'\sum_{\ell\geqslant 0}\frac{(C|\gamma|)^\ell}{\ell!}\,\ell^{\ell/2}\left(\int_z^w \frac{|\,\mathrm{d}u|}{\operatorname{Im} u}\right)^\ell,\end{aligned}$$

where (for $\ell\geqslant 1$) we used Corollary 3.9 in the last inequality. The above integral is minimized by the geodesic distance between z and w for the Poincaré metric in $\mathbb{H}$, so that we can choose a contour such that

$$\int_z^w \frac{|\,\mathrm{d}u|}{\operatorname{Im} u} = \mathrm{d}_{\mathbb{H}}(z,w).$$

We have thus proved that

$$\mathbb{E}_\beta\left[e^{\gamma(\sum_\lambda(\log|z-\lambda|-\log|w-\lambda|)-\pi\mathrm{Im}(z-w))}\right] \leqslant C'\sum_{\ell\geqslant 0}\frac{(C|\gamma|\mathrm{d}_{\mathbb{H}}(z,w))^\ell}{\ell!}\ell^{\ell/2} \leqslant C' e^{C\gamma^2\mathrm{d}_{\mathbb{H}}(z,w)^2}.$$

Then, for any $\gamma_1,\dots,\gamma_m\in\mathbb{R}$, $z_1,\dots,z_m\in\mathbb{H}$, $w_1,\dots,w_m\in\mathbb{H}$, $m\geqslant 1$, applying Hölder's inequality we get

$$\begin{aligned}&\mathbb{E}_\beta\left[e^{\sum_{i=1}^m\gamma_i\left(\sum_\lambda(\log|z_i-\lambda|-\log|w_i-\lambda|)-\pi\,\mathrm{Im}(z_i-w_i)\right)}\right]\\ &\leqslant\left(\prod_{i=1}^m\mathbb{E}_\beta\left[e^{m\gamma_i\left(\sum_\lambda(\log|z_i-\lambda|-\log|w_i-\lambda|)-\pi\,\mathrm{Im}(z_i-w_i)\right)}\right]\right)^{1/m} \leqslant C' e^{Cm\sum_{i=1}^m\gamma_i^2\mathrm{d}_{\mathbb{H}}(z_i,w_i)^2}. \quad (3.24)\end{aligned}$$

Now, recalling that $\sum_{i=1}^m\gamma_i = 0$ and that $\|\boldsymbol{\gamma}\|_\infty\leqslant M$, and applying (3.24) with $w_i = z_1$, $1\leqslant i\leqslant m$, we get

$$\begin{aligned}\log\mathbb{E}_\beta\left[e^{\sum_\lambda h(\lambda)-\pi\sum_{i=1}^m\gamma_i\,\mathrm{Im}\,z_i}\right] &\leqslant C(M^2m^2\max_{1\leqslant i\leqslant m}\mathrm{d}_{\mathbb{H}}(z_i,z_1)^2+1)\\ &\leqslant O_{\beta,m}(1)(M^2\varphi(\Delta/\eta_*)^2+1),\end{aligned}$$

concluding the proof. □

Remark 3.12. The same proof gives a similar bound for the scaled finite N Gaussian β-ensemble:

$$\limsup_{N\to\infty}\log\mathbb{E}_{\beta,N}\left[e^{\sum_{k=1}^N h(N\lambda_k/\pi)-\pi\sum_{i=1}^m\gamma_i\,\mathrm{Im}\,z_i}\right] \leqslant O_{\beta,m}(1)(M^2\varphi(\Delta/\eta_*)^2+1), \quad (3.25)$$

that is, a pre-limit version of (3.22).

3.4.2 ***Biased loop equation.*** Given $\boldsymbol{z} = (z_1, \dots, z_m) \in \mathbb{H}^m$, $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_m) \in \mathbb{R}^m$, with $\sum_{i=1}^m \gamma_i = 0$, and $h(\lambda) = h(\lambda, \boldsymbol{z})$, $H = H(\boldsymbol{z}, \boldsymbol{\gamma})$ as above, define the tilted probability measure

$$\mathbb{P}^h_\beta[A] \stackrel{\text{def}}{=} \frac{\mathbb{E}_\beta\Big[\mathbb{1}_A e^H\Big]}{\mathbb{E}_\beta[e^H]}.$$

Then

$$\mathbb{E}^h_\beta[f] = \frac{\mathbb{E}_\beta[f e^H]}{\mathbb{E}_\beta[e^H]}\,.$$

Lemma 3.13. *For any $w \in \mathbb{H}$ we have*

$$\mathbb{E}^h_\beta\Big[s(w)^2\Big] + \pi^2 + \Big(\frac{2}{\beta} - 1\Big)\mathbb{E}^h_\beta[s'(w)] - \frac{2}{\beta}\mathbb{E}^h_\beta\Big[\sum \frac{h'(\lambda)}{\lambda - w}\Big] = 0.$$

Proof. Our starting point is a finite N version of this equation. For $g \in C^1(\mathbb{R})$ having a subquadratic growth at $\pm\infty$, define the biased measure

$$\mathbb{P}^g_{\beta,N}[A] = \frac{\mathbb{E}_{\beta,N}\Big[\mathbb{1}_A e^{\sum_{i=1}^N g(\lambda_i)}\Big]}{\mathbb{E}_{\beta,N}\Big[e^{\sum_{i=1}^N g(\lambda_i)}\Big]}.$$

Then, also assuming that g' has at most polynomial growth,

$$\mathbb{E}^g_{\beta,N}\Big[s_N(z)^2\Big] + z\mathbb{E}^g_{\beta,N}[s_N(z)] + 1 + \frac{1}{N}\Big(\frac{2}{\beta} - 1\Big)\mathbb{E}^g_{\beta,N}[s'_N(z)] \\ - \frac{2}{\beta N^2}\mathbb{E}^g_{\beta,N}\Big[\sum_{k=1}^N \frac{g'(\lambda_k)}{\lambda_k - z}\Big] = 0, \quad z \in \mathbb{C} \setminus \mathbb{R}, \quad (3.26)$$

which is equation (A.4) in [8]. For the reader's convenience, we provide the proof. The density of the measure $\mathbb{P}^g_{\beta,N}$ equals

$$\rho^g_N(\lambda_1, \dots, \lambda_N) = C \prod_{1 \leqslant k < \ell \leqslant N} |\lambda_k - \lambda_\ell|^\beta \exp\Big[-\frac{\beta N}{4}\sum_{k=1}^N \lambda_k^2 + \sum_{k=1}^N g(\lambda_k)\Big].$$

Consider the chamber $\mathsf{Ch}_N = \{\lambda_1 < \lambda_2 < \ldots < \lambda_N\}$. Inside the chamber the density ρ^g_N is smooth. Then, (formally) integrating by parts, we get

$$\int_{\mathsf{Ch}_N} \sum_{k=1}^N \partial_{\lambda_k}\Big(\frac{\rho^g_N(\lambda_1, \dots, \lambda_N)}{\lambda_k - z}\Big)\,\mathrm{d}\lambda_1 \dots \mathrm{d}\lambda_N = 0.$$

To make this rigorous, we truncate the chamber by $|\lambda_k| \leqslant R$, $\lambda_{k+1} - \lambda_k \geqslant \varepsilon$, and integrate by parts over the truncated chamber. The boundary term at infinity vanishes since ρ^g_N contains

the Gaussian factor while the growth of g is subquadratic. The boundary terms near collision hyperplanes $\{\lambda_k = \lambda_{k+1}\}$ vanish since the density contains the factor $(\lambda_{k+1} - \lambda_k)^\beta$ with $\beta > 0$. Moreover, the terms arising after differentiation, being of order $(\lambda_{k+1} - \lambda_k)^{\beta-1}$ are locally integrable near the collisions. Hence,

$$\int_{\mathsf{Ch}_N} \sum_{k=1}^{N} \Big(\partial_{\lambda_k} \frac{1}{\lambda_k - z} + \frac{1}{\lambda_k - z} \partial_{\lambda_k} \log \rho_N^g \Big) \rho_N^g \, \mathrm{d}\lambda_1 \dots \mathrm{d}\lambda_N = 0,$$

that is,

$$\mathbb{E}_{\beta,N}^g \Big[\sum_{k=1}^{N} \Big(\partial_{\lambda_k} \frac{1}{\lambda_k - z} + \frac{1}{\lambda_k - z} \partial_{\lambda_k} \log \rho_N^g \Big) \Big] = 0.$$

Taking into account that

$$\partial_{\lambda_k} \log \rho_N^g = \beta \sum_{\ell \neq k} \frac{1}{\lambda_k - \lambda_\ell} - \frac{\beta N}{2} \lambda_k + g'(\lambda_k),$$

we arrive at

$$\mathbb{E}_{\beta,N}^g \Big[- \sum_{k=1}^{N} \frac{1}{(\lambda_k - z)^2} + \beta \sum_{k=1}^{N} \sum_{\ell \neq k} \frac{1}{(\lambda_k - z)(\lambda_k - \lambda_\ell)} - \frac{\beta N}{2} \sum_{k=1}^{N} \frac{\lambda_k}{\lambda_k - z} + \sum_{k=1}^{N} \frac{g'(\lambda_k)}{\lambda_k - z} \Big] = 0.$$

Pairing the terms indexed by (k, ℓ) and (ℓ, k) in the second summand, we get

$$\begin{aligned} \sum_{k=1}^{N} \sum_{\ell \neq k} \frac{1}{(\lambda_k - z)(\lambda_k - \lambda_\ell)} &= \sum_{1 \leqslant k < \ell \leqslant N} \frac{1}{\lambda_k - \lambda_\ell} \Big(\frac{1}{\lambda_k - z} - \frac{1}{\lambda_\ell - z} \Big) \\ &= - \sum_{k < \ell} \frac{1}{(\lambda_k - z)(\lambda_\ell - z)} = - \frac{1}{2} \Big[\Big(\sum_{k=1}^{N} \frac{1}{\lambda_k - z} \Big)^2 - \sum_{k=1}^{N} \frac{1}{(\lambda_k - z)^2} \Big]. \end{aligned}$$

Rewriting the third term as

$$\sum_{k=1}^{N} \frac{\lambda_k}{\lambda_k - z} = N + z \sum_{k=1}^{N} \frac{1}{\lambda_k - z},$$

and recalling that

$$s_N(z) = \frac{1}{N} \sum_{k=1}^{N} \frac{1}{\lambda_k - z}, \quad s_N'(z) = \frac{1}{N} \sum_{k=1}^{N} \frac{1}{(\lambda_k - z)^2}$$

(and multiplying the resulting identity by $-2/(\beta N^2)$), we get equation (3.26).

As above, we put

$$\lambda_{k,N} = N\lambda_k/\pi, \ w = Nz/\pi, \ \widetilde{s}_N(w) = \pi s_N(\pi w/N) = \sum_{k=1}^{N} \frac{1}{\lambda_{k,N} - w},$$

and

$$g_N(\lambda) = h(N\lambda/\pi) = \sum_{i=1}^{m} \gamma_i \log|\pi z_i/N - \lambda|.$$

Then $g_N(\lambda_k) = h(\lambda_{k,N})$. Substituting all of this in (3.26) and multiplying by π^2, we get

$$\mathbb{E}^{g_N}_{\beta,N}[\widetilde{s}_N(w)^2] + \frac{\pi^2 w}{N}\,\mathbb{E}^{g_N}_{\beta,N}[\widetilde{s}_N(w)] + \pi^2 + \Big(\frac{2}{\beta} - 1\Big)\mathbb{E}^{g_N}_{\beta,N}[\widetilde{s}'_N(w)] - \frac{2}{\beta}\,\mathbb{E}^{g_N}_{\beta,N}\Big[\sum_{k=1}^{N} \frac{h'(\lambda_{k,N})}{\lambda_{k,N} - w}\Big] = 0. \quad (3.27)$$

It remains to justify the $N \to \infty$ limit.

Set

$$H_N = \sum_{k=1}^{N} h(\lambda_{k,N}) = \sum_{k=1}^{N} g_N(\lambda_k), \quad T_N(w) = \sum_{k=1}^{N} \frac{h'(\lambda_{k,N})}{\lambda_{k,N} - w}\,.$$

By Corollary 3.8, we have convergence in distribution

$$\big\langle H_N, \widetilde{s}_N(w), \widetilde{s}'_N(w), T_N(w)\big\rangle \overset{\mathrm{d}}{\Longrightarrow} \big\langle H, s(w), s'(w), \sum_{\lambda} \frac{h'(\lambda)}{\lambda - w}\,\big\rangle. \quad (3.28)$$

This is obvious for $\widetilde{s}'_N(w)$. To see this for H_N, we represent it in the form

$$H_N = \sum_{i=1}^{m} \gamma_i\,\mathrm{Re}\Big[\int_{z_i}^{z_1} \widetilde{s}_N(u)\ \mathrm{d}u\Big].$$

To see this for T_N, we note that

$$h'(\lambda) = \frac{1}{2}\sum_{i=1}^{m} \gamma_i\Big(\frac{1}{\lambda - z_i} + \frac{1}{\lambda - \bar{z}_i}\Big),$$

and write

$$T_N(w) = \frac{1}{2}\sum_{i=1}^{m} \gamma_i\Big(\frac{\widetilde{s}_N(z_i) - \widetilde{s}_N(w)}{z_i - w} + \frac{\widetilde{s}_N(\bar{z}_i) - \widetilde{s}_N(w)}{\bar{z}_i - w}\Big).$$

In both cases we get continuous functionals of $\widetilde{s}_N$ in the topology of $\mathcal{H}$.

Next, we note that the finite N version of the a priori estimate (Remark 3.12) applied with $q\gamma_i$ gives us, for every finite positive q,

$$\sup_N \mathbb{E}_{\beta,N}[e^{qH_N}] < \infty,$$

while the local law (Theorem 3.2) yields uniform moments of every fixed order for $\widetilde{s}_N(w)$, $\widetilde{s}_N(z_i)$, $\widetilde{s}_N(\bar{z}_i)$, and therefore, also for $\widetilde{s}'_N(w)$ and for $T_N(w)$. Thus, for each of the functions

$$f_N = 1,\ \widetilde{s}_N(w),\ \widetilde{s}_N(w)^2,\ \widetilde{s}'_N(w),\ T_N(w),$$

the family $(f_N e^{H_N})_N$ is $\mathbb{P}_{\beta,N}$-uniformly integrable, which yields

$$\mathbb{E}_{\beta,N}[f_N e^{H_N}] \to \mathbb{E}_\beta[f e^H],$$

in particular,

$$\mathbb{E}_{\beta,N}[e^{H_N}] \to \mathbb{E}_\beta[e^H] > 0.$$

Therefore,

$$\mathbb{E}^{g_N}_{\beta,N}[f_N] = \frac{\mathbb{E}_{\beta,N}[f_N e^{H_N}]}{\mathbb{E}_{\beta,N}[e^{H_N}]} \to \frac{\mathbb{E}_\beta[f e^H]}{\mathbb{E}_\beta[e^H]} = \mathbb{E}^h_\beta[f],$$

completing the proof. □

3.4.3 Proof of Theorem 3.10. By translation-invariance of the sine-beta process $\mathcal{X}_\beta$, we assume throughout the proof that

$$\operatorname{Re} z_j \in [-\Delta/2, \Delta/2], \quad 1 \leqslant j \leqslant m. \tag{3.29}$$

To get started, we fix $0 \leqslant t \leqslant 1$, and write

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \log \mathbb{E}_\beta[e^{tH(\boldsymbol{z},\boldsymbol{\gamma})}] &= \frac{\mathbb{E}_\beta[H(\boldsymbol{z},\boldsymbol{\gamma}) e^{tH(\boldsymbol{z},\boldsymbol{\gamma})}]}{\mathbb{E}_\beta[e^{tH(\boldsymbol{z},\boldsymbol{\gamma})}]} \\
&= \mathbb{E}^{th}_\beta[H(\boldsymbol{z},\boldsymbol{\gamma})] \\
&= \mathbb{E}^{th}_\beta\Big[\operatorname{Re}\Big(\sum_{j=1}^m \gamma_j \int_{z_j}^{Z_j} s(w)\,\mathrm{d}w\Big) + H(\boldsymbol{Z},\boldsymbol{\gamma})\Big] \\
&= \operatorname{Re}\Big(\sum_{j=1}^m \gamma_j \int_{z_j}^{Z_j} \mathbb{E}^{th}_\beta[s(w)]\,\mathrm{d}w\Big) + \mathbb{E}^{th}_\beta[H(\boldsymbol{Z},\boldsymbol{\gamma})].
\end{aligned}$$

Note that the integrand on the RHS is analytic in $\mathbb{H}$, so we are free to modify the integration contour. We take $Z_j = \mathrm{i}\eta_*$, $1 \leqslant j \leqslant m$. Since $\sum_{j=1}^m \gamma_j = 0$, $H(\boldsymbol{Z},\boldsymbol{\gamma}) = 0$, and we get

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \log \mathbb{E}_\beta[e^{tH(\boldsymbol{z},\boldsymbol{\gamma})}] &= \operatorname{Re}\Big(\sum_{j=1}^m \gamma_j \int_{z_j}^{\mathrm{i}\eta_*} \mathbb{E}^{th}_\beta[s(w)]\,\mathrm{d}w\Big) \\
&= \pi \sum_{j=1}^m \gamma_j \operatorname{Im} z_j + \operatorname{Re}\Big(\sum_{j=1}^m \gamma_j \int_{z_j}^{\mathrm{i}\eta_*} \mathbb{E}^{th}_\beta[s(w) - \mathrm{i}\pi]\,\mathrm{d}w\Big).
\end{aligned} \tag{3.30}$$

We aim to show that

$$\mathbb{E}^{th}_\beta[s(w) - \mathrm{i}\pi] = -\frac{t}{\beta} \sum_{i=1}^m \frac{\gamma_i}{\bar{z}_i - w} + \text{error terms},$$

and insert this in (3.30). To see this, we employ the biased loop equation from Lemma 3.13. It yields that

$$\mathbb{E}^{th}_\beta[(s(w) - \mathrm{i}\pi)^2] + 2\mathrm{i}\pi \mathbb{E}^{th}_\beta[s(w) - \mathrm{i}\pi] + \Big(\frac{2}{\beta} - 1\Big)\mathbb{E}^{th}_\beta[s'(w)] = \frac{2t}{\beta}\,\mathbb{E}^{th}_\beta\Big[\sum_\lambda \frac{h'(\lambda)}{\lambda - w}\Big],$$

whence,

$$\mathbb{E}_\beta^{th}[s(w)-\mathrm{i}\pi] = \frac{t}{\mathrm{i}\pi\beta}\,\mathbb{E}_\beta^{th}\Big[\sum_\lambda \frac{h'(\lambda)}{\lambda-w}\Big] - \frac{1}{2\pi\mathrm{i}}\Big(\frac{2}{\beta}-1\Big)\mathbb{E}_\beta^{th}[s'(w)] - \frac{1}{2\pi\mathrm{i}}\mathbb{E}_\beta^{th}[(s(w)-\mathrm{i}\pi)^2]. \quad (3.31)$$

Next,

$$\begin{aligned}\frac{h'(\lambda)}{\lambda-w} &= \frac{1}{2(\lambda-w)}\sum_{i=1}^m \gamma_i\Big(\frac{1}{\lambda-z_i}+\frac{1}{\lambda-\bar z_i}\Big)\\ &= \frac{1}{2}\sum_{i=1}^m \gamma_i\Big(\frac{1}{z_i-w}\Big(\frac{1}{\lambda-z_i}-\frac{1}{\lambda-w}\Big)+\frac{1}{\bar z_i-w}\Big(\frac{1}{\lambda-\bar z_i}-\frac{1}{\lambda-w}\Big)\Big),\end{aligned}$$

whence

$$\mathbb{E}_\beta^{th}\Big[\sum_\lambda \frac{h'(\lambda)}{\lambda-w}\Big] = \frac{1}{2}\,\mathbb{E}_\beta^{th}\Big[\sum_{i=1}^m \gamma_i\Big(\frac{s(z_i)-s(w)}{z_i-w}+\frac{s(\bar z_i)-s(w)}{\bar z_i-w}\Big)\Big].$$

Plugging this in (3.31), we get

$$\begin{aligned}\mathbb{E}_\beta^{th}[s(w)-\mathrm{i}\pi] = \frac{t}{2\mathrm{i}\pi\beta}\,\mathbb{E}_\beta^{th}\Big[\sum_{i=1}^m \gamma_i\Big(\frac{s(z_i)-s(w)}{z_i-w}+\frac{s(\bar z_i)-s(w)}{\bar z_i-w}\Big)\Big]\\ - \frac{1}{2\pi\mathrm{i}}\Big(\frac{2}{\beta}-1\Big)\mathbb{E}_\beta^{th}[s'(w)] - \frac{1}{2\pi\mathrm{i}}\mathbb{E}_\beta^{th}[(s(w)-\mathrm{i}\pi)^2]. \quad (3.32)\end{aligned}$$

Next, we note that

$$\mathbb{E}_\beta^{th}\Big[\sum_{i=1}^m \gamma_i\frac{s(\bar z_i)-s(w)}{\bar z_i-w}\Big] = -2\pi\mathrm{i}\sum_{i=1}^m\frac{\gamma_i}{\bar z_i-w} + \mathbb{E}_\beta^{th}\Big[\sum_{i=1}^m \gamma_i\frac{(s(\bar z_i)+\mathrm{i}\pi)-(s(w)-\mathrm{i}\pi)}{\bar z_i-w}\Big].$$

Inserting this into (3.32), and then the outcome into (3.30), we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\log\mathbb{E}_\beta[e^{tH(\boldsymbol{z},\boldsymbol{\gamma})}] = \pi\sum_{j=1}^m\gamma_j\,\mathrm{Im}\,z_j + \frac{t}{\beta}\sum_{1\leqslant i,j\leqslant m}\gamma_i\gamma_j\log\frac{1}{|z_i-\bar z_j|}\\ + \mathrm{Re}\Big(\sum_{j=1}^m\gamma_j\int_{z_j}^{\mathrm{i}\eta_*}\sum_{\ell=1}^4 J_\ell(w)\,\mathrm{d}w\Big), \quad (3.33)\end{aligned}$$

with

$$\begin{aligned}J_1(w) &= -\frac{1}{2\pi\mathrm{i}}\Big(\frac{2}{\beta}-1\Big)\mathbb{E}_\beta^{th}[s'(w)],\\ J_2(w) &= -\frac{1}{2\pi\mathrm{i}}\mathbb{E}_\beta^{th}[(s(w)-\mathrm{i}\pi)^2],\\ J_3(w) &= \frac{t}{2\pi\mathrm{i}\beta}\sum_{i=1}^m\gamma_i\mathbb{E}_\beta^{th}\Big[\frac{(s(\bar z_i)+\mathrm{i}\pi)-(s(w)-\mathrm{i}\pi)}{\bar z_i-w}\Big]\\ J_4(w) &= \frac{t}{2\pi\mathrm{i}\beta}\sum_{i=1}^m\gamma_i\mathbb{E}_\beta^{th}\Big[\frac{s(z_i)-s(w)}{z_i-w}\Big].\end{aligned}$$

To proceed with estimates of the integrals of these functions, we need the following claim. Recall that $\ell(\Delta) = 1 \vee \log \Delta$.

Claim 3.14. *Let* $\operatorname{Im} w \geqslant 1$, $M \geqslant 1$, *and* $\eta_* \geqslant 1$. *Then*

(i)
$$\mathbb{E}_\beta^{th}[|s(w) - \mathrm{i}\pi|] \lesssim_{\beta,m} \frac{M}{\operatorname{Im} w}\, \ell(\Delta);$$

(ii)
$$\mathbb{E}_\beta^{th}[|s(w) - \mathrm{i}\pi|^2] \lesssim_{\beta,m} \frac{M^2}{(\operatorname{Im} w)^2}\, \ell^2(\Delta);$$

(iii)
$$\mathbb{E}_\beta^{th}[|s'(w)|] \lesssim_{\beta,m} \frac{M}{(\operatorname{Im} w)^2}\, \ell(\Delta).$$

Proof of Claim 3.14:

(i) Given $u > 0$, we have

$$\mathbb{E}_\beta^{th}[|s(w) - \mathrm{i}\pi|] \leqslant u + \mathbb{E}_\beta^{th}\big[\mathbb{1}_{\{|s(w)-\mathrm{i}\pi| \geqslant u\}} |s(w) - \mathrm{i}\pi|\big].$$

Furthermore,

$$\begin{aligned}\mathbb{E}_\beta^{th}\big[\mathbb{1}_{\{|s(w)-\mathrm{i}\pi| \geqslant u\}} |s(w) - \mathrm{i}\pi|\big] &= \frac{\mathbb{E}_\beta\big[\mathbb{1}_{\{|s(w)-\mathrm{i}\pi| \geqslant u\}} |s(w) - \mathrm{i}\pi| e^{tH}\big]}{\mathbb{E}_\beta\big[e^{tH}\big]} \\ &\leqslant \big(\mathbb{E}_\beta\big[|s(w) - \mathrm{i}\pi|^2\big]\big)^{1/2} \cdot \big(\mathbb{P}_\beta[|s(w) - \mathrm{i}\pi| \geqslant u]\big)^{1/4} \cdot \frac{\big(\mathbb{E}_\beta\big[e^{4tH}\big]\big)^{1/4}}{\mathbb{E}_\beta\big[e^{tH}\big]}.\end{aligned}$$

By Corollary 3.9,

$$\big(\mathbb{E}_\beta\big[|s(w) - \mathrm{i}\pi|^2\big]\big)^{1/2} \lesssim_\beta \frac{1}{\operatorname{Im} w},$$

and (optimizing in q)

$$\big(\mathbb{P}_\beta[|s(w) - \mathrm{i}\pi| \geqslant u]\big)^{1/4} \lesssim_\beta e^{-cu^2 (\operatorname{Im} w)^2}.$$

At last,

$$\frac{\big(\mathbb{E}_\beta\big[e^{4tH}\big]\big)^{1/4}}{\mathbb{E}_\beta\big[e^{tH}\big]} = \frac{\big(\mathbb{E}_\beta\big[e^{4t(H(\boldsymbol{z},\boldsymbol{\gamma}) - \pi \sum_{i=1}^m \gamma_i \operatorname{Im} z_i)}\big]\big)^{1/4}}{\mathbb{E}_\beta\big[e^{t(H(\boldsymbol{z},\boldsymbol{\gamma}) - \pi \sum_{i=1}^m \gamma_i \operatorname{Im} z_i)}\big]}.$$

Since $\mathbb{E}_\beta\big[H(\boldsymbol{z},\boldsymbol{\gamma}) - \pi \sum_{i=1}^m \gamma_i \operatorname{Im} z_i\big] = 0$, by Jensen, the denominator of the expression on the RHS is $\geqslant 1$. By the a priori bound (3.23), the numerator is $\lesssim e^{C(m,\beta) M^2 \ell^2(\Delta)}$. Putting all these estimates together, we get

$$\mathbb{E}_\beta^{th}[|s(w) - \mathrm{i}\pi|] \leqslant u + \frac{C}{\operatorname{Im} w} e^{-cu^2 (\operatorname{Im} w)^2 + C(m,\beta) M^2 \ell^2(\Delta)}.$$

Letting $u = AM\ell(\Delta)/\operatorname{Im} w$, with sufficiently large $A = A(m,\beta)$, we get (i).
(ii) Similarly to (i), we start with

$$\mathbb{E}_\beta^{th}[|s(w)-\mathrm{i}\pi|^2] \leqslant u^2 + \mathbb{E}_\beta^{th}\big[\mathbb{1}_{\{|s(w)-\mathrm{i}\pi|\geqslant u\}}|s(w)-\mathrm{i}\pi|^2\big]$$

and get

$$\begin{aligned}\mathbb{E}_\beta^{th}\big[\mathbb{1}_{\{|s(w)-\mathrm{i}\pi|\geqslant u\}}|s(w)-\mathrm{i}\pi|^2\big] &= \frac{\mathbb{E}_\beta\big[\mathbb{1}_{\{|s(w)-\mathrm{i}\pi|\geqslant u\}}|s(w)-\mathrm{i}\pi|^2 e^{tH}\big]}{\mathbb{E}_\beta\big[e^{tH}\big]}\\ &\leqslant \big(\mathbb{E}_\beta\big[|s(w)-\mathrm{i}\pi|^4\big]\big)^{1/2}\cdot\big(\mathbb{P}_\beta[|s(w)-\mathrm{i}\pi|\geqslant u]\big)^{1/4}\cdot\frac{\big(\mathbb{E}_\beta\big[e^{4tH}\big]\big)^{1/4}}{\mathbb{E}_\beta\big[e^{tH}\big]}\\ &\leqslant \frac{C}{(\operatorname{Im} w)^2}\,e^{-cu^2(\operatorname{Im} w)^2+C_{m,\beta}M^2\ell^2(\Delta)},\end{aligned}$$

which, as above, yields (ii).
(iii) By Cauchy's formula, we have

$$|s'(w)| = \Big|\frac{1}{2\pi\mathrm{i}}\int_{|\zeta-w|=\frac{1}{10}\operatorname{Im} w}\frac{s(\zeta)-\mathrm{i}\pi}{(\zeta-w)^2}\,\mathrm{d}\zeta\Big| \lesssim \frac{1}{(\operatorname{Im} w)^2}\int_{|\zeta-w|=\frac{1}{10}\operatorname{Im} w}|s(\zeta)-\mathrm{i}\pi|\,|\,\mathrm{d}\zeta|.$$

Hence,

$$\mathbb{E}_\beta^{th}[|s'(w)|] \lesssim \frac{1}{(\operatorname{Im} w)^2}\int_{|\zeta-w|=\frac{1}{10}\operatorname{Im} w}\mathbb{E}_\beta^{th}[|s(\zeta)-\mathrm{i}\pi|]\,|\,\mathrm{d}\zeta|.$$

Then, applying (i) with ζ instead of w (note that (i) persists in the range $\operatorname{Im} w \geqslant \frac{1}{2}$), we get

$$\mathbb{E}_\beta^{th}[|s'(w)|] \lesssim_{\beta,m} \frac{1}{(\operatorname{Im} w)^2}\cdot\frac{M}{\operatorname{Im} w}\ell(\Delta)\cdot\operatorname{Im} w \lesssim_{\beta,m} \frac{M}{(\operatorname{Im} w)^2}\ell(\Delta),$$

proving the claim. □

Now, we readily estimate each of the four integrals on the RHS of (3.33). By (ii) and (iii) in Claim 3.14, for $\operatorname{Im} w$ large, the functions $J_1(w)$ and $J_2(w)$ decay as $(\operatorname{Im} w)^{-2}$. Hence, deforming the integration contours to vertical lines going upwards and connecting the points z_j and $\mathrm{i}\eta_*$ with infinity, we get

$$\begin{aligned}\Big|\sum_{j=1}^m\gamma_j\int_{z_j}^{\mathrm{i}\eta_*}J_1(w)\,\mathrm{d}w\Big| &= \Big|\frac{1}{2\pi\mathrm{i}}\Big(\frac{2}{\beta}-1\Big)\sum_{j=1}^m\gamma_j\int_{z_j}^{\mathrm{i}\eta_*}\mathbb{E}_\beta^{th}[s'(w)]\,\mathrm{d}w\Big|\\ &\lesssim_{m,\beta} M\sum_{j=1}^m\Big(\int_{\eta_*}^\infty+\int_{\operatorname{Im} z_j}^\infty\Big)\frac{M\ell(\Delta)}{t^2}\,\mathrm{d}t \lesssim_{m,\beta}\frac{M^2\ell(\Delta)}{\eta_*},\end{aligned}$$

and similarly,

$$\Big|\sum_{j=1}^m\gamma_j\int_{z_j}^{\mathrm{i}\eta_*}J_2(w)\,\mathrm{d}w\Big| = \Big|\frac{1}{2\pi\mathrm{i}}\sum_{j=1}^m\gamma_j\int_{z_j}^{\mathrm{i}\eta_*}\mathbb{E}_\beta^{th}[(s(w)-\mathrm{i}\pi)^2]\,\mathrm{d}w\Big| \lesssim_{m,\beta}\frac{M^3\ell^2(\Delta)}{\eta_*}.$$

We will check momentarily that $J_3(w)$ decays as $(\operatorname{Im} w)^{-1}$, so, when dealing with integrals of this function, it is natural to deform the integration contour to the hyperbolic geodesics $\Gamma(z_j, \mathrm{i}\eta_*)$ connecting z_j with $\mathrm{i}\eta_*$. Indeed, using that $|s(\bar{z}) + \mathrm{i}\pi| = |s(z) - \mathrm{i}\pi|$, we get

$$\begin{aligned}
|J_3(w)| &= \Big| \frac{t}{2\pi\mathrm{i}\beta} \sum_{i=1}^m \gamma_i \mathbb{E}_\beta^{th}\Big[\frac{(s(\bar{z}_i) + \mathrm{i}\pi) - (s(w) - \mathrm{i}\pi)}{\bar{z}_i - w} \Big] \Big| \\
&\lesssim_{m,\beta} \frac{M}{\eta_* + \operatorname{Im} w} \big(\max_{1 \leqslant i \leqslant m} \mathbb{E}_\beta^{th}[|s(z_i) - \mathrm{i}\pi|] + \mathbb{E}_\beta^{th}[|s(w) - \mathrm{i}\pi|] \big) \\
&\lesssim_{m,\beta} \frac{M^2}{\eta_* + \operatorname{Im} w} \Big(\frac{1}{\eta_*} + \frac{1}{\operatorname{Im} w} \Big) \ell(\Delta) \qquad \text{(by (i) in Claim 3.14)} \\
&\lesssim_{m,\beta} \frac{M^2}{\eta_* \operatorname{Im} w}\, \ell(\Delta),
\end{aligned}$$

whence,

$$\Big| \sum_{j=1}^m \gamma_j \int_{\Gamma(z_j, \mathrm{i}\eta_*)} J_3(w)\, \mathrm{d}w \Big| \lesssim_{m,\beta} \frac{M^3 \ell(\Delta)}{\eta_*} \sum_{j=1}^m \mathrm{d}_{\mathbb{H}}(z_j, \mathrm{i}\eta_*) \lesssim_{m,\beta} \frac{M^3 \ell^2(\Delta)}{\eta_*}\,.$$

In the remaining integral of $J_4(w)$, we integrate over the interval $[z_j, \mathrm{i}\eta_*]$. To bound the integrand

$$\mathbb{E}_\beta^{th}\Big[\sum_{i=1}^m \gamma_i \frac{s(z_i) - s(w)}{z_i - w} \Big],$$

we separate the cases $|z_i - w| \leqslant \eta_*/2$ and $|z_i - w| \geqslant \eta_*/2$. In the first case we use that

$$\begin{aligned}
\mathbb{E}_\beta^{th}\Big[\Big| \frac{s(z_i) - s(w)}{z_i - w} \Big| \Big] &\leqslant \frac{1}{|z_i - w|} \int_{z_i}^w \mathbb{E}_\beta^{th}[|s'(\zeta)|]\, |\,\mathrm{d}\zeta| \\
&\lesssim_{m,\beta} \frac{M}{|z_i - w|} \int_{z_i}^w \frac{|\,\mathrm{d}\zeta|}{(\operatorname{Im}\zeta)^2}\, \ell(\Delta) \qquad \text{(by (iii) in Claim 3.14)} \\
&\lesssim_{m,\beta} \frac{M}{(\eta_* \wedge \operatorname{Im} w)^2}\, \ell(\Delta). \qquad (3.34)
\end{aligned}$$

In the second case we have

$$\begin{aligned}
\Big| \mathbb{E}_\beta^{th}\Big[\frac{s(z_i) - s(w)}{z_i - w} \Big] \Big| &= \Big| \mathbb{E}_\beta^{th}\Big[\frac{(s(z_i) - \mathrm{i}\pi) - (s(w) - \mathrm{i}\pi)}{z_i - w} \Big] \Big| \\
&\leqslant \frac{1}{|z_i - w|} \big(\mathbb{E}_\beta^{th}[|s(z_i) - \mathrm{i}\pi|] + \mathbb{E}_\beta^{th}[|s(w) - \mathrm{i}\pi|] \big) \\
&\lesssim_{m,\beta} \frac{M}{|z_i - w|} \Big(\frac{1}{\eta_*} + \frac{1}{\operatorname{Im} w} \Big) \ell(\Delta) \qquad \text{(by (i) in Claim 3.14)} \\
&\lesssim_{m,\beta} \frac{M}{|z_i - w|} \cdot \frac{1}{\eta_* \wedge \operatorname{Im} w}\, \ell(\Delta). \qquad (3.35)
\end{aligned}$$

Combining (3.34) and (3.35) and taking into account that on $[z_j, \mathrm{i}\eta_*]$ we have $\operatorname{Im} w \geqslant \eta_*$, we obtain

$$\sum_{i=1}^{m} \Big| \mathbb{E}_\beta^{th}\Big[\frac{s(z_i) - s(w)}{z_i - w} \Big] \Big| \lesssim_{m,\beta} \Big(\sum_{i=1}^{m} \frac{M}{|z_i - w| + \eta_*} \Big) \frac{\ell(\Delta)}{\eta_*}. \tag{3.36}$$

Thus,

$$\Big| \sum_{j=1}^{m} \gamma_j \int_{[z_j, \mathrm{i}\eta_*]} \Big(\sum_{i=1}^{m} \gamma_i \mathbb{E}_\beta^{th}\Big[\frac{s(z_i) - s(w)}{z_i - w} \Big] \Big) \, \mathrm{d}w \Big|$$
$$\lesssim_{m,\beta} M^3 \frac{\ell(\Delta)}{\eta_*} \sum_{j=1}^{m} \sum_{i=1}^{m} \int_{[z_j, \mathrm{i}\eta_*]} \frac{|\,\mathrm{d}w|}{|z_i - w| + \eta_*}. \tag{3.37}$$

To estimate the integral on the RHS, we use the following elementary claim:

Claim 3.15. *For any line segment I of length L, and point z, and any $a > 0$,*

$$\int_I \frac{|\,\mathrm{d}w|}{|w - z| + a} \leqslant 2 \log(1 + \frac{L}{a}).$$

To see this, parametrise the segment I by arclength $w(s)$, $0 \leqslant s \leqslant L$, and let s_0 be the orthogonal projection of z onto the line containing I. Then $|w(s) - z| \geqslant |s - s_0|$. Hence,

$$\int_I \frac{|\,\mathrm{d}w|}{|w - z| + a} \leqslant \int_0^L \frac{\mathrm{d}s}{|s - s_0| + a}.$$

If $s_0 \in [0, L]$, then splitting the integral at s_0, we get

$$\int_0^L \frac{\mathrm{d}s}{|s - s_0| + a} = \int_0^{s_0} \frac{\mathrm{d}s}{(s_0 - s) + a} + \int_{s_0}^L \frac{\mathrm{d}s}{(s - s_0) + a}$$
$$= \log(1 + \frac{s_0}{a}) + \log(1 + \frac{L - s_0}{a}) \leqslant 2 \log(1 + \frac{L}{a}).$$

If $s_0 \notin [0, L]$, the bound is only smaller, proving the claim.

Therefore,

$$\int_{[z_j, \mathrm{i}\eta_*]} \frac{|\,\mathrm{d}w|}{|z_i - w| + \eta_*} \lesssim \log(1 + \frac{|z_j - \mathrm{i}\eta_*|}{\eta_*}).$$

By our normalization, $|z_j - \mathrm{i}\eta_*| \leqslant |\operatorname{Re} z_j| + |\operatorname{Im} z_j - \eta_*| \leqslant \frac{3}{2}\Delta$, and since $\eta_* \geqslant 1$, we get

$$\int_{[z_j, \mathrm{i}\eta_*]} \frac{|\,\mathrm{d}w|}{|z_i - w| + \eta_*} \lesssim \ell(\Delta).$$

Plugging this bound in (3.37), we see that the integral of $J_4(w)$ is bounded by $O_{\beta,m}(1) M^3 \ell^2(\Delta)/\eta_*$. At last, returning to (3.33), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \log \mathbb{E}_\beta[e^{tH(\boldsymbol{z},\boldsymbol{\gamma})}] = \pi \sum_{j=1}^{m} \gamma_j \operatorname{Im} z_j + \frac{t}{\beta} \sum_{1 \leqslant i,j \leqslant m} \gamma_i \gamma_j \log \frac{1}{|z_i - \bar{z}_j|} + O_{\beta,m}(1) M^3 \frac{\ell^2(\Delta)}{\eta_*}.$$

Integrating in t from 0 to 1, we finish off the proof of Theorem 3.10. □

4 Proof of Theorem 1.1

We note that since the distribution of the point process $\mathcal{X}_\beta$ is invariant under the map $\lambda \mapsto -\lambda$, the distribution of the random function $t \mapsto |G_{\mathcal{X}_\beta}(t+\mathrm{i})|$ is invariant under the map $t \mapsto -t$. Hence, to prove Theorem 1.1, it suffices to verify conditions of Propositions 1.3 and 1.4 only for positive t.

4.1 Regularity of $\mathcal{X}_\beta$ and growth of $\log|G_{\mathcal{X}_\beta}(\mathrm{i}\eta)|$. Our goal in this section is to verify Assumption 1.2 and condition (1.6) for the process $\mathcal{X}_\beta$, which are required in order to apply Proposition 1.3. We begin with the following claim.

Claim 4.1. *The random process $\mathcal{X}_\beta$ is almost surely regular, that is, it satisfies Assumption 1.2.*

Proof. Conditions (1.1) and (1.2) follow from the stationarity and ergodicity of the SDE defining the Sine_β process, see, e.g., [30], but can also be read from Lemma 3.6, Part II. Putting there $u = A\sqrt{\log j}$ for integers $j \geqslant 2$, we get

$$\mathbb{P}_\beta\big[\big|\#(\mathcal{X}_\beta \cap [0,j]) - j\big| \geqslant A\log^2 j\big] \leqslant Cj^{-A^2c}.$$

Choosing A so that $A^2c > 1$ and applying Borel–Cantelli, we see that, a.s.,

$$\#(\mathcal{X}_\beta \cap [0,j]) = j + O(\log^2 j), \quad \text{as } j \to \infty.$$

By reflection symmetry, the analogous estimate holds on the negative half-line. This is much stronger than (1.1) and (1.2).

The condition (1.3) readily follows from Lemma 3.7, Part II: since the series $\sum_\lambda (1+|\lambda|^2)^{-1}$ a.s. converges, the existence of the a.s. $K \to \infty$ limit of $s^{(K)}(\mathrm{i}) = \sum_{|\lambda|\leqslant K}(\lambda - \mathrm{i})^{-1}$ implies the existence of the a.s. limit of $\sum_{|\lambda|\leqslant K}\lambda^{-1}$. □

Next, we provide several bounds for the random entire function $G_{\mathcal{X}_\beta}$, which we will be using several times.

Claim 4.2. *Suppose $z_1 = t_1 + \mathrm{i}u_1$, $z_2 = t_2 + \mathrm{i}u_2$ and $|z_1 - z_2| \leqslant 1$ with $u_2 \geqslant u_1 \geqslant 1$. Then*

$$\big|\log|G_{\mathcal{X}_\beta}(z_2)| - \log|G_{\mathcal{X}_\beta}(z_1)| - \pi(u_2-u_1)\big| \leqslant 4|z_2-z_1|\big(|s(z_1)-\mathrm{i}\pi| + \frac{\pi}{u_1}\big).$$

Proof. We have

$$\log|G_{\mathcal{X}_\beta}(z_2)| - \log|G_{\mathcal{X}_\beta}(z_1)| - \pi(u_2-u_1) = -\operatorname{Re}\Big[(z_2-z_1)\int_0^1 [s(z_1+t(z_2-z_1)) - \mathrm{i}\pi]\,\mathrm{d}t\Big],$$

and hence $\big|\log|G_{\mathcal{X}_\beta}(z_2)| - \log|G_{\mathcal{X}_\beta}(z_1)| - \pi(u_2-u_1)\big| \leqslant |z_2-z_1| \max\limits_{t\in[0,1]} |s(z_1+t(z_2-z_1)) - \mathrm{i}\pi|$. Assuming that $|z_2-z_1| \leqslant \frac{1}{2}$ and using that $|s'(z)| \leqslant \operatorname{Im} s(z)/\operatorname{Im} z \leqslant |s(z)-\mathrm{i}\pi| + \pi/u_1$ (since

$s(z)$ has positive imaginary part in $\mathbb{H}$ and $\operatorname{Im} z \geqslant 1$), we get

$$\begin{aligned}\max_{t\in[0,1]} |s(z_1 + t(z_2 - z_1)) - \mathrm{i}\pi| &\leqslant |s(z_1) - \mathrm{i}\pi| + \frac{1}{2}\int_0^1 |s'(z_1 + t(z_2 - z_1))|\,\mathrm{d}t \\ &\leqslant |s(z_1) - \mathrm{i}\pi| + \frac{\pi}{u_1} + \frac{1}{2}\int_0^1 |s(z_1 + t(z_2 - z_1)) - \mathrm{i}\pi|\,\mathrm{d}t \\ &\leqslant |s(z_1) - \mathrm{i}\pi| + \frac{\pi}{u_1} + \frac{1}{2}\max_{t\in[0,1]} |s(z_1 + t(z_2 - z_1)) - \mathrm{i}\pi|,\end{aligned}$$

which yields that $\max_{t\in[0,1]} |s(z_1 + t(z_2 - z_1)) - \mathrm{i}\pi| \leqslant 2[|s(z_1) - \mathrm{i}\pi| + \pi/u_1]$. Applying this bound twice, we prove the claim. □

Claim 4.3. *There exist positive constants C and c such that for all z with $\eta = \operatorname{Im} z > 0$ and all $A > 0$, we have*

$$\mathbb{P}_\beta\big[|s(z) - \mathrm{i}\pi| \geqslant A\big] \leqslant Ce^{-cA^2\eta^2}.$$

Proof. This is a straightforward application of Corollary 3.9. There is nothing to prove if $A\eta < 1$, so we may and will assume that $A\eta \geqslant 1$. For $q \geqslant 1$, we have

$$\mathbb{P}_\beta\big[|s(z) - \mathrm{i}\pi| \geqslant A\big] \leqslant \Big(\frac{1}{A}\Big)^q \mathbb{E}_\beta\big[|s(z) - \mathrm{i}\pi|^q\big] \leqslant \Big(\frac{C}{A\eta}\Big)^q q^{q/2}.$$

The claim follows by optimizing the value of q. □

Claim 4.4. *Let $\gamma \in \mathbb{R}$. Suppose that $1 \leqslant \operatorname{Im} z_1 \leqslant \operatorname{Im} z_2$, $|z_2|$ is sufficiently large, and $|z_1| \leqslant C'(\log|z_2|)^C$. Then*

$$\begin{aligned}&\log \mathbb{E}_\beta\big[e^{\gamma(\log|G_{\mathcal{X}_\beta}(z_2)| - \log|G_{\mathcal{X}_\beta}(z_1)| - \pi(\operatorname{Im} z_2 - \operatorname{Im} z_1))}\big] \\ &\qquad = \frac{\gamma^2}{\beta}\big[\log|z_2| - \frac{1}{2}\log\operatorname{Im} z_2 + O_{C,C'}(\log\log|z_2|)\big] + O_\beta(1)(1 + |\gamma|^3)\,\frac{\log^2|z_2|}{\operatorname{Im} z_1}.\end{aligned}$$

Proof. This is a straightforward consequence of Theorem 3.10 applied with $m = 2$ and $\gamma_1 = -\gamma$, $\gamma_2 = \gamma$, and $M = 1 \vee |\gamma|$. Then $H = \gamma[\log|G_{\mathcal{X}_\beta}(z_2)| - \log|G_{\mathcal{X}_\beta}(z_1)|]$,

$$\begin{aligned}\sigma(\boldsymbol{z}, \boldsymbol{\gamma}) &= -\frac{\gamma^2}{2\beta}\big[\log(2\operatorname{Im} z_2) - 2\log|z_2 - \bar{z}_1| + \log(2\operatorname{Im} z_1)\big] \\ &= \frac{\gamma^2}{\beta}\big[\log|z_2| - \tfrac{1}{2}\log\operatorname{Im} z_2 + O_{C,C'}(\log\log|z_2|)\big],\end{aligned}$$

$\eta_* = \operatorname{Im} z_1$, and $\ell(\Delta) \simeq \log|z_2|$. □

We can finally establish the following.

Claim 4.5. *There exists $A > 0$ such that, almost surely, for η sufficiently large,*

$$\log|G_{\mathcal{X}_\beta}(\mathrm{i}\eta)| \geqslant \pi\eta - A\log\eta.$$

That is, the random entire function $G_{\mathcal{X}_\beta}$ satisfies (1.6), almost surely.

Proof. Since $G_{\mathcal{X}_\beta}(0) = 1$ and the function $\eta \mapsto |G_{\mathcal{X}_\beta}(\mathrm{i}\eta)|$ increases, it suffices to estimate from below the function $\eta \mapsto \log|G_{\mathcal{X}_\beta}(\mathrm{i}\eta)| - \log|G_{\mathcal{X}_\beta}(\mathrm{i}\log\eta)|$. Put

$$\mathfrak{H}(\eta) = \log|G_{\mathcal{X}_\beta}(\mathrm{i}\eta)| - \log|G_{\mathcal{X}_\beta}(\mathrm{i}\log\eta)| - \pi(\eta - \log\eta).$$

Using Claim 4.4 with $\gamma = -1$, $z_1 = \mathrm{i}\log\eta$, $z_2 = \mathrm{i}\eta$, η large, we get

$$\mathbb{E}_\beta[e^{-\mathfrak{H}(\eta)}] \leqslant \eta^C, \quad C = C_\beta$$

for all sufficiently large η. Hence, for every $A > 0$,

$$\mathbb{P}_\beta[\mathfrak{H}(\eta) < -A\log\eta] \leqslant \eta^{-A+C}.$$

Choosing $A > C + 1$ and applying Borel–Cantelli, we see that, a.s., $\mathfrak{H}(k) \geqslant -A\log k$ for all sufficiently large integers k. Therefore, recalling that $\log|G_{\mathcal{X}_\beta}|$ is non-negative on the imaginary axis, after increasing A, we get

$$\log|G_{\mathcal{X}_\beta}(\mathrm{i}k)| \geqslant \pi k - A\log k.$$

By monotonicity of the function $\eta \mapsto |G_{\mathcal{X}_\beta}(\mathrm{i}\eta)|$, the same holds for $k \leqslant \eta \leqslant k+1$ with a bigger value of A. □

4.2 The functions Ψ and Ψ^*. Propositions 1.3 and 1.4 are stated in terms of the function $|G_{\mathcal{X}_\beta}(\cdot)|$. On the other hand, our exponential moment estimates in Claim 4.4 do not directly yield information on the function $\log|G_{\mathcal{X}_\beta}(\cdot)|$, but they do allow for the control of differences of such functions. Recall that by Claim 4.1, we already know that the process $\mathcal{X}_\beta$ is almost surely regular, and therefore the random variable $\log|G_{\mathcal{X}_\beta}(\mathrm{i})|$ is well defined.

We introduce the random function

$$\Psi(t) = 2\sum_{\lambda\in\mathcal{X}_\beta}\left[\log|\mathrm{i}+t-\lambda| - \log|\mathrm{i}-\lambda|\right] = 2\left[\log|G_{\mathcal{X}_\beta}(t+\mathrm{i})| - \log|G_{\mathcal{X}_\beta}(\mathrm{i})|\right]. \tag{4.1}$$

Note that the estimate in Claim 4.4 is insufficient to estimate the Laplace transform $\mathbb{E}\left[e^{\gamma\Psi(t)}\right]$. Nonetheless, if the error terms $O(\,\cdot\,)$ are neglected, we would find that $\Psi(t)$ is Gaussian with asymptotic variance $\frac{8}{\beta}\log|t|$, and in fact (by considering multiple values t in Theorem 3.10) we would have that $\Psi(t)$ is related to a log-correlated Gaussian field. To better control the error terms, we modify the function Ψ, letting

$$\begin{aligned}\Psi^*(t) = 2\sum_{\lambda\in\mathcal{X}_\beta}&\left[\log|\mathrm{i}\lfloor\log|t|\rfloor^4 + t - \lambda| - \log|\mathrm{i}\lfloor\log|t|\rfloor^4 - \lambda|\right]\\ &= 2\left[\log|G_{\mathcal{X}_\beta}(t+\mathrm{i}\lfloor\log|t|\rfloor^4)| - \log|G_{\mathcal{X}_\beta}(\mathrm{i}\lfloor\log|t|\rfloor^4)|\right].\end{aligned} \tag{4.2}$$

Throughout, we will only deal with sufficiently large positive t, so that all logarithms and iterated logarithms we use are well defined and positive.

Let $\mathfrak{T}_j = [e^j, e^{j+1})$. Usually, we assume that $t \in \mathfrak{T}_j$, where j is a large positive integer. For t in this range

$$\Psi^*(t) = 2\big[\log|G_{\mathcal{X}_\beta}(t + \mathrm{i}j^4)| - \log|G_{\mathcal{X}_\beta}(\mathrm{i}j^4)|\big]$$

and, by Claim 4.4, for every fixed $R > 0$,

$$\mathbb{E}_\beta\Big[e^{\gamma\Psi^*(t)}\Big] = e^{\frac{4\gamma^2}{\beta}j + O_{\beta,R}(\log j)}, \tag{4.3}$$

uniformly in $|\gamma| \leqslant R$ and $t \in \mathfrak{T}_j$. The following lemma will allow us to replace Ψ with Ψ^* while applying Proposition 1.3. Fix $\varepsilon > 0$ and introduce the event

$$\mathscr{B}_{j,\Psi,\Psi^*} = \{\sup_{\mathfrak{T}_j}|\Psi^* - \Psi| > j^{1/2+\varepsilon}\}. \tag{4.4}$$

Lemma 4.6. *For any $\varepsilon > 0$,*

$$\sum_j \mathbb{P}_\beta\big(\mathscr{B}_{j,\Psi,\Psi^*}\big) < \infty. \tag{4.5}$$

In particular, the event

$$\mathscr{B}^\varepsilon := \{\exists j_0 : \sup_{\mathfrak{T}_j}|\Psi^* - \Psi| \leqslant j^{1/2+\varepsilon} \quad \forall j > j_0\} \tag{4.6}$$

holds $\mathbb{P}_\beta$-a.s.

Proof. Assume that $t \in \mathfrak{T}_j$ and that j is large. We begin with the bound

$$\begin{aligned}|\Psi^*(t) - \Psi(t)| &\leqslant 2\int_1^{j^4}|s(\mathrm{i}u + t) - s(\mathrm{i}u)|du \\ &\leqslant 2\int_1^{j^4}|s(\mathrm{i}u) - \mathrm{i}\pi|du + 2\int_1^{j^4}|s(t + \mathrm{i}u) - \mathrm{i}\pi|du =: I_1 + I_2.\end{aligned}$$

Using Minkowski's integral inequality and then Corollary 3.9, we get

$$\begin{aligned}\mathbb{E}_\beta[I_1^{2p}] &\leqslant 2^{2p}\Big(\int_1^{j^4}\big(\mathbb{E}_\beta[|s(\mathrm{i}u) - \mathrm{i}\pi|^{2p}]\big)^{1/2p}du\Big)^{2p} \\ &\leqslant C^p p^p\left(\int_1^{j^4}\Big(\frac{1}{u^{2p}}\Big)^{1/2p}du\right)^{2p} \leqslant C^p p^p(\log j)^{2p}.\end{aligned}$$

With x given, choosing $p = c(x/\log j)^2$, we obtain that

$$\mathbb{P}_\beta(I_1 > x) \leqslant e^{-c(x/\log j)^2}.$$

A similar estimate holds for I_2. It follows that

$$\mathbb{P}_\beta(|\Psi^*(t) - \Psi(t)| > \tfrac{1}{4}\,j^{1/2+\varepsilon}) \leqslant 2e^{-cj^{1+2\varepsilon}/\log^2 j} \leqslant e^{-cj^{1+\varepsilon}},$$

provided that j is sufficiently large. Hence, by the union bound,

$$\mathbb{P}_\beta\big[\max_{k\in\mathfrak{T}_j\cap\mathbb{Z}}|\Psi^*(k)-\Psi(k)|>\tfrac{1}{4}\,j^{1/2+\varepsilon}\big]<e^{j+1}\cdot e^{-cj^{1+\varepsilon}}<e^{-cj^{1+\varepsilon}}. \tag{4.7}$$

To make this estimate uniform in t, we note that by Claim 4.2 at heights 1 and j^4,

$$\begin{aligned}&\sup_{t\in\mathfrak{T}_j}|\Psi^*(t)-\Psi(t)|\\ &\qquad\leqslant \max_{k\in\mathfrak{T}_j\cap\mathbb{Z}}|\Psi^*(k)-\Psi(k)|+8\max_{k\in\mathfrak{T}_j\cap\mathbb{Z}}|s(k+\mathrm{i})-\mathrm{i}\pi|+8\max_{k\in\mathfrak{T}_j\cap\mathbb{Z}}|s(k+\mathrm{i}j^4)-\mathrm{i}\pi|+O(1).\end{aligned}$$

Now, (4.7) estimates the first term on the RHS, while Claim 4.3, combined with the union bound, estimates the second and the third terms, yielding

$$\mathbb{P}_\beta(\sup_{\mathfrak{T}_j}|\Psi^*-\Psi|>j^{1/2+\varepsilon})\leqslant Ce^{-cj^{1+\varepsilon}}$$

and completing the proof. □

4.3 A second regularization: decomposition of $\Psi^*(t)$ and upper bounds. We next introduce a decomposition of the field $\Psi^*(t)$ for $t\in\mathfrak{T}_j$.

$$V_{j,t}=2[\log|G_{\mathcal{X}_\beta}(t+\mathrm{i}j^4)|-\log|G_{\mathcal{X}_\beta}(t+\mathrm{i}e^j)|-\pi(j^4-e^j)], \tag{4.8}$$

$$Y_{j,2}=2[\log|G_{\mathcal{X}_\beta}(\mathrm{i}e^j)|-\log|G_{\mathcal{X}_\beta}(\mathrm{i}j^4)|-\pi(e^j-j^4)], \tag{4.9}$$

$$Y_{j,3,t}=2[\log|G_{\mathcal{X}_\beta}(t+\mathrm{i}e^j)|-\log|G_{\mathcal{X}_\beta}(\mathrm{i}e^j)|], \tag{4.10}$$

(We will later, see (4.28) and (4.29), introduce variables $Y_{\alpha,j,1,t}$ and use a finer decomposition of $\Psi^*(t)$.) Note that

$$\Psi^*(t)=V_{j,t}+Y_{j,2}+Y_{j,3,t},\qquad t\in\mathfrak{T}_j. \tag{4.11}$$

Note that by stationarity, at any given point $t\in\mathfrak{T}_j$, the random variables $V_{j,t}$ and $-Y_{j,2}$ are equidistributed. So, by Claim 4.4, we have uniformly in $t\in\mathfrak{T}_j$ and for bounded θ,

$$\mathbb{E}_\beta[e^{\theta V_{j,t}}]=\mathbb{E}_\beta[e^{-\theta Y_{j,2}}]=e^{\frac{2\theta^2}{\beta}j+O(\log j)}. \tag{4.12}$$

First, we will show that the terms $Y_{j,2}$ and $Y_{j,3,t}$ are negligible with respect to the maximum of $V_{j,t}$ over $t\in\mathfrak{T}_j$. Fix $0<\varepsilon<\frac{1}{2}$.

Lemma 4.7. *For $j\geqslant 2$, let*

$$\mathscr{B}_{j,23}=\{|Y_{j,2}|>j^{1/2+\varepsilon}\}\cup\{\sup_{t\in\mathfrak{T}_j}|Y_{j,3,t}|>\log j\}. \tag{4.13}$$

Then

$$\sum_j\mathbb{P}_\beta(\,\mathscr{B}_{j,23}\,)<\infty. \tag{4.14}$$

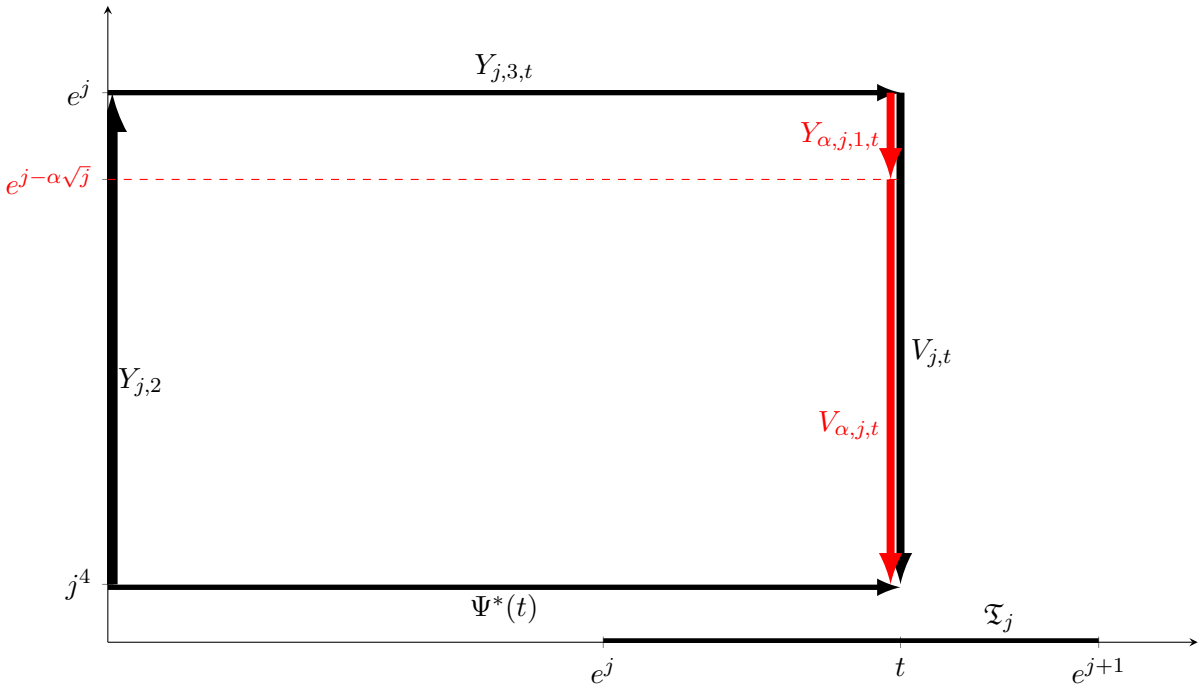


Figure 1: The interval $\mathfrak{T}_j$ and decompositions of $\Psi^*(t)$. In black, the decomposition of (4.11). In red, the modification for the decomposition of (4.29).

Proof. By Claim 4.4, $\mathbb{E}_\beta[e^{\theta Y_{j,2}}] = e^{\frac{2\theta^2}{\beta}j+O(\log j)}$, uniformly for $|\theta| \leqslant 1$. Then

$$\mathbb{P}_\beta(Y_{j,2} > j^{1/2+\varepsilon}) \leqslant e^{-\theta j^{1/2+\varepsilon}+\frac{2}{\beta}\theta^2 j+O(\log j)}.$$

Optimizing θ, we get $\mathbb{P}_\beta(Y_{j,2} > j^{1/2+\varepsilon}) \leqslant e^{-cj^{2\varepsilon}}$, provided that j is sufficiently large. Applying the same exponential moment estimate with negative θ, we see that the same bound holds for $\mathbb{P}_\beta(Y_{j,2} < -j^{1/2+\varepsilon})$.

To estimate the probability of the event that $Y_{j,3,t}$ is large somewhere on $\mathfrak{T}_j$, we write

$$Y_{j,3,t} = -2\,\mathrm{Re}\int_0^t (s(u+\mathrm{i}e^j)-\mathrm{i}\pi)\,\mathrm{d}u,$$

whence

$$\sup_{t\in\mathfrak{T}_j}|Y_{j,3,t}| \leqslant 2\int_0^{e^{j+1}} |s(u+\mathrm{i}e^j)-\mathrm{i}\pi|\,\mathrm{d}u.$$

Using, as above, Minkowski's integral inequality and then the limiting local law from Corollary 3.9, we get, for every $p \geqslant 1$,

$$\Big(\mathbb{E}_\beta\big[(\sup_{t\in\mathfrak{T}_j}|Y_{j,3,t}|)^p\big]\Big)^{1/p} \leqslant 2\int_0^{e^{j+1}} \big(\mathbb{E}_\beta\big[|s(u+\mathrm{i}e^j)-\mathrm{i}\pi|^p\big]\big)^{1/p}\,\mathrm{d}u \lesssim \int_0^{e^{j+1}} (\sqrt{p}\,e^{-j})\,\mathrm{d}u \lesssim \sqrt{p}.$$

Hence, $\mathbb{P}_\beta(\sup_{t\in\mathfrak{T}_j}|Y_{j,3,t}|>x)\lesssim e^{-cx^2}$. Taking $x=\log j$, we get

$$\mathbb{P}_\beta\big(\sup_{t\in\mathfrak{T}_j}|Y_{j,3,t}|>\log j\big)\lesssim e^{-c(\log j)^2},$$

completing the proof. □

We can now prove our first upper bound on $G_{\mathcal{X}_\beta}$ (which will provide an upper bound on the excess).

Proposition 4.8. *Assume $\beta>2$. Then*

$$G_{\mathcal{X}_\beta}(t+\mathrm{i})(1+|t|)^{-1}\in L^2(\mathbb{R}),\quad a.s.$$

Proof. By Lemmas 4.6 and 4.7 and the Borel–Cantelli lemma, almost surely, for sufficiently large j, we have

$$\Psi(t)\leqslant\Psi^*(t)+j^{1/2+\varepsilon}\leqslant V_{j,t}+r_j,\quad r_j=2j^{1/2+\varepsilon}+\log^2 j=o(j).$$

Recalling that $|G_{\mathcal{X}_\beta}(t+\mathrm{i})|^2=|G_{\mathcal{X}_\beta}(\mathrm{i})|^2e^{\Psi(t)}$, we get

$$\int_{\mathfrak{T}_j}\frac{|G_{\mathcal{X}_\beta}(t+\mathrm{i})|^2}{(1+t)^2}\,\mathrm{d}t\leqslant|G_{\mathcal{X}_\beta}(\mathrm{i})|^2e^{-2j+r_j}\int_{\mathfrak{T}_j}e^{V_{j,t}}\,\mathrm{d}t.$$

Therefore, it remains to show that

$$J_\beta:=\sum_{j\geqslant1}e^{-2j+r_j}\int_{\mathfrak{T}_j}e^{V_{j,t}}\,\mathrm{d}t<\infty,\qquad\text{a.s.},$$

which follows from the computation of the mean:

$$\mathbb{E}_\beta[J_\beta]=\sum_{j\geqslant1}e^{-2j+r_j}\int_{\mathfrak{T}_j}\mathbb{E}_\beta\big[e^{V_{j,t}}\big]\,\mathrm{d}t\overset{(4.12)}{\lesssim}\sum_{j\geqslant1}e^{-(1-\frac{2}{\beta})j+o(j)}<\infty$$

since $\beta>2$. □

The argument for $\beta\leqslant2$ is slightly more involved. The key is the following lemma.

Lemma 4.9. *There exists $C=C_\beta>0$ independent of j so that*

$$\max_{t\in\mathfrak{T}_j}V_{j,t}\leqslant\sqrt{\frac{8}{\beta}}\,j+C\log j\qquad\text{eventually a.s.}\tag{4.15}$$

Proof. To simplify notation, put

$$m_j = \sqrt{\frac{8}{\beta}}\, j, \quad \lambda = \sqrt{\frac{\beta}{2}}\,, \quad m_j\lambda = 2j.$$

By (4.12), $\mathbb{E}_\beta[e^{\lambda V_{j,t}}] \leqslant e^{j+O(\log j)}$. Then, by the union bound and Chebyshev's inequality, for a sufficiently large constant C', we have

$$\mathbb{P}_\beta\big(\max_{t\in\mathfrak{T}_j\cap\mathbb{Z}} V_{j,t} \geqslant m_j + C'\log j\big) \lesssim e^j \cdot e^{-\lambda(m_j+C'\log j)} \cdot e^{j+C_0\log j} \lesssim j^{C_0-\lambda C'}. \tag{4.16}$$

Choosing C' so that $C_0 - \lambda C' < -1$, we get summable probabilities.

Given $t \in \mathfrak{T}_j$, choose $k(t) \in \mathfrak{T}_j \cap \mathbb{Z}$ so that $|t - k(t)| \leqslant 1$. By Claim 4.2, we have

$$\sup_{t\in\mathfrak{T}_j} |V_{j,t} - V_{j,k(t)}| \leqslant 8\big(\max_{k\in\mathfrak{T}_j\cap\mathbb{Z}} |s(k+\mathrm{i}j^4) - \mathrm{i}\pi| + \max_{k\in\mathfrak{T}_j\cap\mathbb{Z}} |s(k+\mathrm{i}e^j) - \mathrm{i}\pi| + \pi(j^{-4}+e^{-j})\big).$$

Then, combining Claim 4.3 with the Borel–Cantelli lemma, we get

$$\sup_{t\in\mathfrak{T}_j} |V_{j,t} - V_{j,k(t)}| = o_j(1) \qquad \text{a.s.} \tag{4.17}$$

Putting together estimates (4.16) and (4.17), we get (4.15).

□

We can now prove an upper bound on the excess for $\beta \leqslant 2$.

Proposition 4.10. *Assume $\beta \leqslant 2$ and that $k = \lfloor\sqrt{2/\beta}\rfloor$. Then*

$$G_{\mathcal{X}_\beta}(t+\mathrm{i})(1+|t|)^{-(k+1)} \in L^2(\mathbb{R}), \quad \text{a.s.} \tag{4.18}$$

Proof. We use the same notation m_j and λ as above. We also set

$$\nu = 1/\lambda = \sqrt{2/\beta}, \quad m_j = 2\nu j,$$

so that $k = \lfloor\nu\rfloor$. As in the proof of Proposition 4.8, we choose a deterministic sequence $r_j = o(j)$ such that, almost surely, for all sufficiently large j, $\Psi(t) \leqslant V_{j,t} + r_j$, $t \in \mathfrak{T}_j$.

Set $A_j = m_j + C\log j$ with $C = C_\beta$ the constant from Lemma 4.9. Then, by Lemma 4.9, almost surely, for all sufficiently large j, $\max_{t\in\mathfrak{T}_j} V_{j,t} \leqslant A_j$. Thus, it suffices to prove that

$$\widehat{J}_\beta := \sum_{j\geqslant 1} e^{-2(k+1)j+r_j} \int_{\mathfrak{T}_j} e^{V_{j,t}} \mathbb{1}_{\{V_{j,t}\leqslant A_j\}}\, \mathrm{d}t < \infty \qquad \text{a.s.}$$

(The truncated series differs from the non-truncated one only for finitely many indices j.) Since $\beta \leqslant 2$, $0 < \lambda \leqslant 1$. Then, on the event $\{V_{j,t} \leqslant A_j\}$, we have $e^{V_{j,t}} \leqslant e^{(1-\lambda)A_j} e^{\lambda V_{j,t}}$. Therefore,

$$\mathbb{E}_\beta[e^{V_{j,t}}\mathbb{1}_{\{V_{j,t}\leqslant A_j\}}] \leqslant e^{(1-\lambda)A_j}\mathbb{E}_\beta[e^{\lambda V_{j,t}}] \leqslant e^{2\nu(1-\lambda)j+j+O(\log j)} = e^{(2\nu-1)j+O(\log j)}.$$

Hence,

$$\mathbb{E}_\beta[\widehat{J}_\beta] = \sum_{j\geqslant 1} e^{-2(k+1)j+r_j} \int_{\mathfrak{T}_j} \mathbb{E}_\beta\big[e^{V_{j,t}} \mathbb{1}_{\{V_{j,t}\leqslant A_j\}}\big]\, \mathrm{d}t \lesssim \sum_j e^{-2(k+1-\nu)j+o(j)}.$$

Finally, $k = \lfloor \nu \rfloor$, so $k+1-\nu > 0$. Thus, the last series converges, and $\widehat{J}_\beta$ is finite almost surely. □

4.4 Lower bounds for Ψ^*. We begin by noting that in proving the lower bound on $|G_{\mathcal{X}_\beta}|$ needed for verifying (1.10) in Proposition 1.4, it suffices to consider the function Ψ^*.[6]

Lemma 4.11. *Let A be the constant from Claim 4.5. Assume that for some $k_* \in \mathbb{Z}$ and all $j_0 > 0$,*

$$\sum_{j\geqslant j_0} j^{-8A} \int_{\mathfrak{T}_j} e^{\Psi^*(t)} (1+t)^{-2k_*}\, \mathrm{d}t = \infty. \tag{4.19}$$

Then, (1.10) holds with $\eta(t) = \lfloor \log|t| \rfloor^4$ for sufficiently large t.

Proof. From the definitions, we have that

$$|G_{\mathcal{X}_\beta}(t+\mathrm{i}j^4)|^2 = |G_{\mathcal{X}_\beta}(\mathrm{i}j^4)|^2 e^{\Psi^*(t)}.$$

On the other hand, by Claim 4.5 (applied with $\eta = j^4$), we have that for j large,

$$|G_{\mathcal{X}_\beta}(\mathrm{i}j^4)|^2 \geqslant e^{2\pi j^4}(1+j^4)^{-2A} \geqslant \frac{1}{2} e^{2\pi j^4} j^{-8A}.$$

Combining the last two displays gives that for j large,

$$|G_{\mathcal{X}_\beta}(t+\mathrm{i}j^4)|^2 (1+t)^{-2k_*} e^{-2\pi j^4} \geqslant \frac{1}{2} e^{\Psi^*(t)} j^{-8A} (1+t)^{-2k_*}.$$

Thus, the assumption (4.19) implies that (1.10) holds. □

Recall the decomposition of Ψ^* given by (4.11). The main step in proving (4.19) will be the following control of

$$V_j^* \stackrel{\mathrm{def}}{=} \sup_{t\in\mathfrak{T}_j} V_{j,t}.$$

Its proof, which is rather involved, is provided in Section 4.5.

Proposition 4.12. *Fix $\delta > 0$. Then*

$$\mathbb{P}_\beta\Big(V_j^* \geqslant \sqrt{\frac{8}{\beta}}\, j - \delta\sqrt{j}\Big) \to_{j\to\infty} 1. \tag{4.20}$$

[6] We note that in case $\sqrt{2/\beta}$ is not an integer, an alternative route which verifies (1.7) of Proposition 1.3 and controls the error $|\Psi(t) - \Psi^*(t)|$ by $j^{1/2+\delta}$ for some $\delta > 0$ small is possible. This cannot work in the critical cases, and therefore we chose to present a unified argument that works in all cases.

Equipped with Proposition 4.12 and Lemma 4.11, we can verify (1.10) with $k_* = \lfloor \sqrt{2/\beta} \rfloor$.

Proposition 4.13. *Let $k_* = \lfloor \sqrt{2/\beta} \rfloor$. Then (1.10) holds.*

Proof. By Lemma 4.11, it suffices to check (4.19).

The proof is somewhat simpler when $\sqrt{2/\beta}$ is not an integer. So we consider that case first. By Lemma 4.7, it suffices to check that

$$\sum_{j \geqslant j_0} j^{-8A} \int_{\mathfrak{T}_j} e^{V_{j,t}} e^{-O(j^{1/2+\varepsilon})} (1+t)^{-2k_*} \, \mathrm{d}t = \infty. \tag{4.21}$$

By Claims 4.2 and 4.3, a union bound, and the Borel–Cantelli lemma, possibly increasing j_0 if needed, we have that for all $j \geqslant j_0$,

$$\int_{\mathfrak{T}_j} e^{V_{j,t}} \, \mathrm{d}t \geqslant e^{-O(j^{1/2+\varepsilon})} e^{V_j^*},$$

and therefore it suffices to check that for all j_0,

$$\sum_{j \geqslant j_0} j^{-8A} e^{V_j^*} e^{-O(j^{1/2+\varepsilon})} e^{-2jk_*} = \infty. \tag{4.22}$$

By Proposition 4.12, we have that, almost surely, the event $\{V_j^* \geqslant \sqrt{8/\beta}\, j - \delta \sqrt{j}\}$ holds for infinitely many j. Hence, (4.22) holds almost surely provided that

$$\lim_{j \to \infty} j^{-8A} e^{\sqrt{8/\beta} j - O(j^{1/2+\varepsilon}) - 2jk_*} = \infty. \tag{4.23}$$

Since $2k_* < \sqrt{8/\beta}$ when $\sqrt{2/\beta}$ is not an integer, we have that (4.23) holds, which completes the proof in that case.

We next turn to the case where $\sqrt{2/\beta}$ is an integer. In that case, Lemma 4.7 can still be used to control $Y_{j,3,t}$. Therefore, to check (4.19), it suffices to check that for any fixed j_0,

$$\sum_{j \geqslant j_0} j^{-8A} \int_{\mathfrak{T}_j} e^{V_{j,t} + Y_{j,2} - (\log j)^2} (1+t)^{-2k_*} \, \mathrm{d}t = \infty, \quad a.s. \tag{4.24}$$

As before, by Claims 4.2 and 4.3 and the Borel–Cantelli lemma, it in turn suffices to check that

$$\sum_{j \geqslant j_0} j^{-8A} e^{V_j^* + Y_{j,2} - (\log j)^2} e^{-2jk_*} = \infty. \tag{4.25}$$

Fix $K > 0, L > 0$ integers (we first let $L \to \infty$ with K fixed, and then let $K \to \infty$). Set $a_\ell = a_{\ell,L} = L^{L+\ell}$, $\ell = 1, \dots, K$. Consider in Theorem 3.10 $z_{2\ell} = \mathrm{i} e^{a_\ell}$, $z_{2\ell-1} = \mathrm{i} a_\ell^4$, $\gamma_{2\ell} = -\gamma_{2\ell-1} = 2\theta_\ell / \sqrt{a_\ell}$, and θ_ℓ independent of L. Note that with this choice, we have in the notation and assumptions of Theorem 3.10 with $m = 2K$, that with K fixed and L large, $\eta_* = a_1^4$ and $\log \Delta \leqslant 2a_K \ll a_1^2$, so $(\log \Delta)^2 / \eta_* \leqslant 4a_K^2 / a_1^4 = o_L(1)$, as needed for the

application of the theorem. On the other hand, for $\ell' < \ell$, $|z_{2\ell} - \bar{z}_{2\ell'}| = (1 + o_L(1))|z_{2\ell}|$, $|z_{2\ell-1} - \bar{z}_{2\ell'-1}| = (1 + o_L(1))|z_{2\ell-1}|$ and, for $\ell \neq \ell'$, $|z_{2\ell} - z_{2\ell'-1}| = (1 + o_L(1))|z_{2\ell}|$. Further,

$$\sigma(\boldsymbol{z}, \boldsymbol{\gamma}) = \frac{2}{\beta} \sum_{\ell=1}^{K} \theta_\ell^2 (1 + o_L(1)) + o_L(1).$$

Therefore, the random vector $\{Y_{a_\ell,2}/\sqrt{a_\ell}\}_{\ell=1}^K$ converges in distribution as $L \to \infty$ to a centered Gaussian vector with i.i.d. entries of variance $4/\beta$. It follows that there exists a constant $c_{\beta,\delta} \in (0,1)$ so that

$$\lim_{L\to\infty} \mathbb{P}_\beta(Y_{a_\ell,2} < 2\delta\sqrt{a_\ell}, \ell = 1, \dots, K) \leqslant c_{\beta,\delta}^K. \tag{4.26}$$

Fix now $\varepsilon > 0$. It follows from the above that we can find $K = K(\varepsilon, \delta)$ so that for all L large,

$$\mathbb{P}_\beta(Y_{a_\ell,2} < 2\delta\sqrt{a_\ell}, \ell = 1, \dots, K) \leqslant \varepsilon/2.$$

On the other hand, by Proposition 4.12, we have that for all L large,

$$\mathbb{P}_\beta(V^*_{a_\ell} > \sqrt{8/\beta} a_\ell - \delta\sqrt{a_\ell}, \ell = 1, \dots, K) > 1 - \varepsilon/2.$$

It follows that for any fixed L large, with probability at least $1 - \varepsilon$,

$$\max_{j \in \{a_1, \dots, a_K\}} j^{-8A} e^{V^*_j + Y_{j,2} - (\log j)^2 - 2jk_*} > e^{\delta\sqrt{a_1} - (\log a_K)^2} a_K^{-8A}.$$

Taking now L large, it follows that the left hand side of (4.25) is larger than $1/\varepsilon$, say, with probability larger than $1 - \varepsilon$. Since ε is arbitrary, (4.25) holds a.s. This completes the proof also when $\sqrt{2/\beta}$ is an integer. □

4.5 Proof of Proposition 4.12: control of the maximum V_j^*. The proof will be somewhat long. We split it into several parts. In Section 4.5.1, we introduce two auxiliary results, Lemmas 4.14 and 4.15, which control respectively the maximum of $V_{\alpha,j,t}$ and the minimum of $Y_{\alpha,j,1,t}$ over $\mathfrak{T}_j$, and we show how Proposition 4.12 readily follows from these lemmas, whose proofs are postponed to subsequent sections. Most of the work is in the proof of Lemma 4.14: we provide some preliminaries and a sketch of the proof in Section 4.5.2. Section 4.5.3 introduces $X_{j,t}(\cdot)$, a rescaled and linearly interpolated version of $\widehat{V}_{\alpha,j,t}(\cdot)$, as well as the auxiliary measures $\mu_{j,t}$ and $\mu_{j,t,t'}$. We then, in Lemma 4.16, state the convergence, under $\mu_{j,t}$ and $\mu_{j,t,t'}$, of $X_{j,t}(\cdot)$ and $(X_{j,t}(\cdot), X_{j,t'}(\cdot))$, toward a Brownian motion and a pair of independent Brownian motions, respectively. We also state in Lemma 4.17 a crucial second moment estimate on M_j. The rest of Section 4.5.3 then quickly proves Lemma 4.14. Lemma 4.16 is proved in Section 4.5.5, after some preliminary work done in Section 4.5.4. Lemma 4.17 is proved in Section 4.5.6. Finally, Lemma 4.15 is proved in Section 4.5.7.

4.5.1 Proof of Proposition 4.12 (assuming Lemmas 4.14 and 4.15). Put $s^* = s^*(j) = j - 4\log j$. Theorem 3.10 hints (after disregarding error terms) that the process $V_{j,t}(s^*)$ satisfies

$$\mathbb{E}_\beta(V_{j,t}(s^*)V_{j,t'}(s^*)) = \frac{4}{\beta}\log\frac{e^j}{|t-t'|\vee j^4} + O(1), \quad t,t' \in \mathfrak{T}_j.$$

That is, $V_{j,t}(s^*)$ is a logarithmically correlated field, albeit only asymptotically Gaussian. We will use techniques from the theory of logarithmically correlated fields in order to provide a lower bound on its maximum. Toward this end, it is useful to introduce a scale decomposition (this is the role of the parameter s below) and also to treat separately the large scales (this is the role of α below).

Fix $\alpha \in (0,1/2)$, set $s^*_\alpha = s^*_\alpha(j) = j - 4\log j - \alpha\sqrt{j}$. Then $s^* = s^*_0$. For $s \in [0, s^*_\alpha]$, set

$$V_{\alpha,j,t}(s) = 2[\log|G\mathcal{X}_\beta(t+\mathrm{i}e^{j-\alpha\sqrt{j}-s})| - \log|G\mathcal{X}_\beta(t+\mathrm{i}e^{j-\alpha\sqrt{j}})| + \pi e^{j-\alpha\sqrt{j}}(1-e^{-s})], \quad (4.27)$$

$$Y_{\alpha,j,1,t} = 2[\log|G\mathcal{X}_\beta(t+\mathrm{i}e^{j-\alpha\sqrt{j}})| - \log|G\mathcal{X}_\beta(t+\mathrm{i}e^{j})| + \pi e^{j}(1-e^{-\alpha\sqrt{j}})], \quad (4.28)$$

Note that by (4.8),

$$V_{j,t} = V_{j,t}(s^*) = V_{\alpha,j,t}(s^*_\alpha) + Y_{\alpha,j,1,t}$$

and therefore (4.11) can be rewritten as

$$\Psi^*(t) = V_{\alpha,j,t}(s^*_\alpha) + Y_{\alpha,j,1,t} + Y_{j,2} + Y_{j,3,t}, \quad (4.29)$$

with $Y_{j,2}$ and $Y_{j,3,t}$ introduced in (4.9) and (4.10), see also Figure 1.

To simplify notation, put $V_{\alpha,j,t} = V_{\alpha,j,t}(s^*_\alpha)$, so

$$V_{j,t}(s^*) = V_{\alpha,j,t} + Y_{\alpha,j,1,t}. \quad (4.30)$$

Set $V^*_{\alpha,j} := \sup_{t\in\mathfrak{T}_j} V_{\alpha,j,t}$. Our main technical work will be to show the following lemma, whose proof is deferred.

Lemma 4.14. *Fix $0 < \eta \leqslant \alpha/2$. Then*

$$\mathbb{P}_\beta\Big(V^*_{\alpha,j} \geqslant \sqrt{\frac{8}{\beta}}\, s^*_\alpha - \sqrt{\frac{2}{\beta}}\,\eta\sqrt{s^*_\alpha}\Big) = 1 - o_j(1). \quad (4.31)$$

The following lemma, which controls $Y^*_{\alpha,j,1} := \min_{t\in[e^j,e^{j+1}]} Y_{\alpha,j,1,t}$, will be used in conjunction with (4.31), to complete the proof of Proposition 4.12. Its proof, based on Theorem 3.10, is given in Section 4.5.7.

Lemma 4.15. *There exists a constant c^*_β so that*

$$\mathbb{P}_\beta(Y^*_{\alpha,j,1} \leqslant -c^*_\beta\alpha\sqrt{j}\,) = o_j(1). \quad (4.32)$$

Proof of Proposition 4.12 (assuming Lemmas 4.14 and 4.15). Using (4.31) and (4.32), and choosing $\eta = \alpha/2$, we obtain that

$$\mathbb{P}_\beta\Big(V^*_{\alpha,j} \geqslant \sqrt{\frac{8}{\beta}}\, s^*_\alpha - \sqrt{\frac{2}{\beta}}\eta\, \sqrt{s^*_\alpha}, Y^*_{\alpha,j,1} > -c^*_\beta \alpha \sqrt{j}\Big) = 1 - o_j(1).$$

In particular, using (4.30),

$$\mathbb{P}_\beta\Big(V^*_{j} \geqslant \sqrt{\frac{8}{\beta}}\, s^*_\alpha - \sqrt{\frac{2}{\beta}}\, \alpha \sqrt{s^*_\alpha} - c^*_\beta \alpha \sqrt{j}\Big) = 1 - o_j(1).$$

Taking now α small enough (as a function of δ) yields the proposition. □

4.5.2 Proof sketch of Lemma 4.14. We now turn to the proof of Lemma 4.14, providing here some definitions and a sketch. Most of our work will involve the process $V_{\alpha,j,t}(s)$, $s \in [0, s^*_\alpha]$ defined in (4.27). Introduce the process

$$\widehat{V}_{\alpha,j,t}(u) = \sqrt{\frac{\beta}{2}}\, V_{\alpha,j,t}(u s^*_\alpha) - 2u s^*_\alpha, \quad u \in [0,1].$$

Note that $\widehat{V}_{\alpha,j,t}(0) = 0$, while $\widehat{V}_{\alpha,j,t}(1) = \sqrt{\beta/2}\, V_{\alpha,j,t} - 2s^*_\alpha$.

Introduce the mesh

$$N = \lceil s^*_\alpha \rceil,\ u_k = \frac{k}{N},\ 0 \leqslant k \leqslant N.$$

Fix $\eta > 0$ with $\eta < \alpha/2$ and introduce the event

$$\mathcal{A}_j(t) = \{ \max_{0 \leqslant k \leqslant N} \widehat{V}_{\alpha,j,t}(u_k) \leqslant \eta \sqrt{s^*_\alpha},\ \widehat{V}_{\alpha,j,t}(1) \geqslant -\eta \sqrt{s^*_\alpha}\}, \tag{4.33}$$

and the random variable

$$M_j = e^{-j} \int_{e^j}^{e^{j+1}} e^{\sqrt{\frac{\beta}{2}} V_{\alpha,j,t}(s^*_\alpha) - s^*_\alpha} \mathbf{1}_{\mathcal{A}_j(t)}\, \mathrm{d}t = e^{-j+s^*_\alpha} \int_{e^j}^{e^{j+1}} e^{\widehat{V}_{\alpha,j,t}(1)} \mathbf{1}_{\mathcal{A}_j(t)}\, \mathrm{d}t. \tag{4.34}$$

Note that

$$\mathbb{E}_\beta(M_j) = e^{-j+s^*_\alpha} \int_{e^j}^{e^{j+1}} \mu_{j,t}(\mathcal{A}_j(t))\, \mathbb{E}_\beta(e^{\widehat{V}_{\alpha,j,t}(1)})\, \mathrm{d}t, \tag{4.35}$$

and

$$\mathbb{E}_\beta(M_j^2) = e^{-2j+2s^*_\alpha} \iint_{[e^j, e^{j+1}]^2} \mu_{j,t,t'}(\mathcal{A}_j(t) \cap \mathcal{A}_j(t'))\, \mathbb{E}_\beta(e^{\widehat{V}_{\alpha,j,t}(1) + \widehat{V}_{\alpha,j,t'}(1)})\, \mathrm{d}t\, \mathrm{d}t'. \tag{4.36}$$

We will show that with probability $1 - o_j(1)$, the random variable M_j is strictly positive, which in turn shows that there exists a $t \in \mathfrak{T}_j$ for which $\mathcal{A}_j(t)$ occurs, which implies the lemma. To prove the strict positivity of M_j, we will show that $\liminf_{j\to\infty} \mathbb{E}_\beta(M_j) > 0$ and

$\mathbb{E}_\beta(M_j^2)/(\mathbb{E}_\beta(M_j))^2 = 1 + o_j(1)$, which proves by the Paley-Zygmund inequality [20, Sect 3.3] that $\mathbb{P}_\beta(M_j > 0) = 1 - o_j(1)$.

The motivation for the construction of M_j is that the fields $V_{\alpha,j,t}$ and $V_{\alpha,j,t'}$ are essentially independent if $|t-t'| > e^{j-\alpha\sqrt{j}/2}$; this will be quantified in Lemma 4.16(b) below. Thus, M_j is essentially an average of independent random variables, which should concentrate around its mean. Part (a) of the same lemma will allow us to compute the mean and show it is strictly positive. (In the theory of Gaussian multiplicative chaos (GMC), this is related to the total mass of a GMC. We will avoid using this terminology here.)

The event $\mathcal{A}_j(t)$ is the intersection of the *barrier event* $\{\max_{0\leqslant k\leqslant N} \widehat{V}_{\alpha,j,t}(u_k) \leqslant \eta\sqrt{s_\alpha^*}\}$ with the *endpoint event* $\{\widehat{V}_{\alpha,j,t}(1) \geqslant -\eta\sqrt{s_\alpha^*}\}$. Their role is quite distinct: the barrier event will allow us to control the contribution to the second moment of M_j of pairs t, t' of distance smaller than $e^{j-\alpha\sqrt{j}}$, while the endpoint event will ensure that if $M_j > 0$ then there is a $t \in \mathfrak{T}_j$ with the desired property. We now proceed with the actual proof.

4.5.3 Proof of Lemma 4.14 (assuming Lemmas 4.16 and 4.17). Define the random process $X_{j,t}(u)$, $u \in [0,1]$, as the polygonal interpolation of a sequence of points

$$X_{j,t}(u_k) = \widehat{V}_{\alpha,j,t}(u_k)/\sqrt{2s_\alpha^*}, \quad 0 \leqslant k \leqslant N.$$

Clearly, $X_{j,t} \in C_0[0,1]$ (the space of continuous functions on $[0,1]$ which vanish at 0).

Let $(\Omega, \mathcal{F}, \mathbb{P}_\beta)$ be the underlying probability space. Similarly to Sect. 3.4.2, we define the tilted probability measures $\mu_{j,t}$ and $\mu_{j,t,t'}$ on $(\Omega, \mathcal{F})$ by their Radon–Nikodym derivatives

$$\frac{\mathrm{d}\mu_{j,t}}{\mathrm{d}\mathbb{P}_\beta} = \frac{e^{\widehat{V}_{\alpha,j,t}(1)}}{\mathbb{E}_\beta[e^{\widehat{V}_{\alpha,j,t}(1)}]},$$
$$\frac{\mathrm{d}\mu_{j,t,t'}}{\mathrm{d}\mathbb{P}_\beta} = \frac{e^{\widehat{V}_{\alpha,j,t}(1)+\widehat{V}_{\alpha,j,t'}(1)}}{\mathbb{E}_\beta[e^{\widehat{V}_{\alpha,j,t}(1)+\widehat{V}_{\alpha,j,t'}(1)}]}$$

(note that both denominators are finite and positive by Theorem 3.10). The interest in the measures $\mu_{j,t}$ and $\mu_{j,t,t'}$ comes from the following lemma, which gives a scale decomposition of the field $X_{j,t}(u)$ in (u,t) which resembles that of a collection of $e^{\alpha\sqrt{j}/2}$ independent branching random walks.

Lemma 4.16.

(a) *Let $t \in \mathfrak{T}_j$. Under the law $\mu_{j,t}$, the process $X_{j,t}$ converges weakly in $C_0[0,1]$ to a standard Brownian motion. Further,*

$$4e^{s_\alpha^*}\mathbb{E}_\beta(e^{\widehat{V}_{\alpha,j,t}(1)}) = 1 + o_j(1). \tag{4.37}$$

(b) *Assume $t, t' \in \mathfrak{T}_j$, $|t-t'| \geqslant e^{j-\alpha\sqrt{j}/2}$. Then, under the measure $\mu_{j,t,t'}$, the pair of processes $(X_{j,t}, X_{j,t'})$ converges weakly in $C_0[0,1] \times C_0[0,1]$ to a pair of independent Brownian motions.*

The convergence is uniform over all pairs $t, t' \in \mathfrak{T}_j$ satisfying $|t - t'| \geqslant e^{j-\alpha\sqrt{j}/2}$. Further,

$$16e^{2s^*_\alpha}\mathbb{E}_\beta(e^{\widehat{V}_{\alpha,j,t}(1)+\widehat{V}_{\alpha,j,t'}(1)}) = 1 + o_j(1) \tag{4.38}$$

with the same uniformity in t, t'.

Note that due to the translation-invariance of $\mathbb{P}_\beta$, the statement in (a) is automatically uniform in $t \in \mathfrak{T}_j$.

We postpone the proof of Lemma 4.16 and proceed with the proof of Lemma 4.14. With W_t denoting a standard Brownian motion, let

$$p_\eta = \mathbb{P}(\max_{t\in[0,1]} W_t \leqslant \eta/\sqrt{2}, W_1 \geqslant -\eta/\sqrt{2}), \tag{4.39}$$

and note that $p_\eta > 0$ since, for every $\tau > 0$, standard Brownian motion has positive probability of remaining in the interval $(-\tau, \tau)$ throughout $[0, 1]$. Next, we note that

$$\begin{aligned}\mu_{j,t}(\mathcal{A}_j(t)) &= \mu_{j,t}\{\max_{0\leqslant k\leqslant N} X_{j,t}(u_k) \leqslant \eta/\sqrt{2},\ X_{j,t}(1) \geqslant -\eta/\sqrt{2}\}\\ &= \mu_{j,t}\{\max_{u\in[0,1]} X_{j,t}(u) \leqslant \eta/\sqrt{2},\ X_{j,t}(1) \geqslant -\eta/\sqrt{2}\}\end{aligned}$$

since the polygonal function $X_{j,t}$ attains its maximal values at the mesh points. By the weak convergence in Lemma 4.16(a), we conclude that

$$\mu_{j,t}(\mathcal{A}_j(t)) = p_\eta(1 + o_j(1))$$

(here, we tacitly used that the boundary of the event $\{W_t \leqslant \eta/\sqrt{2}, t \in [0, 1], W_1 \geqslant -\eta/\sqrt{2}\}$ has zero measure under the Brownian motion law). Hence,

$$\begin{aligned}\mathbb{E}_\beta M_j &\overset{(4.35)}{=} e^{-j+s^*_\alpha}\int_{e^j}^{e^{j+1}} \mu_{j,t}(\mathcal{A}_j(t))\mathbb{E}_\beta(e^{\widehat{V}_{\alpha,j,t}(1)})\,\mathrm{d}t\\ &\overset{(4.37)}{=} \frac{1}{4}(1 + o_j(1))e^{-j}\int_{e^j}^{e^{j+1}} \mu_{j,t}(\mathcal{A}_j(t))\,\mathrm{d}t = \frac{e-1}{4}\,p_\eta(1 + o_j(1))\,.\end{aligned} \tag{4.40}$$

The following crucial lemma, whose proof will be also postponed, provides us with the second moment of M_j.

Lemma 4.17. *We have*

$$\mathbb{E}_\beta M_j^2 = \frac{(e-1)^2}{16}p_\eta^2(1 + o_j(1)). \tag{4.41}$$

Now, equipped with (4.40), Lemma 4.17, and the Paley–Zygmund inequality, we conclude that

$$\mathbb{P}_\beta(M_j > 0) \geqslant 1 - o_j(1). \tag{4.42}$$

By the definition of M_j in (4.34), the event $M_j > 0$ implies that

$$V^*_{\alpha,j} := \max_{t\in[e^j,e^{j+1}]} V_{\alpha,j,t} \geqslant \sqrt{\frac{8}{\beta}}\, s^*_\alpha - \sqrt{\frac{2}{\beta}}\, \eta\sqrt{s^*_\alpha}.$$

Therefore, we have shown that

$$\mathbb{P}_\beta\Big(V^*_{\alpha,j} \geqslant \sqrt{\frac{8}{\beta}}\, s^*_\alpha - \sqrt{\frac{2}{\beta}}\, \eta\sqrt{s^*_\alpha}\Big) \geqslant 1 - o_j(1), \tag{4.43}$$

completing the proof of Lemma 4.14 □

4.5.4 Auxiliary claims. We start with two claims which provide us with versions of Theorem 3.10 tuned for the proofs of Lemmas 4.16 and 4.17.

Claim 4.18. *Fix $m \geqslant 1$. Then, uniformly in*

$$t \in \mathfrak{T}_j,\ \ s_1,\dots,s_m \in [0, s^*_\alpha],\ \ \theta_1,\dots,\theta_m \in [-M, M], M \geqslant 1,$$

we have

$$\log \mathbb{E}_\beta \exp\Big\{\sqrt{\frac{\beta}{2}} \sum_{\ell=1}^m \theta_\ell V_{\alpha,j,t}(s_\ell)\Big\} = \sum_{\ell,\ell'=1}^m \theta_\ell\theta_{\ell'} \mathsf{K}(s_\ell, s_{\ell'}) + O_{\beta,m}(M^3 j^{-2}),$$

where

$$\mathsf{K}(s,r) = \log \frac{(1+e^{-s})(1+e^{-r})}{2(e^{-s}+e^{-r})}.$$

Proof. We have

$$\sqrt{\frac{\beta}{2}} \sum_{\ell=1}^m \theta_\ell V_{\alpha,j,t}(s_\ell) = \sqrt{2\beta} \sum_{\ell=1}^m \theta_\ell \Big[\sum_\lambda \log\Big|\frac{t+\mathrm{i}ae^{-s_\ell}-\lambda}{t+\mathrm{i}a-\lambda}\Big| + \pi a(1-e^{-s_\ell})\Big],$$

with $a = e^{j-\alpha\sqrt{j}}$, so we apply Theorem 3.10 with

$$\gamma_\ell = \sqrt{2\beta}\,\theta_\ell,\ \ \gamma_{\ell+m} = -\sqrt{2\beta}\,\theta_\ell,\ \ z_\ell = t + \mathrm{i}ae^{-s_\ell},\ \ z_{m+\ell} = t + \mathrm{i}a.$$

The corresponding quadratic form equals

$$\begin{aligned}\sigma(\mathbf{z},\boldsymbol{\gamma}) &= \sum_{\ell,\ell'}^m \theta_\ell\theta_{\ell'}\Big\{\log\frac{1}{|ae^{-s_\ell}+ae^{-s_{\ell'}}|} - \log\frac{1}{|a+ae^{-s_\ell}|} - \log\frac{1}{|a+ae^{-s_{\ell'}}|} + \log\frac{1}{2a}\Big\}\\ &= \sum_{\ell,\ell'}^m \theta_\ell\theta_{\ell'} \log\frac{(1+e^{-s_\ell})(1+e^{-s_{\ell'}})}{2(e^{-s_\ell}+e^{-s_{\ell'}})} = \sum_{\ell,\ell'}^m \theta_\ell\theta_{\ell'}\mathsf{K}(s_\ell,s_{\ell'}).\end{aligned}$$

To estimate the remainder, we note that the smallest imaginary part of z_ℓ is at least j^4, while the logarithm of the diameter of the set $\{z_\ell\colon 1 \leqslant \ell \leqslant 2m\}$ is $O(j)$. □

We record several properties of the kernel K, which will be needed below:

(K_1) $\mathsf{K}(s,r) = s \wedge r + O(1)$ uniformly on $(s,r) \in [0, s^*_\alpha]^2$.

(K_2) $\mathsf{K}(s,s) = s - \log 4 + O(e^{-s})$, $s \to \infty$,

(K_3) $\mathsf{K}(s,s) - 2\mathsf{K}(s,r) + \mathsf{K}(r,r) = 2 \log\cosh \dfrac{s-r}{2}$.

In particular, $0 \leqslant \mathsf{K}(s,s) - 2\mathsf{K}(s,r) + \mathsf{K}(r,r) \leqslant |s-r|$.

The statement of the second claim needs a more general kernel (and looks a bit formidable). For $\xi \geqslant 0$, $a > 0$ and $s, r \in [0, s^*_\alpha]$, we set

$$\mathsf{K}_{\xi,a}(s,r) = \log \frac{|\xi + \mathrm{i}a(1+e^{-s})|\,|\xi + \mathrm{i}a(1+e^{-r})|}{|\xi + \mathrm{i}a(e^{-s}+e^{-r})|\,|\xi + 2\mathrm{i}a|}.$$

In particular, $\mathsf{K}_{0,a} = \mathsf{K}$.

Claim 4.19. *Fix $m, m' \geqslant 1$. Then, uniformly in*

$$t, t' \in \mathfrak{T}_j,\ s_1, \dots, s_m, r_1, \dots, r_{m'} \in [0, s^*_\alpha],\ \theta_1, \dots, \theta_m, \theta'_1, \dots, \theta'_{m'} \in [-M, M], M \geqslant 1,$$

we have

$$\begin{aligned}
&\log \mathbb{E}_\beta \exp\Big\{ \sqrt{\frac{\beta}{2}} \Big[\sum_{\ell=1}^m \theta_\ell V_{\alpha,j,t}(s_\ell) + \sum_{\ell'=1}^{m'} \theta'_{\ell'} V_{\alpha,j,t'}(r_{\ell'}) \Big] \Big\} \\
&= \sum_{\ell,\ell'=1}^m \theta_\ell \theta_{\ell'}\, \mathsf{K}(s_\ell, s_{\ell'}) + \sum_{\ell,\ell'=1}^{m'} \theta'_\ell \theta'_{\ell'}\, \mathsf{K}(r_\ell, r_{\ell'}) + 2 \sum_{\ell=1}^m \sum_{\ell'=1}^{m'} \theta_\ell \theta'_{\ell'} \mathsf{K}_{\xi,a}(s_\ell, r_{\ell'}) + O_{\beta,m,m'}(M^3 j^{-2}),
\end{aligned}$$

where $\xi = |t - t'|$ and $a = e^{j - \alpha\sqrt{j}}$.

Proof. To simplify notation, we assume that $m' = m$. Then

$$\begin{aligned}
&\Big\{ \sqrt{\frac{\beta}{2}} \sum_{\ell=1}^m \big[\theta_\ell V_{\alpha,j,t}(s_\ell) + \theta'_\ell V_{\alpha,j,t'}(r_\ell) \big] \Big\} \\
&= \sqrt{2\beta} \sum_{\ell=1}^m \Big[\theta_\ell \Big(\sum_\lambda \log \Big| \frac{t + \mathrm{i}ae^{-s_\ell} - \lambda}{t + \mathrm{i}a - \lambda} \Big| + \pi a (1 - e^{-s_\ell}) \Big) + \theta'_\ell \Big(\sum_\lambda \log \Big| \frac{t' + \mathrm{i}ae^{-r_\ell} - \lambda}{t' + \mathrm{i}a - \lambda} \Big| + \pi a (1 - e^{-r_\ell}) \Big) \Big].
\end{aligned}$$

This time we apply Theorem 3.10 to $4m$ points

$$z_\ell = t + \mathrm{i}ae^{-s_\ell},\ z_{m+\ell} = t + \mathrm{i}a,\ z'_{\ell'} = t' + \mathrm{i}ae^{-r_{\ell'}},\ z'_{m+\ell'} = t' + \mathrm{i}a,$$

with coefficients $\pm\sqrt{2\beta}\,\theta_\ell$, $\pm\sqrt{2\beta}\,\theta'_\ell$. We need to compute the corresponding quadratic form σ. It has two parts involving only points z with real part t, or only points with real part t'. These are

$$\sum_{\ell,\ell'=1}^m \theta_\ell \theta_{\ell'} \mathsf{K}(s_\ell, s_{\ell'}), \qquad \sum_{\ell,\ell'=1}^m \theta'_\ell \theta'_{\ell'} \mathsf{K}(r_\ell, r_{\ell'}).$$

The four mixed distances are

$$|\xi + \mathrm{i}a(e^{-s_\ell} + e^{-r_{\ell'}})|,\ |\xi + \mathrm{i}a(1 + e^{-s_\ell})|,\ |\xi + \mathrm{i}a(1 + e^{-r_{\ell'}})|,\ |\xi + 2\mathrm{i}a|.$$

The corresponding linear combination of their logarithms gives us the kernel $\mathsf{K}_{\xi,a}$. By symmetry of the quadratic form, the mixed contribution becomes

$$2\sum_{\ell,\ell'=1}^{m} \theta_\ell \theta'_{\ell'} \mathsf{K}_{\xi,a}(s_\ell, r_{\ell'}).$$

This completes the proof of Claim 4.19. □

In our applications, $\xi = |t - t'|$ will be $\geqslant e^{j - \frac{1}{2}\alpha\sqrt{j}}$, which is much bigger than $a = e^{j - \alpha\sqrt{j}}$, and, due to the following claim, the mixed terms become negligible.

Claim 4.20. *For $\xi > 0$, $a > 0$, and $s, r \geqslant 0$, $|\mathsf{K}_{\xi,a}(s,r)| \leqslant (a/\xi)^2$.*

Proof. Put $f_\xi(x) = \frac{1}{2}\log(\xi^2 + x^2)$, $b = ae^{-s}$, $c = ae^{-r}$. Then

$$\mathsf{K}_{\xi,a}(s,r) = f_\xi(a+b) + f_\xi(a+c) - f_\xi(b+c) - f_\xi(2a) = -\int_b^a \int_c^a f''_\xi(x+y)\,\mathrm{d}x\,\mathrm{d}y.$$

Furthermore, $f''_\xi(x) = (\xi^2 - x^2)/(\xi^2 + x^2)^2$, and $|f''_\xi(x)| \leqslant 1/\xi^2$. Therefore,

$$|\mathsf{K}_{\xi,a}(s,r)| \leqslant \frac{(a-b)(a-c)}{\xi^2} = \frac{a^2}{\xi^2}(1 - e^{-s})(1 - e^{-r}) \leqslant \frac{a^2}{\xi^2},$$

completing the proof. □

4.5.5 Proof of Lemma 4.16. We start with the asymptotics (4.37) and (4.38) for the exponential moments $\mathbb{E}_\beta\big[e^{\widehat{V}_{\alpha,j,t}(1)}\big]$ and $\mathbb{E}_\beta\big[e^{\widehat{V}_{\alpha,j,t}(1) + \widehat{V}_{\alpha,j,t'}(1)}\big]$. By Claim 4.18,

$$\log \mathbb{E}_\beta\big[e^{\widehat{V}_{\alpha,j,t}(1)}\big] = \log \mathbb{E}_\beta\big[e^{\sqrt{\beta/2}\,V_{\alpha,j,t}(s^*_\alpha) - 2s^*_\alpha}\big] = \mathsf{K}(s^*_\alpha, s^*_\alpha) - 2s^*_\alpha + O_\beta(j^{-2}) \overset{(\mathsf{K}_2)}{=} -s^*_\alpha - \log 4 + o_j(1),$$

which gives us (4.37). Similarly, by Claims 4.19 and 4.20 with $\xi = |t - t'| \geqslant e^{j - \frac{1}{2}\alpha\sqrt{j}}$, $a = e^{j - \alpha\sqrt{j}}$,

$$\begin{aligned}\log \mathbb{E}_\beta\big[e^{\widehat{V}_{\alpha,j,t}(1) + \widehat{V}_{\alpha,j,t'}(1)}\big] &= \log \mathbb{E}_\beta\big[e^{\sqrt{\beta/2}\,V_{\alpha,j,t}(s^*_\alpha) + \sqrt{\beta/2}\,V_{\alpha,j,t'}(s^*_\alpha) - 4s^*_\alpha}\big] \\ &= 2\mathsf{K}(s^*_\alpha, s^*_\alpha) + 2\mathsf{K}_{\xi,a}(s^*_\alpha, s^*_\alpha) - 4s^*_\alpha + O_\beta(j^{-2}) = -2s^*_\alpha - \log 16 + o_j(1),\end{aligned}$$

yielding (4.38).

To prove convergence of $X_{j,t}$ to Brownian motion, we employ a version of Kolmogorov's theorem, see for instance, [16, Section 9.2, Theorem 2]. According to this theorem, it suffices

to verify two conditions: (i) convergence of finite-dimensional distributions of $X_{j,t}$ to the ones of the Brownian motions, and (ii) the moment condition

$$\mathbb{E}_{\mu_{j,t}}\big[|X_{j,t}(u)-X_{j,t}(v)|^A\big]\leqslant C|u-v|^B,\qquad u,v\in[0,1].$$

with some $A,C>0$ and $B>1$, all independent of t, which is needed for the tightness of the distribution of $X_{j,t}$ and which we will verify with $A=4$, $B=2$. Similarly, we will argue for convergence of the pair $(X_{j,t},X_{j,t'})$ to the pair of independent Brownian motions.

To proceed, we will slightly rewrite the previous claims putting them into a form directly applicable here.

(a) Let $t\in\mathfrak{T}_j$. Fix $M,m\geqslant 1$. For $s_1,\dots,s_m\in[0,s^*_\alpha]$, $\theta_1,\dots,\theta_m\in[-M,M]$, we have

$$\log\mathbb{E}_{\mu_{j,t}}\exp\sum_{\ell=1}^m\theta_\ell(\sqrt{\beta/2}\,V_{\alpha,j,t}(s_\ell)-2s_\ell)$$
$$=2\sum_{\ell=1}^m\theta_\ell[\mathsf{K}(s^*_\alpha,s_\ell)-s_\ell]+\sum_{\ell,\ell'=1}^m\theta_\ell\theta_{\ell'}\mathsf{K}(s_\ell,s_{\ell'})+o_j(1).\quad(4.44)$$

The implicit constant in $o_j(1)$ depends neither on s_ℓ nor t.

(b) Let $t,t'\in\mathfrak{T}_j$, $|t-t'|\geqslant e^{j-\frac12\alpha\sqrt{j}}$. Then, for $m,M\geqslant 1$, $s_1,\dots s_m,r_1,\dots r_m\in[0,s^*_\alpha]$, $\theta_1,\dots\theta_m,\theta'_1,\dots\theta'_m\in[-M,M]$, we have

$$\log\mathbb{E}_{\mu_{j,t,t'}}\exp\Big[\sum_{\ell=1}^m\theta_\ell(\sqrt{\beta/2}\,V_{\alpha,j,t}(s_\ell)-2s_\ell)+\sum_{\ell'=1}^m\theta'_{\ell'}(\sqrt{\beta/2}\,V_{\alpha,j,t'}(r_{\ell'})-2r_{\ell'})\Big]$$
$$=2\sum_{\ell=1}^m\theta_\ell[\mathsf{K}(s^*_\alpha,s_\ell)-s_\ell]+2\sum_{\ell'=1}^m\theta'_{\ell'}[\mathsf{K}(s^*_\alpha,r_{\ell'})-r_{\ell'}]$$
$$+\sum_{\ell,\ell'=1}^m\theta_\ell\theta_{\ell'}\mathsf{K}(s_\ell,s_{\ell'})+\sum_{\ell,\ell'=1}^m\theta'_\ell\theta'_{\ell'}\mathsf{K}(r_\ell,r_{\ell'})+o_j(1).\quad(4.45)$$

The implicit constant in $o_j(1)$ does not depend on s_ℓ, r_ℓ and t,t'.

We only sketch the proof of (b), the proof of (a) follows the same lines. The LHS of (4.45) equals

$$\log\mathbb{E}_\beta\exp\Big[\sqrt{\beta/2}V_{\alpha,j,t}(s^*_\alpha)+\sqrt{\beta/2}V_{\alpha,j,t'}(s^*_\alpha)+\sum_{\ell=1}^m\theta_\ell\sqrt{\beta/2}\,V_{\alpha,j,t}(s_\ell)+\sum_{\ell'=1}^m\theta'_{\ell'}\sqrt{\beta/2}\,V_{\alpha,j,t'}(r_{\ell'})\Big]$$
$$-\log\mathbb{E}_\beta\exp\big[\sqrt{\beta/2}V_{\alpha,j,t}(s^*_\alpha)+\sqrt{\beta/2}V_{\alpha,j,t'}(s^*_\alpha)\big]-2\sum_{\ell=1}^m(\theta_\ell s_\ell+\theta'_\ell r_\ell).\quad(4.46)$$

Applying Claim 4.19 with the coefficients $1,1,\theta_1,\dots\theta_m,\theta'_1\dots\theta'_m$, and discarding the mixed terms (by Claim 4.20), we see that the first term on the RHS equals

$$2\mathsf{K}(s^*_\alpha,s^*_\alpha)+2\sum_{\ell=1}^m[\theta_\ell\mathsf{K}(s^*_\alpha,s_\ell)+\theta'_\ell\mathsf{K}(s^*_\alpha,r_\ell)]+\sum_{\ell,\ell'=1}^m[\theta_\ell\theta_{\ell'}\mathsf{K}(s_\ell,s_{\ell'})+\theta'_\ell\theta'_{\ell'}\mathsf{K}(r_\ell,r_{\ell'})]+o_j(1).$$

The second term on the RHS of (4.46) equals $2\mathsf{K}(s^*_\alpha, s^*_\alpha) + o_j(1)$. Cancelling $2\mathsf{K}(s^*_\alpha, s^*_\alpha)$, we get (4.45).

Now, we turn to the proof of the finite-dimensional convergence at mesh points, starting with the one-point case. Fix $m \geqslant 1$ and points $u_1, \dots, u_m \in [0,1]$. Choose mesh-points $u_\ell(j) = k_\ell(j)/N \longrightarrow u_\ell$ and set $s_\ell(j) = u_\ell(j) s^*_\alpha$. Using (4.44) with $\theta_\ell/\sqrt{2s^*_\alpha}$, we get

$$\log \mathbb{E}_{\mu_{j,t}} \exp\Big[\sum_{\ell=1}^m \theta_\ell X_{j,t}(u_\ell(j))\Big] \\ = \frac{2}{\sqrt{2s^*_\alpha}} \sum_{\ell=1}^m \theta_\ell [\mathsf{K}(s^*_\alpha, s_\ell(j)) - s_\ell(j)] + \frac{1}{2s^*_\alpha} \sum_{\ell,\ell'=1}^m \theta_\ell \theta_{\ell'} \mathsf{K}(s_\ell(j), s_{\ell'}(j)) + o_j(1).$$

By (K_1), the linear term is $o_j(1)$, and

$$\frac{1}{s^*_\alpha} \mathsf{K}(s_\ell(j), s_{\ell'}(j)) = u_\ell(j) \wedge u_{\ell'}(j) + o_j(1) \longrightarrow u_\ell \wedge u_{\ell'}.$$

Therefore,

$$\log \mathbb{E}_{\mu_{j,t}} \exp\Big[\sum_{\ell=1}^m \theta_\ell X_{j,t}(u_\ell(j))\Big] \longrightarrow \frac{1}{2} \sum_{\ell,\ell'=1}^m \theta_\ell \theta_{\ell'} (u_\ell \wedge u_{\ell'}).$$

The RHS is the logarithm of the moment-generating function of $(W(u_1), \dots W(u_m))$, for standard Brownian motion W, and convergence of moment-generating functions in a neighbourhood of the origin yields convergence in distribution of the corresponding vectors. The two-point case follows from the same argument based on (4.45).

Next, we will prove a fourth moment estimate at mesh points, starting again with the one-point case. We take two mesh points $u_k > u_\ell$, put $s_k = u_k s^*_\alpha$, $s_\ell = u_\ell s^*_\alpha$, $d = s_k - s_\ell$ (note that $d \geqslant s^*_\alpha/N \geqslant \frac{1}{2}$) and

$$D = \widehat{V}_{\alpha,j,t}(u_k) - \widehat{V}_{\alpha,j,t}(u_\ell) = \sqrt{\beta/2}(V_{\alpha,j,t}(s_k) - V_{\alpha,j,t}(s_\ell)) - 2d.$$

Applying (4.44) with $m = 2$ and coefficients $\lambda, -\lambda$, we get

$$\log \mathbb{E}_{\mu_{j,t}}[e^{\lambda D}] \\ = 2\lambda[\mathsf{K}(s^*_\alpha, s_k) - \mathsf{K}(s^*_\alpha, s_\ell) - d] + \lambda^2[\mathsf{K}(s_k, s_k) - 2\mathsf{K}(s_k, s_\ell) + \mathsf{K}(s_\ell, s_\ell)] + o_j(1).$$

By (K_1),

$$|\mathsf{K}(s^*_\alpha, s_k) - \mathsf{K}(s^*_\alpha, s_\ell) - d| \leqslant O(1),$$

while by (K_3),

$$0 \leqslant \mathsf{K}(s_k, s_k) - 2\mathsf{K}(s_k, s_\ell) + \mathsf{K}(s_\ell, s_\ell) \leqslant d.$$

We conclude that

$$\log \mathbb{E}_{\mu_{j,t}}[e^{\lambda D}] \leqslant O(1)|\lambda| + \lambda^2 d + o_j(1).$$

The rest is straightforward. Applying this with $\lambda = \pm d^{-1/2}$, we get

$$\log \mathbb{E}_{\mu_{j,t}}\left[e^{\pm D/\sqrt{d}}\right] \leqslant \frac{O(1)}{\sqrt{d}} + O(1) + o_j(1) \overset{d\geqslant \frac{1}{2}}{=} O(1).$$

Hence $\mathbb{E}_{\mu_{j,t}}\left[e^{\pm D/\sqrt{d}}\right] = O(1)$ and then $\mathbb{E}_{\mu_{j,t}}\left[e^{|D|/\sqrt{d}}\right] = O(1)$. Thus, $\mathbb{E}_{\mu_{j,t}}[|D|^4] \leqslant Cd^2$, i.e.,

$$\mathbb{E}_{\mu_{j,t}}[|\widehat{V}_{\alpha,j,t}(u_k) - \widehat{V}_{\alpha,j,t}(u_\ell)|^4] \leqslant C s_\alpha^{*\,2} |u_k - u_\ell|^2,$$

and then

$$\mathbb{E}_{\mu_{j,t}}[|X_{j,t}(u_k) - X_{j,t}(u_\ell)|^4] \leqslant C|u_k - u_\ell|^2.$$

The two-point case is quite similar. The only difference is that now, in order to estimate the exponential moment $\mathbb{E}_{\mu_{j,t,t'}}[e^{|D|/\sqrt{d}}]$ we should start with (4.45), so we omit the details.

Having at hand the fourth moment estimate at mesh points, we easily complete the proof of Lemma 4.16. First, we extend the fourth moment estimate to arbitrary points $u, v \in [0,1]$. If u and v lie in the same mesh interval $[u_k, u_{k+1}]$, then

$$X_{j,t}(v) - X_{j,t}(u) = (v-u)N[X_{j,t}(u_{k+1}) - X_{j,t}(u_k)],$$

and since $|v-u|N \leqslant 1$, we are done. If $u < v$ lie in different mesh intervals, let u_ℓ be the first mesh point to the right of u and u_k be the first mesh point to the left of v. Then,

$$X_{j,t}(v) - X_{j,t}(u) = [X_{j,t}(v) - X_{j,t}(u_k)] + [X_{j,t}(u_k) - X_{j,t}(u_\ell)] + [X_{j,t}(u_\ell) - X_{j,t}(u)],$$

and the triangle inequality in $L^4(\mu_{j,t})$ combined with the previous argument implies the required estimate. The two-point case follows the same lines.

At last, to check finite-dimensional convergence at arbitrary points, given $u_1, \dots, u_m \in [0,1]$ we choose mesh points $u_\ell(j)$ with $|u_\ell(j) - u_\ell| \leqslant 1/N$. Then the fourth moment estimate yields

$$\mathbb{E}_{\mu_{j,t}}[|X_{j,t}(u_\ell(j)) - X_{j,t}(u_\ell)|^4] = O(N^{-2}) = o_j(1).$$

Therefore, $X_{j,t}(u_\ell(j)) - X_{j,t}(u_\ell) \to 0$ in probability. Since we already know that

$$(X_{j,t}(u_1(j)), \dots X_{j,t}(u_m(j))) \longrightarrow (W(u_1), \dots, W(u_m)) \qquad \text{in distribution,}$$

so does $(X_{j,t}(u_1), \dots X_{j,t}(u_m))$. The same argument applies simultaneously to both coordinates under the two-point tilt, uniformly over the separated pairs (t, t'). □

4.5.6 *Proof of Lemma 4.17.*

We aim to show that

$$\mathbb{E}_\beta M_j^2 = \frac{(e-1)^2}{16}\, p_\eta^2 + o_j(1),$$

where, see (4.33), (4.34) and (4.39),

$$M_j^2 = e^{-2j+2s_\alpha^*} \iint_{\mathfrak{T}_j\times\mathfrak{T}_j} e^{\widehat{V}_{\alpha,j,t}(1)+\widehat{V}_{\alpha,j,t'}(1)} \mathbb{1}_{\mathcal{A}_j(t)}\mathbb{1}_{\mathcal{A}_j(t')}\, \mathrm{d}t\,\mathrm{d}t',$$

$$\mathcal{A}_j(t) = \{\max_{0\leqslant k\leqslant N} \widehat{V}_{\alpha,j,t}(u_k) \leqslant \eta\sqrt{s_\alpha^*}, \widehat{V}_{\alpha,j,t}(1) \geqslant -\eta\sqrt{s_\alpha^*}\}$$

is the barrier event,

$$p_\eta = \mathbb{P}\{\max_{t\in[0,1]} W_t \leqslant \eta/\sqrt{2}, W_1 \geqslant -\eta/\sqrt{2}\},$$

and the parameters $0<\alpha<\frac{1}{2}$ and $0<\eta<\frac{1}{2}\,\alpha$ are fixed.

The main contribution to the asymptotics of $\mathbb{E}_\beta M_j^2$ is given by the well-separated pairs $|t-t'|\geqslant e^{j-\frac{1}{2}\alpha\sqrt{j}}$ in the double integral over $\mathfrak{T}_j\times\mathfrak{T}_j$. In that range, by Lemma 4.16(b),

$$\mu_{j,t,t'}(\mathcal{A}_j(t)\cap\mathcal{A}_j(t')) \longrightarrow p_\eta^2$$

and, by (4.38) in the same Lemma,

$$\mathbb{E}_\beta\big[e^{\widehat{V}_{\alpha,j,t}(1)+\widehat{V}_{\alpha,j,t'}(1)}\big] = \frac{1+o_j(1)}{16}\, e^{-2s_\alpha^*},$$

and both convergences are uniform in (t,t'). We split the rest of the double integral into three ranges:

$$\text{(i) } e^{j-\alpha\sqrt{j}} \leqslant |t-t'| \leqslant e^{j-\frac{1}{2}\alpha\sqrt{j}}, \quad \text{(ii) } j^4 \leqslant |t-t'| \leqslant e^{j-\alpha\sqrt{j}}, \quad \text{(iii) } 0<|t-t'|\leqslant j^4,$$

and show that each of them contributes by $o_j(1)$, i.e., is negligible. As above, we set $\xi=|t-t'|$ and $a=e^{j-\alpha\sqrt{j}}$.

(i) $e^{j-\alpha\sqrt{j}} \leqslant |t-t'| \leqslant e^{j-\frac{1}{2}\alpha\sqrt{j}}$. Here the barrier is not needed. By Claim 4.19 applied with $m=m'=1$ and the coefficients $1,1$, we have

$$\begin{aligned}\log\mathbb{E}_\beta\big[e^{\widehat{V}_{\alpha,j,t}(1)+\widehat{V}_{\alpha,j,t'}(1)}\big] &= \log\mathbb{E}_\beta\big[e^{\sqrt{\beta/2}(V_{\alpha,j,t}(s_\alpha^*)+V_{\alpha,j,t'}(s_\alpha^*))-4s_\alpha^*}\big]\\ &= 2\mathsf{K}(s_\alpha^*,s_\alpha^*)+2\mathsf{K}_{\xi,a}(s_\alpha^*,s_\alpha^*)-4s_\alpha^*+O(j^{-2}).\end{aligned}$$

By (K_2), $\mathsf{K}(s_\alpha^*,s_\alpha^*) = s_\alpha^* - \log 4 + o_j(1)$, while by Claim 4.20, $|\mathsf{K}_{\xi,a}(s_\alpha^*,s_\alpha^*)| \leqslant (a/\xi)^2 \leqslant 1$. Thus,

$$\mathbb{E}_\beta\big[e^{\widehat{V}_{\alpha,j,t}(1)+\widehat{V}_{\alpha,j,t'}(1)}\big] \leqslant Ce^{-2s_\alpha^*}.$$

The area of this distance range is $O(e^j\cdot e^{j-\frac{1}{2}\alpha\sqrt{j}})$, so, the contribution of this range does not exceed

$$e^{-2j+2s_\alpha^*}\cdot Ce^{-2s_\alpha^*}\cdot e^{2j-\frac{1}{2}\alpha\sqrt{j}} = O(e^{-\frac{1}{2}\alpha\sqrt{j}}) = o_j(1).$$

(ii) $j^4\leqslant|t-t'|\leqslant e^{j-\alpha\sqrt{j}}$. As above, we set $N=\lceil s_\alpha^*\rceil$, $u_k=k/N$, $0\leqslant k\leqslant N$, $\xi=|t-t'|$, $a=e^{j-\alpha\sqrt{j}}$, and put $s_k=u_k s_\alpha^*$. By $r=r(\xi)$ we denote the largest s_k satisfying $s_k\leqslant j-\alpha\sqrt{j}-\log\xi$.

Since the spacing of the mesh (s_k) is $s^*_\alpha/N \leqslant 1$, we have $j-\alpha\sqrt{j}-\log\xi-1 \leqslant r \leqslant j-\alpha\sqrt{j}-\log\xi$, that is, $\xi \leqslant ae^{-r} \leqslant e\xi$.

On the event $\mathcal{A}_j(t')$ we have

$$e^{\widehat{V}_{\alpha,j,t}(1)+\widehat{V}_{\alpha,j,t'}(1)} \leqslant e^{\widehat{V}_{\alpha,j,t}(1)+\widehat{V}_{\alpha,j,t'}(1)-\widehat{V}_{\alpha,j,t'}(r/s^*_\alpha)+\eta\sqrt{s^*_\alpha}},$$

so, we are after

$$e^{-2j+2s^*_\alpha+\eta\sqrt{s^*_\alpha}} \iint_{j^4\leqslant|t-t'|\leqslant e^{j-\alpha\sqrt{j}}} \mathbb{E}_\beta\big[e^{\widehat{V}_{\alpha,j,t}(1)+\widehat{V}_{\alpha,j,t'}(1)-\widehat{V}_{\alpha,j,t'}(r/s^*_\alpha)}\big]\,\mathrm{d}t\,\mathrm{d}t'.$$

We claim that

$$\log\mathbb{E}_\beta\big[e^{\widehat{V}_{\alpha,j,t}(1)+\widehat{V}_{\alpha,j,t'}(1)-\widehat{V}_{\alpha,j,t'}(r/s^*_\alpha)}\big] = -2s^*_\alpha + r + O(1). \tag{4.47}$$

We will prove this estimate momentarily, but first, let us finish off the argument. Having (4.47), we bound the quantity of interest by

$$\begin{aligned} &Ce^{-2j+\eta\sqrt{s^*_\alpha}} \iint_{j^4\leqslant|t-t'|\leqslant e^{j-\alpha\sqrt{j}}} e^r\,\mathrm{d}t\,\mathrm{d}t' \\ &\qquad \overset{e^r\leqslant a/\xi}{\leqslant} Cae^{-2j+\eta\sqrt{s^*_\alpha}}e^j\int_{j^4}^a\frac{\mathrm{d}\xi}{\xi} \overset{a=e^{j-\alpha\sqrt{j}}}{\leqslant} O(j)e^{-\alpha\sqrt{j}+\eta\sqrt{s^*_\alpha}} \overset{\eta<\alpha/2}{=} o_j(1). \end{aligned}$$

It remains to prove (4.47). To do this, we use Claim 4.19. The "pure terms" are

$$\begin{aligned} &\mathsf{K}(s^*_\alpha,s^*_\alpha)+[\mathsf{K}(s^*_\alpha,s^*_\alpha)-2\mathsf{K}(s^*_\alpha,r)+\mathsf{K}(r,r)] \\ &\qquad = s^*_\alpha + 2\log\cosh\tfrac{1}{2}(s^*_\alpha-r)+O(1) = s^*_\alpha+(s^*_\alpha-r)+O(1) = 2s^*_\alpha-r+O(1). \end{aligned}$$

The mixed terms are

$$2[\mathsf{K}_{\xi,a}(s^*_\alpha,s^*_\alpha)-\mathsf{K}_{\xi,a}(s^*_\alpha,r)].$$

This expression remains bounded uniformly in $j^4\leqslant\xi\leqslant a$. Indeed,

$$\mathsf{K}_{\xi,a}(s^*_\alpha,s^*_\alpha)-\mathsf{K}_{\xi,a}(s^*_\alpha,r) = \log\Big[\frac{|\xi+\mathrm{i}a(1+e^{-s^*_\alpha})|}{|\xi+\mathrm{i}a(1+e^{-r})|}\cdot\frac{|\xi+\mathrm{i}a(e^{-s^*_\alpha}+e^{-r})|}{|\xi+2\mathrm{i}ae^{-s^*_\alpha}|}\Big].$$

Using that $ae^{-s^*_\alpha}=j^4\leqslant\xi\leqslant ae^{-r}\leqslant e\xi$ and $\xi\leqslant a$, we find that both factors on the RHS are bounded from above and below by positive numerical constants. Thus, the mixed terms give a bounded contribution, proving (4.47).

(iii) $0<|t-t'|\leqslant j^4$. In this case, the range is short, which makes the argument simple. By the end-point upper bound in the event $\mathcal{A}_j(t)$, we have $\widehat{V}_{\alpha,j,t}(1)\leqslant\eta\sqrt{s^*_\alpha}$. Thus, we are after

$$e^{-2j+2s^*_\alpha+\eta\sqrt{s^*_\alpha}} \iint_{|t-t'|\leqslant j^4} \mathbb{E}_\beta\big[e^{\widehat{V}_{\alpha,j,t'}(1)}\big]\,\mathrm{d}t\,\mathrm{d}t',$$

which, by Claim 4.18, is bounded by

$$Ce^{-2j+s^*_\alpha+\eta\sqrt{s^*_\alpha}}\,j^4e^j = Cj^4e^{-j+j-4\log j-\alpha\sqrt{j}+\eta\sqrt{s^*_\alpha}} = Ce^{-\alpha\sqrt{j}+\eta\sqrt{s^*_\alpha}} = o_j(1),$$

completing the proof of Lemma 4.17. □

4.5.7 Proof of Lemma 4.15. We need to show that there exists a positive constant $c^*_\beta > 0$ such that

$$\mathbb{P}_\beta(\min_{t\in\mathfrak{T}_j} Y_{\alpha,j,1,t} \leqslant -c^*_\beta \alpha\sqrt{j}) = o_j(1), \tag{4.48}$$

where $\alpha \in (0, \frac{1}{2})$ is fixed and, see (4.28),

$$Y_{\alpha,j,1,t} = 2[\log|G_{\mathcal{X}_\beta}(t+\mathrm{i}a)| - \log|G_{\mathcal{X}_\beta}(t+\mathrm{i}b)| + \pi(b-a)], \qquad t \in \mathfrak{T}_j,$$

with $a = e^{j-\alpha\sqrt{j}}$, $b = e^j$. As above, we use a mesh-discretisation argument.

First, we show that, for sufficiently large $j \geqslant j_\beta$ and for any positive A,

$$\sup_{t\in\mathfrak{T}_j} \mathbb{P}_\beta\{Y_{\alpha,j,1,t} \leqslant -A\alpha\sqrt{j}\} \leqslant C_\beta e^{-(A-\frac{2}{\beta})\alpha\sqrt{j}}. \tag{4.49}$$

To see this, we apply Theorem 3.10 with $z_1 = t + \mathrm{i}a$, $z_2 = t + \mathrm{i}b$, $\gamma_1 = -2$, $\gamma_2 = 2$, $\eta_* = a$, $\Delta = b - a$, and $\ell(\Delta) = O(j)$. In the expression for the logarithmic exponential moment $\log\mathbb{E}_\beta[e^{-Y_{\alpha,j,1,t}}]$, the linear term cancels, while the quadratic term equals

$$\frac{2}{\beta}\,(2\log(a+b) - \log(2a) - \log(2b)) = \frac{2}{\beta}\,\log\frac{(a+b)^2}{4ab}.$$

Thus, the theorem gives us

$$\log\mathbb{E}_\beta[e^{-Y_{\alpha,j,1,t}}] = \frac{2}{\beta}\,\log\frac{(a+b)^2}{4ab} + O_\beta(j^2/a),$$

uniformly in $t \in \mathfrak{T}_j$. Hence, for sufficiently large j, we get

$$\sup_{t\in\mathfrak{T}_j} \mathbb{E}_\beta[e^{-Y_{\alpha,j,1,t}}] \leqslant C_\beta e^{\frac{2}{\beta}\,\alpha\sqrt{j}},$$

and Chebyshev's inequality yields (4.49) with any A.

Next, we set

$$K = \Big\lceil \frac{|\mathfrak{T}_j|}{\alpha a} \Big\rceil,\ t_k = e^j + k\,\frac{|\mathfrak{T}_j|}{K},\ 0 \leqslant k \leqslant K.$$

Noting that $K \simeq \alpha^{-1}e^{\alpha\sqrt{j}}$ and applying the union bound, we get

$$\mathbb{P}_\beta\{\min_{0\leqslant k\leqslant K} Y_{\alpha,j,1,t_k} \leqslant -A\alpha\sqrt{j}\} \leqslant C_{\beta,\alpha} e^{-(A-1-\frac{2}{\beta})\alpha\sqrt{j}} = o_j(1), \tag{4.50}$$

provided that $A > 1 + \frac{2}{\beta}$.

To upgrade this bound to (4.48), we observe that

$$\partial_t Y_{\alpha,j,1,t} = 2\partial_t(\log|G_{\mathcal{X}_\beta}(t+\mathrm{i}a)| - \log|G_{\mathcal{X}_\beta}(t+\mathrm{i}b)|)$$

and that

$$\partial_t \log |G_{\mathcal{X}_\beta}(t+\mathrm{i}y)| = -\operatorname{Re}(s(t+\mathrm{i}y)-\mathrm{i}\pi).$$

To start with, we let $N_j = \mathbb{Z} \cap [e^j - 1, e^{j+1}+1]$ and introduce the event

$$\mathscr{E}_j = \{\forall n \in N_j \ \forall y \in \{a,b\} \colon |s(n+\mathrm{i}y) - \mathrm{i}\pi| \leqslant B\sqrt{j}/y\},$$

with a positive constant $B = B_\beta$ to be chosen. We claim that

$$\mathbb{P}_\beta(\mathscr{E}_j) \geqslant 1 - o_j(1). \tag{4.51}$$

Indeed, by Claim 4.3, for a given $n \in N_j$, we have

$$\mathbb{P}_\beta\{|s(n+\mathrm{i}y) - \mathrm{i}\pi| > B\sqrt{j}/y\} \leqslant C_\beta e^{-c_\beta B^2 j},$$

whence, by the union bound, $\mathbb{P}_\beta(\mathscr{E}_j^{\mathsf{c}}) \leqslant C_\beta e^{-(c_\beta B^2 - 1)j}$. Choosing B so that $c_\beta B^2 \geqslant 2$, we get (4.51).

To extend the estimate from the integers to the whole interval $\mathfrak{T}_j$, for $y \in \{a,b\}$, we argue as in the proof of Claim 4.2. Given n and $y \geqslant 1$, we put $M = \max\limits_{|x-n|\leqslant 1/2} |s(x+\mathrm{i}y) - \mathrm{i}\pi|$. Then, using the estimate $|s'(z)| \leqslant (|s(z) - \mathrm{i}\pi| + \pi)/\operatorname{Im} z$, we get $M \leqslant 2|s(n+\mathrm{i}y) - \mathrm{i}\pi| + \dfrac{\pi}{y}$. Hence, on $\mathscr{E}_j$,

$$\sup_{t\in\mathfrak{T}_j} |s(t+\mathrm{i}y) - \mathrm{i}\pi| \leqslant \frac{2B\sqrt{j}+\pi}{y} \leqslant \frac{C_\beta\sqrt{j}}{y}, \qquad y = a, b.$$

Therefore, on $\mathscr{E}_j$ we have

$$\sup_{t\in\mathfrak{T}_j} |\partial_t Y_{\alpha,j,1,t}| \leqslant 2 \sup_{t\in\mathfrak{T}_j} (|s(t+\mathrm{i}a) - \mathrm{i}\pi| + |s(t+\mathrm{i}b) - \mathrm{i}\pi|) \leqslant C_\beta\sqrt{j}\,(\frac{1}{a} + \frac{1}{b}) \leqslant \frac{C_\beta\sqrt{j}}{a}. \tag{4.52}$$

Each $t \in \mathfrak{T}_j$ is at distance at most αa from some t_k. So, on $\mathscr{E}_j$, (4.52) implies

$$|Y_{\alpha,j,1,t} - Y_{\alpha,j,1,t_k}| \leqslant \alpha a \sup_{t\in\mathfrak{T}_j} |\partial_t Y_{\alpha,j,1,t}| \leqslant C_\beta \alpha\sqrt{j}.$$

Hence, on $\mathscr{E}_j \cap \{\min\limits_{0\leqslant k\leqslant K} Y_{\alpha,j,1,t_k} > -A\alpha\sqrt{j}\}$, we get $\min\limits_{t\in\mathfrak{T}_j} Y_{\alpha,j,1,t} > -(A + C_\beta)\alpha\sqrt{j}$, proving the lemma. □

4.6 Proof of Theorem 1.1. By Propositions 4.8 and 4.10, we have that (1.5) holds with $k_* = \lfloor\sqrt{2/\beta}\rfloor$. This implies, by Proposition 1.3, that the excess is at most k_*. On the other hand, Proposition 4.13 together with Lemma 4.11 imply, using Proposition 1.4, that the excess is at least $k_* = \lfloor\sqrt{2/\beta}\rfloor$. This completes the proof of Theorem 1.1. □

5 Loop equations and local laws

In this section, we prove the local law for the $\mathcal{X}_\beta$-process with explicit dependence of the RHS on β, thus improving Theorem 3.2 and Corollary 3.9. The proof is based on the loop equation hierarchy for the $\mathcal{X}_\beta$-process, which might be of independent interest, and closely follows ideas from [8].

In this section by C and c we always denote various positive numerical constants.

5.1 Loop equations for the β-ensemble. We start with classical loop equations for the β-ensemble [4, Section 3] and [8, Proposition A.3].

Lemma 5.1. *For every $n \geqslant 1$ and every $z, z_1, \dots, z_{n-1} \in \mathbb{C} \setminus \mathbb{R}$, we have*

$$\mathbb{E}_N\Big[(s_N(z)^2 + z s_N(z) + 1)\prod_{i=1}^{n-1} s_N(z_i) + \frac{1}{N}\Big(\frac{2}{\beta} - 1\Big)s_N'(z)\prod_{i=1}^{n-1} s_N(z_i) + \frac{2}{N^2\beta}\sum_{j=1}^{n-1}\partial_{z_j}\Big[\frac{s_N(z) - s_N(z_j)}{z - z_j}\Big]\prod_{i\neq j} s_N(z_i)\Big] = 0. \quad (5.1)$$

Note that if $z = z_j$ for some j, the expression

$$\partial_{z_j}\Big[\frac{s_N(z) - s_N(z_j)}{z - z_j}\Big]$$

is understood as $s_N''(z_j)/2$. In other words, proving the loop identities, we can assume first that $z \neq z_j$ for all j, and then extend by analyticity.

We also note that for $n = 1$ the product and the sum are empty, so the first equation simplifies to

$$\mathbb{E}_N\Big[s_N(z)^2 + z s_N(z) + 1 + \frac{1}{N}\Big(\frac{2}{\beta} - 1\Big)s_N'(z)\Big] = 0$$

For the reader's convenience, we present a sketch of the derivation of the loop equations by integration by parts.

Proof. Let

$$p_N(\lambda) = \prod_{1\leqslant k<\ell\leqslant N} |\lambda_k - \lambda_\ell|^\beta e^{-\frac{\beta N}{4}\sum_{k=1}^N \lambda_k^2}$$

be the (unnormalized) density of the β-ensemble,

$$g(\lambda) = \prod_{i=1}^{n-1} s_N(z_i),$$

and

$$f_k(\lambda) = \frac{g(\lambda)}{\lambda_k - z}, \quad 1 \leqslant k \leqslant N.$$

We are going to use the integration by parts formula

$$\mathbb{E}_N\Big[\sum_{k=1}^{N}(\partial_{\lambda_k} f_k + f_k \partial_{\lambda_k}\log p_N)\Big] = 0 \tag{5.2}$$

(the expression inside the sum equals $p_N(\lambda)^{-1}\partial_{\lambda_k}[f_k(\lambda)p_N(\lambda)]$, and p_N is a.s. positive). The justification of (5.2) is similar to the one that we used in the proof of the biased loop equation in Lemma 3.13.

Computing the logarithmic derivative, we get

$$\partial_{\lambda_k}\log p_N(\lambda) = \beta\sum_{\ell\neq k}\frac{1}{\lambda_k-\lambda_\ell} - \frac{\beta N}{2}\lambda_k.$$

Hence,

$$\mathbb{E}_N\Big[\sum_{k=1}^{N}\partial_{\lambda_k}\Big(\frac{g}{\lambda_k - z}\Big) + \beta g\sum_{k=1}^{N}\sum_{\ell\neq k}\frac{1}{(\lambda_k-z)(\lambda_k-\lambda_\ell)} - \frac{\beta N}{2}\,g\sum_{k=1}^{N}\frac{\lambda_k}{\lambda_k - z}\Big] = 0. \tag{5.3}$$

Next, we simplify each of the three terms inside the expectation sign, which we denote by I, II, and III.

First, we claim that

$$\mathrm{I} = -Ns_N'(z)\prod_{i=1}^{n-1}s_N(z_i) - \sum_{j=1}^{n-1}\partial_{z_j}\Big[\frac{s_N(z)-s_N(z_j)}{z-z_j}\Big]\prod_{i\neq j}s_N(z_i). \tag{5.4}$$

Indeed,

$$\partial_{\lambda_k}\Big(\frac{g}{\lambda_k-z}\Big) = -\frac{g}{(\lambda_k-z)^2} + \frac{1}{\lambda_k-z}\partial_{\lambda_k}g,$$

and

$$-\sum_{k=1}^{N}\frac{g}{(\lambda_k-z)^2} = -Ns_N'(z)g.$$

Next,

$$\partial_{\lambda_k}g = \sum_{j=1}^{n-1}\partial_{\lambda_k}s_N(z_j)\prod_{i\neq j}s_N(z_i) = -\frac{1}{N}\sum_{j=1}^{n-1}\frac{1}{(\lambda_k-z_j)^2}\prod_{i\neq j}s_N(z_i),$$

whence

$$\sum_{k=1}^{N}\frac{1}{\lambda_k-z}\,\partial_{\lambda_k}g = -\frac{1}{N}\sum_{j=1}^{n-1}\Big[\sum_{k=1}^{N}\frac{1}{(\lambda_k-z)(\lambda_k-z_j)^2}\Big]\prod_{i\neq j}s_N(z_i).$$

Differentiating the identity

$$\frac{s_N(z)-s_N(z_j)}{z-z_j} = \frac{1}{N}\sum_{k=1}^{N}\frac{1}{(\lambda_k-z)(\lambda_k-z_j)}$$

in z_j, we get

$$\sum_{k=1}^{N} \frac{1}{\lambda_k - z}\, \partial_{\lambda_k} g = -\sum_{j=1}^{n-1} \partial_{z_j} \Big[\frac{s_N(z) - s_N(z_j)}{z - z_j}\Big] \prod_{i \neq j} s_N(z_i),$$

proving (5.4).

Next, we treat the interaction term II. We claim that

$$\mathrm{II} = -\frac{\beta}{2}\left[N^2 s_N(z)^2 - N s_N'(z)\right] \prod_{i=1}^{n-1} s_N(z_i). \tag{5.5}$$

We symmetrize the double sum II in k and ℓ, using that

$$\begin{aligned}\sum_{k=1}^{N} \sum_{\ell \neq k} \frac{1}{(\lambda_k - z)(\lambda_k - \lambda_\ell)} &= -\frac{1}{2}\Big[\Big(\sum_{k=1}^{N} \frac{1}{\lambda_k - z}\Big)^2 - \sum_{k=1}^{N} \frac{1}{(\lambda_k - z)^2}\Big] \\ &= -\frac{1}{2}\big[N^2 s_N(z)^2 - N s_N'(z)\big],\end{aligned}$$

which proves (5.5).

At last,

$$\mathrm{III} = -\frac{\beta N^2}{2}\,(1 + z s_N(z)) \prod_{i=1}^{n-1} s_N(z_i). \tag{5.6}$$

Indeed,

$$\frac{1}{N}\sum_{k=1}^{N} \frac{\lambda_k}{\lambda_k - z} = \frac{1}{N}\sum_{k=1}^{N} \Big(1 + \frac{z}{\lambda_k - z}\Big) = 1 + z s_N(z).$$

Plugging all three contributions in (5.3), dividing the result by $\beta N^2/2$, and regrouping the terms, we get the loop equations (5.1). □

5.2 Moment estimates for the Stieltjes transform of the β-ensemble. Next, from the loop equations, we get moment estimates of the rescaled Stieltjes transform $s_N(w/N)$. In what follows, to simplify notation, we put

$$\{s_N\}(z, z^*) = \begin{cases} \partial_{z^*}\big(\frac{s_N(z) - s_N(z^*)}{z - z^*}\big) & z^* \neq z, \\ \frac{1}{2}\partial_{zz}^2 s_N(z) & z^* = z. \end{cases}$$

We rescale the variables, letting $z = \pi w/N$, $|w| \leqslant M$, $N \geqslant N_0(M)$,

$$\widetilde{s}_N(w) = \pi s_N(\pi w/N) = \sum_{k=1}^{N} \frac{1}{(N\lambda_k/\pi) - w},$$

and $m_N(w) = \pi m_{\mathrm{sc}}(\pi w/N)$. Note that m_N satisfies the quadratic equation

$$m_N^2 + \frac{\pi^2 w}{N}\, m_N + \pi^2 = 0,$$

and that

$$|m_N(w) - \mathrm{i}\pi\mathrm{sgn}(\operatorname{Im} w)| = O(M/N).$$

Theorem 5.2 (compare with Theorem 3.2)**.** *For every $w \in \mathbb{C} \setminus \mathbb{R}$, $|w| \leqslant M$, $N \geqslant N_0(M)$, and for every $q \geqslant 1$,*

$$\|\widetilde{s}_N(w) - m_N(w)\|_{L^q(\mathbb{P}_N)} \leqslant \Big(\Big|1 - \frac{2}{\beta}\Big| + \sqrt{\frac{q}{\beta}}\Big)\,\frac{C}{|\operatorname{Im} w|}.$$

Proof. Proving the theorem, *it suffices to assume that $w \in \mathbb{H}$ and that $q \geqslant 2$ is an integer.* To simplify notation, we set $\|\,.\,\|_q = \|\,.\|_{L^q(\mathbb{P}_N)}$.

Introduce the function

$$P_N = \widetilde{s}_N^2 + \frac{\pi^2 w}{N}\,\widetilde{s}_N + \pi^2,$$

and notice that

$$\begin{aligned} P_N &= (\widetilde{s}_N^2 + \frac{\pi^2 w}{N}\,\widetilde{s}_N + \pi^2) - (m_N^2 + \frac{\pi^2 w}{N}\,m_N + \pi^2) \\ &= (\widetilde{s}_N - m_N)(\widetilde{s}_N + m_N + \frac{\pi^2 w}{N}\,). \end{aligned} \tag{5.7}$$

Since

$$|\widetilde{s}_N + m_N + \frac{\pi^2 w}{N}\,| \geqslant \operatorname{Im}(\widetilde{s}_N + m_N + \frac{\pi^2 w}{N}\,) \geqslant \operatorname{Im}(m_N + \frac{\pi^2 w}{N}\,) \geqslant \pi - O(M/N), \tag{5.8}$$

we see that

$$|\widetilde{s}_N - m_N| \leqslant \frac{1}{3}\,|P_N|, \tag{5.9}$$

in $|w| \leqslant M$, $N \geqslant N_0$, provided that N_0 was chosen sufficiently large. Thus, we are looking for the moment bounds for P_N.

Next, we rescale the loop equations. Letting $z_i = \pi w_i/N$ and noting that

$$s'_N(z) = \frac{N}{\pi^2}\,\widetilde{s}\,'_N(w),$$

and that

$$\{s_N\}(z, z_j) = \frac{N^2}{\pi^3}\,\{\widetilde{s}_N\}(w, w_j),$$

and multiplying the equations (5.1) by π^{n+1}, we get

$$\mathbb{E}_N\Big[P_N(w)\prod_{i=1}^{n-1}\widetilde{s}_N(w_i)\Big] = \Big(1-\frac{2}{\beta}\Big)\mathbb{E}_N\Big[\widetilde{s}\,'_N(w)\prod_{i=1}^{n-1}\widetilde{s}_N(w_i)\Big] - \frac{2}{\beta}\sum_{j=1}^{n-1}\mathbb{E}_N\Big[\{\widetilde{s}_N\}(w, w_j)\prod_{i\neq j}\widetilde{s}_N(w_i)\Big].$$

Next, we split $n-1 = r+t$, and let $w_i = w_1$ for $1 \leqslant i \leqslant r$ and $w_i = \bar{w}_1$ for $r+1 \leqslant i \leqslant n-1$. Letting $S = \widetilde{s}_N(w_1)$, and noting that $\bar{S} = \widetilde{s}_N(\bar{w}_1)$, we get

$$\mathbb{E}_N[P_N(w)S^r\bar{S}^t] = \Big(1-\frac{2}{\beta}\Big)\mathbb{E}_N[\widetilde{s}\,'_N(w)S^r\bar{S}^t] - \frac{2}{\beta}\,\mathbb{E}_N\Big[r\,\{\widetilde{s}_N\}(w, w_1)S^{r-1}\bar{S}^t + t\,\{\widetilde{s}_N\}(w, \bar{w}_1)S^r\bar{S}^{t-1}\Big].$$

Now, take any polynomial $Q \in \mathbb{C}[S, \bar{S}]$,

$$Q(S, \bar{S}) = \sum_{r,t} a_{r,t} S^r \bar{S}^t,$$

multiply the identity above by $a_{r,t}$ and sum over r and t. The LHS becomes $\mathbb{E}_N[P_N(w)Q(S, \bar{S})]$. The first term on the RHS becomes $(1 - 2\beta^{-1})\mathbb{E}_N[\widetilde{s}'_N(w)Q(S, \bar{S})]$. At last, observing that

$$\sum_{r,t} a_{r,t} r S^{r-1} \bar{S}^t = \partial_S Q(S, \bar{S}) \quad \text{and} \quad \sum_{r,t} a_{r,t} t S^r \bar{S}^{t-1} = \partial_{\bar{S}} Q(S, \bar{S}),$$

we arrive at the identity

$$\mathbb{E}_N[P_N(w)Q] = \Big(1 - \frac{2}{\beta}\Big)\, \mathbb{E}_N[\widetilde{s}'_N(w)Q] - \frac{2}{\beta}\, \mathbb{E}_N\big[\{\widetilde{s}_N\}(w, w_1)\partial_S Q + \{\widetilde{s}_N\}(w, \bar{w}_1)\partial_{\bar{S}} Q\big]. \quad (5.10)$$

Now, for a given integer $q \geqslant 2$, we choose Q so that $P_N(w_1)^{q-1}\overline{P_N(w_1)}^{\,q} = Q(S, \bar{S})$ (the polynomial obtained by expansion of $P_N(w_1)^{q-1}\,\overline{P_N(w_1)}^{\,q}$ in S and $\bar{S}$). Recalling that $P_N(w_1) = S^2 + (\pi^2 w_1/N)S + \pi^2$, we compute the partial derivatives

$$\partial_S Q = (q-1)(2S + \pi^2 w_1/N)P_N(w_1)^{q-2}\,\overline{P_N(w_1)}^{\,q},$$

and

$$\partial_{\bar{S}} Q = q(2\bar{S} + \pi^2 \bar{w}_1/N)P_N(w_1)^{q-1}\,\overline{P_N(w_1)}^{\,q-1}.$$

Plugging this into (5.10), and then letting $w_1 = w$, we get the identity

$$\begin{aligned}
\mathbb{E}_N[|P_N(w)|^{2q}] = \Big(1 - \frac{2}{\beta}\Big)\, &\mathbb{E}_N[\widetilde{s}'_N(w)P_N(w)^{q-1}\overline{P_N(w)}^{\,q}] \\
&- \frac{2(q-1)}{\beta}\, \mathbb{E}_N\big[\{\widetilde{s}_N\}(w, w)(2\widetilde{s}_N(w) + \pi^2 w/N)P_N(w)^{q-2}\,\overline{P_N(w)}^{\,q}\big] \\
&- \frac{2q}{\beta}\, \mathbb{E}_N\big[\{\widetilde{s}_N\}(w, \bar{w})(2\overline{\widetilde{s}_N(w)} + \pi^2 \bar{w}/N)P_N(w)^{q-1}\,\overline{P_N(w)}^{\,q-1}\big],
\end{aligned}$$

which becomes the starting point for the moment estimates.

We have

$$\begin{aligned}
\mathbb{E}_N[|P_N(w)|^{2q}] &\leqslant \mathbb{E}_N[|1 - \tfrac{2}{\beta}|\,|\widetilde{s}'_N(w)|\,|P_N(w)|^{2q-1}] \\
&\quad + \mathbb{E}_N\big[\tfrac{4(q-1)}{\beta}\,|\{\widetilde{s}_N\}(w, w)|\,(|\widetilde{s}_N(w)| + O(\tfrac{M}{N}))\,|P_N(w)|^{2q-2}\big] \\
&\quad + \mathbb{E}_N\big[\tfrac{4q}{\beta}\,|\{\widetilde{s}_N\}(w, \bar{w})|\,(|\overline{\widetilde{s}_N(w)}| + O(\tfrac{M}{N}))\,|P_N(w)|^{2q-2}\big] \\
&= \mathrm{I} + \mathrm{II} + \mathrm{III}\,.
\end{aligned}$$

To proceed, we fix a small $0 < \delta \leqslant 1$, to be chosen a bit later, and use the Young inequality

$$xy \leqslant \Big(\frac{1}{\delta}\Big)^a \frac{x^a}{a} + \delta^b \frac{y^b}{b}, \quad x, y \geqslant 0,\ a^{-1} + b^{-1} = 1,$$

to estimate each term on the RHS. Estimating the first term, we take $a = 2q$, $b = (2q)/(2q-1)$. Then

$$\mathrm{I} \leqslant \frac{1}{2q}\left(\frac{1}{\delta}\right)^{2q}\left|1-\frac{2}{\beta}\right|^{2q}\mathbb{E}_N\big[|\widetilde{s}'_N(w)|^{2q}\big] + \frac{2q-1}{2q}\,\delta^{2q/(2q-1)}\mathbb{E}_N\big[|P_N(w)|^{2q}\big].$$

Estimating the second and the third terms, we recall that $q \geqslant 2$ and take $a = q$, $b = q/(q-1)$. Then

$$\mathrm{II} \leqslant \frac{1}{q}\left(\frac{1}{\delta}\right)^{q}\left(\frac{4(q-1)}{\beta}\right)^{q}\mathbb{E}_N\big[|\{\widetilde{s}_N\}(w,w)|^q\,(|\widetilde{s}_N(w)|+O(\tfrac{M}{N})\,)^q\big]+\frac{q-1}{q}\,\delta^{q/(q-1)}\mathbb{E}_N\big[|P_N(w)|^{2q}\big],$$

and

$$\mathrm{III} \leqslant \frac{1}{q}\left(\frac{1}{\delta}\right)^{q}\left(\frac{4q}{\beta}\right)^{q}\mathbb{E}_N\big[|\{\widetilde{s}_N\}(w,\bar{w})|^q\,(|\widetilde{s}_N(w)|+O(\tfrac{M}{N})\,)^q\big] + \frac{q-1}{q}\,\delta^{q/(q-1)}\mathbb{E}_N\big[|P_N(w)|^{2q}\big].$$

Combining these three estimates, we get

$$\begin{aligned}\mathbb{E}_N[|P_N(w)|^{2q}] &\leqslant \frac{1}{2q}\left(\frac{1}{\delta}\right)^{2q}\left|1-\frac{2}{\beta}\right|^{2q}\mathbb{E}_N\big[|\widetilde{s}'_N(w)|^{2q}\big]\\ &\quad+\frac{1}{q}\left(\frac{1}{\delta}\right)^{q}\left(\frac{4q}{\beta}\right)^{q}\mathbb{E}_N\big[(|\{\widetilde{s}_N\}(w,w)|^q+|\{\widetilde{s}_N\}(w,\bar{w})|^q)\,(|\widetilde{s}_N(w)|+O(\tfrac{M}{N})\,)^q\big]\\ &\qquad+\Big(\frac{2q-1}{2q}\,\delta^{2q/(2q-1)}+2\frac{q-1}{q}\,\delta^{q/(q-1)}\Big)\,\mathbb{E}_N\big[|P_N(w)|^{2q}\big].\end{aligned}$$

For $\delta = \frac{1}{4}$, the last term on the RHS does not exceed $(1-\frac{1}{2q})\mathbb{E}_N\big[|P_N(w)|^{2q}\big]$, so we get

$$\begin{aligned}\frac{1}{2q}\mathbb{E}_N[|P_N(w)|^{2q}] &\leqslant \frac{1}{2q}\,4^{2q}\left|1-\frac{2}{\beta}\right|^{2q}\mathbb{E}_N\big[|\widetilde{s}'_N(w)|^{2q}\big]\\ &\quad+\frac{1}{q}\,4^q\left(\frac{4q}{\beta}\right)^{q}\mathbb{E}_N\big[(|\{\widetilde{s}_N\}(w,w)|^q+|\{\widetilde{s}_N\}(w,\bar{w})|^q)\,(|\widetilde{s}_N(w)|+O(\tfrac{M}{N})\,)^q\big],\end{aligned}$$

whence

$$\begin{aligned}\mathbb{E}_N[|P_N(w)|^{2q}] &\leqslant 4^{2q}\left|1-\frac{2}{\beta}\right|^{2q}\mathbb{E}_N\big[|\widetilde{s}'_N(w)|^{2q}\big]\\ &\quad+2\left(\frac{16q}{\beta}\right)^{q}\mathbb{E}_N\big[(|\{\widetilde{s}_N\}(w,w)|^q+|\{\widetilde{s}_N\}(w,\bar{w})|^q)\,(|\widetilde{s}_N(w)|+O(\tfrac{M}{N})\,)^q\big].\end{aligned}$$

We proceed with estimates of the RHS. Recall that we write $w = E + \mathrm{i}\eta$. Then, for $\eta > 0$, we have

$$|\widetilde{s}'_N(w)| = \Big|\sum_{k=1}^{N}\frac{1}{((N\lambda_k/\pi)-w)^2}\Big| \leqslant \sum_{k=1}^{N}\frac{1}{|(N\lambda_k/\pi)-w|^2} = \frac{\operatorname{Im}\widetilde{s}_N(w)}{\eta},$$

and recalling the definition of $\{\widetilde{s}_N\}$ and proceeding similarly,

$$|\{\widetilde{s}_N\}(w,w)| = \frac{1}{2}\,|\widetilde{s}''_N(w)| \leqslant \frac{\operatorname{Im}\widetilde{s}_N(w)}{\eta^2},$$

and

$$|\{\widetilde{s}_N\}(w,\bar{w})| = \Big|\frac{\widetilde{s}_N(w)-\widetilde{s}_N(\bar{w})}{(w-\bar{w})^2} - \frac{\widetilde{s}\,'_N(\bar{w})}{w-\bar{w}}\Big| \leqslant 2\,\frac{\operatorname{Im}\widetilde{s}_N(w)}{\eta^2}.$$

Hence, we get

$$\begin{aligned}\mathbb{E}_N[|P_N(w)|^{2q}] &\leqslant \Big(\frac{4}{\eta}\Big)^{2q}\Big|1-\frac{2}{\beta}\Big|^{2q}\,\mathbb{E}_N\big[|\operatorname{Im}\widetilde{s}_N(w)|^{2q}\big]\\ &\qquad + 4\Big(\frac{4}{\eta}\Big)^{2q}\Big(\frac{2q}{\beta}\Big)^{q}\,\mathbb{E}_N\big[|\operatorname{Im}\widetilde{s}_N(w)|^{q}\big(|\widetilde{s}_N(w)|+O\big(\tfrac{M}{N}\big)\big)^{q}\big] = A+B.\end{aligned}$$

To simplify notation, set

$$a = \frac{1}{\eta}\Big(\Big|1-\frac{2}{\beta}\Big| + \sqrt{\frac{q}{\beta}}\Big)$$

We claim that the previous inequality, after taking the $2q$-th root, gives us

$$\|P_N\|_{2q} \leqslant Ca(1+\|\widetilde{s}_N - m_N\|_{2q}). \tag{5.11}$$

Indeed,

$$\|P_N\|_{2q} \leqslant (A+B)^{1/(2q)} \leqslant A^{1/(2q)} + B^{1/(2q)},$$

and

$$A^{1/(2q)} = \frac{4}{\eta}\Big|1-\frac{2}{\beta}\Big|\,\|\operatorname{Im}\widetilde{s}_N(w)\|_{2q}\,.$$

Furthermore,

$$\begin{aligned}B^{1/(2q)} &= 4^{1/(2q)}\,\frac{4}{\eta}\sqrt{\frac{2q}{\beta}}\,\big(\mathbb{E}_N\big[(\operatorname{Im}\widetilde{s}_N(w))^{q}\big(|\widetilde{s}_N(w)|+O\big(\tfrac{M}{N}\big)\big)^{q}\big]\big)^{1/(2q)}\\ &\leqslant \frac{8}{\eta}\sqrt{\frac{q}{\beta}}\,\big(\mathbb{E}_N\big[(\operatorname{Im}\widetilde{s}_N(w))^{q}\big(|\widetilde{s}_N(w)|+O\big(\tfrac{M}{N}\big)\big)^{q}\big]\big)^{1/(2q)}.\end{aligned}$$

Now,

$$\begin{aligned}&\Big(\mathbb{E}_N\big[(\operatorname{Im}\widetilde{s}_N(w))^{q}\cdot\big(|\widetilde{s}_N(w)|+O\big(\tfrac{M}{N}\big)\big)^{q}\big]\Big)^{1/(2q)}\\ &\qquad\leqslant \big(\mathbb{E}_N[(\operatorname{Im}\widetilde{s}_N(w))^{2q}]\big)^{1/(4q)}\cdot\big(\mathbb{E}_N[(|\widetilde{s}_N(w)|+O\big(\tfrac{M}{N}\big))^{2q}]\big)^{1/(4q)}\\ &\qquad= \|\operatorname{Im}\widetilde{s}_N(w)\|_{2q}^{1/2}\cdot\||\widetilde{s}_N(w)|+O\big(\tfrac{M}{N}\big)\|_{2q}^{1/2}.\end{aligned}$$

Hence

$$B^{1/(2q)} \leqslant \frac{8}{\eta}\sqrt{\frac{q}{\beta}}\,\|\operatorname{Im}\widetilde{s}_N(w)\|_{2q}^{1/2}\cdot\||\widetilde{s}_N(w)|+O\big(\tfrac{M}{N}\big)\|_{2q}^{1/2},$$

whence,

$$\|P_N\|_{2q} \leqslant \frac{4}{\eta}\Big|1-\frac{2}{\beta}\Big|\,\|\operatorname{Im}\widetilde{s}_N(w)\|_{2q} + \frac{8}{\eta}\sqrt{\frac{q}{\beta}}\,\|\operatorname{Im}\widetilde{s}_N(w)\|_{2q}^{1/2}\cdot\||\widetilde{s}_N(w)|+O\big(\tfrac{M}{N}\big)\|_{2q}^{1/2}.$$

Recalling that $m_N(w) = \mathrm{i}\pi + O(M/N)$ where $|w| \leqslant M$ and that N is sufficiently large, we see that $\operatorname{Im} \widetilde{s}_N(w) \leqslant C + |\widetilde{s_N}(w) - m_N(w)|$, and therefore,

$$\|\operatorname{Im} \widetilde{s}_N(w)\|_{2q} \leqslant C + \|\widetilde{s_N}(w) - m_N(w)\|_{2q}.$$

Similarly,

$$\||\widetilde{s}_N(w)| + O(\tfrac{M}{N})\|_{2q} \leqslant C(1 + \|\widetilde{s_N}(w) - m_N(w)\|_{2q}).$$

Combining these estimates, we arrive at (5.11).

Our next step is to show the bound

$$|\widetilde{s}_N - m_N|^2 \leqslant C|P_N| \tag{5.12}$$

which complements (5.9). As in the proof of (5.9), we use factorization (5.7) combined with the lower bound (5.8). Since

$$\widetilde{s}_N - m_N = (\widetilde{s}_N + m_N + \frac{\pi^2 w}{N}) - (2m_N + \frac{\pi^2 w}{N}),$$

and the absolute value of the first term is $\geqslant c$, while the second term is bounded, we get

$$|\widetilde{s}_N - m_N| \leqslant |\widetilde{s}_N + m_N + \frac{\pi^2 w}{N}| + C \leqslant C|\widetilde{s}_N + m_N + \frac{\pi^2 w}{N}|.$$

Therefore,

$$|\widetilde{s}_N - m_N|^2 \leqslant C|\widetilde{s}_N - m_N| \cdot |\widetilde{s}_N + m_N + \frac{\pi^2 w}{N}| \overset{(5.7)}{=} C|P_N|,$$

proving (5.12).

At last, we return to the bound (5.11) and combine it with (5.9) and (5.12). First, we obtain

$$\|\widetilde{s}_N - m_N\|_{2q} \leqslant \|P_N\|_{2q} \leqslant Ca(1 + \|\widetilde{s}_N - m_N\|_{2q}).$$

If $Ca \leqslant 1/2$, we get what we wanted $\|\widetilde{s}_N - m_N\|_{2q} \leqslant Ca$. If $Ca \geqslant 1/2$, using (5.12), we get

$$\|\widetilde{s}_N - m_N\|_{2q}^2 \leqslant C\|P_N\|_{2q} \leqslant Ca(1 + \|\widetilde{s}_N - m_N\|_{2q}).$$

If $\|\widetilde{s}_N - m_N\|_{2q} \leqslant 1$, then trivially $\|\widetilde{s}_N - m_N\|_{2q} \leqslant 2Ca$. If $\|\widetilde{s}_N - m_N\|_{2q} \geqslant 1$, then $1 + \|\widetilde{s}_N - m_N\|_{2q} \leqslant 2\|\widetilde{s}_N - m_N\|_{2q}$, so $\|\widetilde{s}_N - m_N\|_{2q}^2 \leqslant 2Ca\|\widetilde{s}_N - m_N\|_{2q}$, whence $\|\widetilde{s}_N - m_N\|_{2q} \leqslant 2Ca$.

By monotonicity of the L^q-norms, this implies the bound $\|\widetilde{s}_N - m_N\|_q \leqslant C'a$ for every $q \geqslant 1$. □

5.3 Loop equations and local law for the Sine_β ***process.*** Now, we readily get the limiting versions of Lemma 5.1 and Theorem 5.2.

Lemma 5.3 (limiting loop equations)**.** *For every* $n \geqslant 1$*, every* $z \in \mathbb{C} \setminus \mathbb{R}$*, and every* $z_1, \dots, z_{n-1} \in \mathbb{C} \setminus \mathbb{R}$*,*

$$\mathbb{E}_\beta\Big[(s(z)^2+\pi^2)\prod_{i=1}^{n-1} s(z_i) + \Big(\frac{2}{\beta}-1\Big)s'(z)\prod_{i=1}^{n-1} s(z_i) + \frac{2}{\beta}\sum_{j=1}^{n-1}\{s\}(z,z_j)\prod_{i\neq j} s(z_i)\Big] = 0, \qquad (5.13)$$

where

$$\{s\}(z,w) = \begin{cases} \partial_w\Big[\frac{s(z)-s(w)}{z-w}\Big], & w \neq z, \\ \frac{1}{2}s''(z), & w = z. \end{cases}$$

Proof. We will pass to the $N \to \infty$-limit in the rescaled finite-N loop equations of Lemma 5.1. The argument will be the same as in the $N \to \infty$ passage in the biased loop equation in Section 3.4.2.

By Corollary 3.8, $\widetilde{s}_N \xrightarrow{\mathrm{d}} s$ as elements of the space $\mathcal{H}$ of holomorphic functions on $\mathbb{C} \setminus \mathbb{R}$. Hence, jointly,

$$\widetilde{s}_N(z_i) \xrightarrow{\mathrm{d}} s(z_i), \quad \widetilde{s}'_N(z) \xrightarrow{\mathrm{d}} s'(z), \quad \{\widetilde{s}_N\}(z,z_i) \xrightarrow{\mathrm{d}} \{s\}(z,z_i) \qquad (i = 1, \dots, n-1),$$

and

$$P_N(z) = \widetilde{s}_N(z)^2 + \frac{\pi^2 z}{N}\widetilde{s}_N(z) + \pi^2 \xrightarrow{\mathrm{d}} s(z)^2 + \pi^2.$$

By Theorem 5.2, the moments of $\widetilde{s}_N$ are bounded uniformly in N. Together with the elementary bounds $|\widetilde{s}'_N(z)| \leqslant \operatorname{Im}\widetilde{s}_N(z)/\operatorname{Im} z$ and

$$|\{\widetilde{s}_N\}(z,z_i)| \leqslant \frac{|\operatorname{Im}\widetilde{s}_N(z)| + |\operatorname{Im}\widetilde{s}_N(z_i)|}{|\operatorname{Im} z| \cdot |\operatorname{Im} z_i|},$$

this gives uniform integrability of all the products that appear in the loop equations. Therefore, we can pass to the limit in their expectations, obtaining (5.13). □

Corollary 5.4 (limiting local law, compare with Corollary 3.9)**.** *For every* $z \in \mathbb{C} \setminus \mathbb{R}$ *and every* $q \geqslant 1$*,*

$$\|s(z) - \pi\mathrm{i}\operatorname{sgn}(\operatorname{Im} z)\|_{L^q(\mathbb{P}_\beta)} \leqslant \Big(\Big|1-\frac{2}{\beta}\Big| + \sqrt{\frac{q}{\beta}}\Big)\frac{C}{|\operatorname{Im} z|}.$$

Proof. We repeat the argument used in the proof of Corollary 3.9. Fix $z \in \mathbb{C} \setminus \mathbb{R}$. By Corollary 3.8, $\widetilde{s}_N \xrightarrow{\mathrm{d}} s$, while $m_N(z) = \pi m_{\mathrm{sc}}(\pi z/N) \to \pi\mathrm{i}\operatorname{sgn}(\operatorname{Im} z)$. Applying this convergence to the bounded continuous functions $x \mapsto |x|^q \wedge M$, using Theorem 5.2, and then letting $M \to \infty$, gives the result. □

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