Fluctuations for non-Hermitian dynamics

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We prove that under the Brownian evolution on large non-Hermitian matrices the log-determinant converges in distribution to a 2+1 dimensional Gaussian field in the Edwards-Wilkinson regularity class, namely it is logarithmically correlated for the parabolic distance. This dynamically extends a seminal result by Rider and Virág about convergence to the Gaussian free field. The convergence holds out of equilibrium for centered, i.i.d. matrix entries as an initial condition.

A remarkable aspect of the limiting field is its non-Markovianity, due to long range correlations of the eigenvector overlaps, for which we identify the exact space-time polynomial decay.

In the proof, we obtain a quantitative, optimal relaxation at the hard edge, for a broad extension of the Dyson Brownian motion, with a driving noise arbitrarily correlated in space.

1 Introduction

1.1 Random matrices and logarithmically correlated fields, dimensions one and two. The eigenvalues of large random matrices exhibit universal, anomalous small fluctuations. These are log-correlated and originate from equal superposition of randomness on all length scales. This phenomenon first appeared in dimension one, the prominent example being Haar-distributed unitary random matrices of size N, with eigenangles $\boldsymbol{\theta}$: If f, g are smooth with mean zero on the torus, then $\sum f(\theta_k)$ and $\sum g(\theta_k)$ converge with no normalization to Gaussian random variables ℓ_f, ℓ_g as $N \to \infty$, and

$$\mathrm{Cov}(\ell_f, \ell_g) = -\frac{2}{\pi^2} \iint f'(\theta) g'(\varphi) \log |e^{\mathrm{i}\theta} - e^{\mathrm{i}\varphi}| \mathrm{d}\theta \mathrm{d}\varphi. \tag{1.1}$$

This limiting covariance was derived first by Dyson and Mehta [49]. It has then appeared for multiple matrix models in dimension one, L-functions and high genus hyperbolic surfaces, often with techniques from integrable systems, representation theory, mathematical physics (loop equations) and probability theory (moments method). We refer to the end of this introduction for a partial review of this immense literature.

In dimension two, logarithmic correlations appeared much later, first in the context of time-dependent 1d spectra in Spohn [87]. Consider, for example, B a Brownian motion on $N \times N$ Hermitian matrices, or more precisely its Ornstein-Uhlenbeck version H which has the Gaussian unitary ensemble as equilibrium, and the Dyson Brownian motion dynamics induced on its spectrum $\boldsymbol{\lambda}$:

$$\mathrm{d}H_t=\frac{\mathrm{d}B_t}{\sqrt{N}}-\frac{1}{2}H_t\mathrm{d}t,\qquad \mathrm{d}\lambda_k(t)=\frac{\mathrm{d}b_k(t)}{\sqrt{N}}+\Big(\frac{1}{N}\sum_{\ell\neq k}\frac{1}{\lambda_k(t)-\lambda_\ell(t)}-\frac{1}{2}\lambda_k(t)\Big)\mathrm{d}t, \tag{1.2}$$

where $b_1,\dots,b_n$ are independent standard Brownian motions. Log-correlations for these dynamics hold in the following sense: For any centered, smooth f supported in the bulk of the spectrum, the linear statistics $\mathrm{Tr} f(H_s)$ converge in distribution in the limit of large dimension, jointly in f and s, to Gaussian random variables $\ell_{f,s}$ with covariance

$$\mathrm{Cov}(\ell_{f,s},\ell_{g,t})=\frac{2}{\pi^2}\iint f'(x)g'(y)\,k((x,s),(y,t))\mathrm{d}x\mathrm{d}y\quad\text{where}\quad k(u,v)\underset{u\to v}{\sim}-\log|u-v|. \tag{1.3}$$

Here k is an explicit kernel which is smooth off the diagonal, and $|\cdot|$ is the Euclidean distance in $\mathbb{R}^2$.

The derivation of this limiting 2d log-correlated field relies on the self-adjointness of H: It makes the eigenvalues dynamics in (1.2) autonomous, in the sense that it does not depend on the eigenvectors. Actually the Dyson Brownian motion also coincides with Brownian motions conditioned not to intersect, so from the Karlin-McGregor formula the spectrum $(\boldsymbol{\lambda}_t)_{t\geqslant 0}$ forms a determinantal point process, which is fully integrable.

Another natural context for bidimensional random matrix theory is non-Hermitian models. The paradigmatic example is the Ginibre ensemble, an $N\times N$ random matrix G with independent complex Gaussian entries with covariance $\mathbf{1}_2/(2N)$[1]. The eigenvalues for this model also form a determinantal point process, which implies the following analogue of (1.3), by Rider and Virág [81]: For smooth enough f,g supported in the unit disk, $\mathrm{Tr} f(G)$ and $\mathrm{Tr} g(G)$ converges to Gaussian random variables L_f,L_g with covariance

$$\mathrm{Cov}(L_f,L_g)=\int\nabla f(z)\cdot\nabla g(z)\frac{\mathrm{d}z}{4\pi}=-\frac{1}{8\pi^2}\iint\Delta f(z)\Delta g(w)\log|z-w|\,\mathrm{d}z\mathrm{d}w, \tag{1.4}$$

where $\mathrm{d}z$ is the Lebesgue measure on $\mathbb{C}$. For $f=\log|v-\cdot|$ we have $\Delta f=2\pi\delta_v$ in distribution, so the above central limit theorem means distributional convergence of $(\log|\det(G-z)|)_{|z|<1}$ to a $2d$ log-correlated field.

1.2 Result. The Ginibre ensemble is the stationary distribution for the non-Hermitian Ornstein-Uhlenbeck process. At the matrix and eigenvalues level these dynamics are

$$\mathrm{d}X_t=\frac{\mathrm{d}B_t}{\sqrt{N}}-\frac{1}{2}X_t\mathrm{d}t,\qquad \mathrm{d}\sigma_i(t)=\frac{\mathrm{d}M_i(t)}{\sqrt{N}}-\frac{1}{2}\sigma_i(t)\mathrm{d}t, \tag{1.5}$$

where the entries of B are independent complex standard Brownian motions, i.e. of type $\frac{1}{\sqrt{2}}(B_1+\mathrm{i}B_2)$ with B_1,B_2 independent standard real Brownian motions. Here the martingales M_i are correlated, with joint bracket $\mathrm{d}\langle M_i,M_j\rangle_t=\mathscr{O}_{ij}(t)\mathrm{d}t$, $\mathscr{O}_{ij}=(R_j^*R_i)(L_j^*L_i)$, where the left and right eigenvectors $(L_i,R_i)_{i=1}^N$ of X_t are a biorthogonal basis sets, normalized by $L_i^tR_j=\delta_{ij}$. Therefore, contrary to the Hermitian setting (1.2), the dynamics for non-Hermitian spectra fundamentally differs from the Langevin dynamics for the log-gas. In particular the evolution is now non-autonomous, and the eigenvalues distribution is determinantal only at equilibrium, for one fixed time. Nevertheless, our result below gives $3d$-logarithmic correlations for these non-Hermitian dynamics, now with respect to the parabolic distance

$$d((z,s),(w,t))=(|z-w|^2+|t-s|)^{1/2}. \tag{1.6}$$

Theorem. *Consider (1.5) at equilibrium and let f,g be smooth functions supported in the open unit disk $\mathbb{D}$. Then for any fixed s,t, as $N\to\infty$ the linear statistics* $\mathrm{Tr} f(G_s)-N\int\frac{f}{\pi}$ *and* $\mathrm{Tr} g(G_t)-N\int\frac{g}{\pi}$ *converge jointly to Gaussian random variables $L_{f,s},L_{g,t}$, which are centered with covariance*

$$\mathbb{E}[L_{f,s}L_{g,t}]=\frac{1}{16\pi^2}\iint\Delta f(z)\Delta g(w)K(z,w,|t-s|)\mathrm{d}z\mathrm{d}w, \tag{1.7}$$

$$K(z,w,\tau):=-\log\big((1-e^{-\tau})(1-|z|^2)+|z-e^{-\tau/2}w|^2\big). \tag{1.8}$$

One can easily check that the above kernel K is symmetric in z,w. Moreover it satisfies

$$K(z,w,t-s)\asymp-\log d((z,s),(w,t))$$

[1]Here $\mathbf{1}_2$ denotes the 2×2 identity matrix.

as $(z,s)\to(w,t)$, uniformly in compact subsets of $\mathbb{D}\times\mathbb{R}$.

The above theorem follows from a general decomposition theorem which allows to treat out of equilibrium dynamics with i.i.d. initial condition, test functions overlapping the edge of the spectrum and mesoscopic scales. We refer to Section 2 for these results. There, we will also give consequences on the space-time correlations between overlaps of non-Hermitian random matrices, and the main ideas of the proof, circumventing the lack of autonomous spectral evolution and determinantal representations.

1.3 Related literature. We conclude this introduction with more context on logarithmically correlated fields, in random matrix theory and other settings.

In dimension one, log-correlations are now considered a core result in random matrix theory, as manifested by the books [77, Chapter 16] and [54, Chapter 14]. Formulas similar to (1.1) hold for a multitude of matrix models, including Wigner matrices [74] and their deformations [73], covariance matrices [8], products of random matrices [62], band matrices [7], random graphs [47], free sums of random matrices [9]. This covariance also appears in the limit of large dimension for general log-gases [66], determinantal point processes [29], and particle systems emerging from special functions [19]. The most common techniques towards such results are the method of moments [6, Chapter 2], Schur-Weyl duality [43], loop equations [67], transport maps [85, 11, 12] and the analysis of Toeplitz determinants [65]. Moreover, in a few cases, the distributional convergence to a logarithmically correlated field has been upgraded to pointwise convergence [20, 26]. In completely different contexts, the covariance (1.1) was also exhibited for linear statistics related to zeros of L-functions [25, 82] and eigenvalues of the Laplacian on random hyperbolic surfaces [84]. It is impossible to mention all such contributions and we refer to [55] for a recent review.

In dimension two, Spohn's original space-time fluctuations (1.3) were first proved for non-intersecting Brownian motions at equilibrium [87], and then extended to arbitrary temperature on the circle, still at equilibrium [88]. The space-time fluctuations of Dyson's Brownian motion are now understood out of equilibrium, on any mesoscopic scale [46, 64, 1]. The $2d$ Gaussian free field also arises in discrete setting, often in connection with non-intersecting paths; important examples include domino tilings [68], dimer models [69, 45], and many more discrete particle systems, see e.g. [61].

Still in dimension two, but in the non-Hermitian setting, the Rider-Virág central limit theorem (1.4) is actually more general as it allows functions overlapping the edge of the spectrum. It fully verified a prediction from [53], after partial results in [79, 80] for radial or angular functions. It has been generalized in three major directions, first to arbitrary normal random matrices, i.e. to non-quadratic confining external potential [4, 5], then to $2d$ Coulomb gases at arbitrary temperature [72, 10], and finally to non-Hermitian random matrices with arbitrary independent entries [40, 39, 41].

In dimension three, before (1.7) the only example connected to random matrices came from minors of Hermitian matrices along the matrix dynamics (1.2), thanks to the method of moments [14]. The covariance kernel of the limiting $3d$ Gaussian field depends on the spectral variable, the size of the submatrix, and time. It is log-correlated for the Euclidean distance in $\mathbb{R}^3$. A similar $3d$ limiting field was then proved in connection to representations of $U(\infty)$ [15], which is also log-correlated for the L^2 norm.

By contrast, non-Hermitian dynamics present the same singular behavior as the so-called Edwards-Wilkinson fluctuations in dimension 2+1, i.e. logarithmic correlations for the parabolic distance. The canonical model for these fluctuations is the $2d$ additive stochastic heat equation [63]. Despite the conjectured breadth of the Edwards-Wilkinson universality class, only a few models have been proved to be part of it.

In the continuous setting, the following stochastic partial differential equations have been shown to exhibit Edwards-Wilkinson fluctuations in the weak coupling regime: the multiplicative stochastic heat equation (or the directed polymer model in random environment) [31, 48], and the KPZ equation driven by space-time white noise, may it be isotropic subcritical [32] or anisotropic [30]. Langevin dynamics for Liouville quantum gravity measures also present log-correlated fluctuations for the parabolic distance [58].

In the discrete setting, fluctuations for the Ginzburg-Landau interface model were shown to converge to a $2d$ additive stochastic heat equation, at equilibrium [59]. Other stochastic growth models in $2+1$ dimensions, that belong to the anisotropic KPZ class, fluctuate like the additive stochastic heat equation, for specific initial conditions. This was first proved through convergence to the free field at fixed time [18], and then extended to space-time correlations [16, 17].

As described above, Edwards-Wilkinson fluctuations have been proved mostly at equilibrium or for a restricted range of initial conditions. Theorem 2.2 below works out of equilibrium, for arbitrary initial random matrix with centered i.i.d. entries. In fact, the methods we rely on should be robust enough to cover broad classes of deterministic initial data, including the possibility of time-dependent hydrodynamic profiles. The assumptions of centered independent entries is for convenience and it gives a particularly simple formula for correlations of the limiting field (1.8). In the bulk of the spectrum, the local singularity of the limiting kernel should be independent of the initial condition, only its long range behavior may change.

The obtained fluctuations for non-Hermitian dynamics raise other questions, for which some techniques developed in this paper may apply. In view of the singularity of the covariance (1.8), it would be interesting to develop branching methods for logarithmically-correlated fields for the parabolic distance, to understand extreme statistics. In particular the Fyodorov-Hiary-Keating analogies [56, 57] ($d = 1$) and the universality proved in [44] (any dimension) give broad classes of models with the same extreme values, in the case of log-correlations for the Euclidean distance. Similarly, the recent convergences of random matrix statistics to Gaussian multiplicative chaos hold with respect to the L^2 distance, in dimension 1 [91, 78, 35] and 2 [24].

Finally, we note that there is no first principle explanation for logarithmic correlations for so many random matrix models, despite the enormous literature on this topic in dimensions 1 and 2, and the new occurrence in dimension 3 proved in this paper.

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2 Main Results.

2.1 Multi-time central limit theorems for eigenvalues. Our initial condition consists in $N \times N$ non-Hermitian matrices X with i.i.d. complex entries, i.e. $x_{ab} \overset{d}{=} N^{-1/2}\chi$, with χ satisfying the following assumptions.

Assumption 2.1. *The random variable χ is centered and has unit variance, i.e. $\mathbb{E}\chi = 0$, $\mathbb{E}|\chi|^2 = 1$, and we also assume that $\mathbb{E}\chi^2 = 0$. Additionally, high moments of χ are finite, i.e. there exist a constants $C_p > 0$, for any $p \in \mathbb{N}$, such that*

$$\mathbb{E}|\chi|^p \leqslant C_p.$$

We now consider the matrix dynamics (1.5), where we remind that B_t is a matrix whose entries are i.i.d. standard complex Brownian motions. Note that

$$X_t \overset{(\mathrm{d})}{=} e^{-t/2}X + \sqrt{1-e^{-t}}\widetilde{X},$$

with $\widetilde{X}$ being a complex Ginibre matrix independent of X. The first two moments of X_t are preserved along the flow, and the Ginibre ensemble is the unique equilibrium for these dynamics. Denoting the eigenvalues of X_t by $\{\sigma_i(t)\}_{1\leqslant i\leqslant N}$, our main interest is to study the space–time correlation of the linear statistics

$$L_N(f,t) = \sum_{i=1}^{N} f(\sigma_i(t)) - \mathbb{E}\sum_{i=1}^{N} f(\sigma_i(t)). \tag{2.1}$$

Here f is a test function supported on Ω, a fixed disk with center 0 and arbitrary radius greater than 1. Without loss of generality, by polarization, it is enough to consider real test functions f, which will be assumed to be in the Sobolev space $\mathrm{H}_0^{2+\varepsilon}(\Omega)$, for a fixed small $\varepsilon > 0$, which is defined as the completion of the smooth compactly supported functions $C_c^\infty(\Omega)$ under the norm

$$\|f\|_{\mathrm{H}^{2+\varepsilon}(\Omega)} = \|(1+|\xi|)^{2+\varepsilon}\widehat{f}(\xi)\|_{\mathrm{L}^2(\Omega)},$$

where $\widehat{f}$ denotes the Fourier transform of f. Furthermore, for h defined on the boundary of the unit disk $\partial\mathbb{D}$, our convention for its Fourier transform is

$$\widehat{h}_k = \frac{1}{2\pi}\int_0^{2\pi} h(e^{\mathrm{i}\theta})e^{-\mathrm{i}\theta k}\,\mathrm{d}\theta, \qquad k \in \mathbb{Z},$$

and we denote

$$\langle g, f\rangle_{\mathrm{H}^{1/2}(\partial\mathbb{D})} = \sum_{k\in\mathbb{Z}} |k| \widehat{f}_k \overline{\widehat{g}_k}, \qquad \|f\|^2_{\mathrm{H}^{1/2}(\partial\mathbb{D})} = \langle f, f\rangle_{\mathrm{H}^{1/2}(\partial\Omega)}.$$

We now first state the CLT for linear statistics (2.1) with macroscopic test functions in Theorem 2.2, and then we present the statement of the mesoscopic case separately in Theorem 2.3. Before stating these results we introduce some useful short–hand notations. We define the averages

$$\langle f\rangle_{\mathbb{D}} := \frac{1}{\pi}\int_{\mathbb{D}} f(z)\,\mathrm{d}z, \qquad \langle f\rangle_{\partial\mathbb{D}} := \frac{1}{2\pi}\int_0^{2\pi} f(e^{\mathrm{i}\theta})\,\mathrm{d}\theta.$$

With the notation (1.8) for the limiting covariance kernel K at equilibrium in the bulk of the spectrum, and defining $P_t f(z) = \sum_{\mathbb{Z}} e^{-|k|\frac{t}{2}} \widehat{f}_k z^k$ for the Poisson kernel, we denote

$$\begin{aligned}\Gamma(f,g,\tau,\kappa) = \frac{1}{\pi^2}\int_{\mathbb{D}^2} \partial_{\overline{z}} f(z)\partial_w g(w)\partial_z\partial_{\overline{w}} K(z,w,\tau)\,\mathrm{d}z\mathrm{d}w + \frac{1}{2}\langle f, P_\tau g\rangle_{H^{1/2}(\partial\mathbb{D})} \\ + \kappa e^{-\tau}\big(\langle f\rangle_{\mathbb{D}} - \langle f\rangle_{\partial\mathbb{D}}\big)\big(\langle g\rangle_{\mathbb{D}} - \langle g\rangle_{\partial\mathbb{D}}\big).\end{aligned} \tag{2.2}$$

In the following main result, $\kappa_{4,t}$ is the fourth cumulant of the matrix entries after time t:

$$\kappa_{4,t} = \mathbb{E}|\chi_t|^4 - 2 \quad \text{where} \quad \chi_t \overset{(\mathrm{d})}{=} e^{-t/2}\chi + \sqrt{1-e^{-t}}g,$$

with g a standard complex Gaussian random variable independent of χ, and χ satisfying Assumption 2.1.

Theorem 2.2 (Macroscopic CLT)**.** *Let X_t be the solution of* (1.5)*, with X satisfying Assumption 2.1. Consider real valued $f, g \in H_0^{2+\varepsilon}(\Omega)$. Then for any fixed $s \leqslant t$, $L_N(f,t), L_N(g,s)$ converge jointly in distribution to centered Gaussian random variables $(L(f,t),\, L(g,s))$ with covariance*

$$\mathbb{E}|L(f,t)|^2 = \Gamma(f,f,0,\kappa_{4,t}), \qquad \mathbb{E}L(f,t)L(g,s) = \Gamma(f,g,t-s,\kappa_{4,s}), \qquad \mathbb{E}|L(g,s)|^2 = \Gamma(g,g,0,\kappa_{4,s}). \tag{2.3}$$

Additionally, we have the decomposition

$$L_N(f,t) = L_{N,1}(f,s,t) + L_{N,2}(f,s,t), \tag{2.4}$$

with $L_{N,1}(f,s,t)$ depending only on X_s, and $L_{N,1}(f,s,t)$, $L_{N,2}(f,s,t)$ converging to independent Gaussian random variables $\mathcal{L}_1(f,s,t), \mathcal{L}_2(f,s,t)$ with

$$\mathbb{E}|\mathcal{L}_1(f,s,t)|^2 = \Gamma(f,f,2(t-s),\kappa_{4,s}), \qquad \mathbb{E}|\mathcal{L}_2(f,s,t)|^2 = \Gamma(f,f,0,0) - \Gamma(f,f,2(t-s),0). \tag{2.5}$$

The above theorem and Equation (2.3) naturally extend to an arbitrary fixed number of test functions and times. In particular (2.5) identifies the limiting distribution of the field through its increments, and indeed the proof relies on the decomposition (2.4).

Moreover, for $\kappa_{4,t} = 0$ and functions supported in the bulk of the spectrum one recovers the special case (1.7) stated in the introduction. Theorem 2.2 also generalizes the static result from [40], as (2.2) gives

$$\Gamma(f,f,0,\kappa_4) = \frac{1}{4\pi}\int_{\mathbb{D}} \big|\nabla f\big|^2\,\mathrm{d}z + \frac{1}{2}\|f\|^2_{\mathrm{H}^{1/2}(\partial\mathbb{D})} + \kappa_4\left|\frac{1}{\pi}\int_{\mathbb{D}} f(z)\,\mathrm{d}z - \frac{1}{2\pi}\int_0^{2\pi} f(e^{\mathrm{i}\theta})\mathrm{d}\theta\right|^2$$

which agrees with [40, Equation (2.6)]. Note also that precise asymptotics are known for the centering term $\mathbb{E}\sum_{i=1}^N f(\sigma_i(t))$ in (2.1), see [40, Equation (2.8)].

We now state the mesoscopic version of Thereom 2.2. For this purpose given a function f we define its rescaled version around a point $v \in \mathbb{C}$ as

$$f_{v,a}(z) = f(N^a(z-v)), \qquad a \in (0, 1/2). \tag{2.6}$$

The following result shows that after a proper time rescaling, a universal Gaussian limiting field emerges, with a simpler covariance structure denoted by

$$\Gamma_v(f,g,\tau) = -\frac{1}{\pi^2}\int_{\mathbb{C}^2} \partial_{\overline{z}} f(z)\partial_w g(w)\partial_z\partial_{\overline{w}} \log\big(\tau(1-|v|^2) + |z-w|^2\big)\,\mathrm{d}z\mathrm{d}w. \tag{2.7}$$

Theorem 2.3 (Mesoscopic CLT)**.** *Fix $T \geqslant 0$, a small $c > 0$, $|v| \leqslant 1 - c$, $a \in (0, 1/2)$, $\Omega \subset \mathbb{C}$ an open set, and let X_t be as above. Consider real valued $f, g \in H_0^{2+\varepsilon}(\Omega)$ and let $s_a = T + n^{-2a}s$, $t_a = T + n^{-2a}t$ with $0 \leqslant s \leqslant t \lesssim 1$ fixed. Then $(L_N(f_{v,a}, t_a), L_N(g_{v,a}, s_a))$ converge jointly in distribution to centered Gaussian random variables $L(f, t), L(g, s)$ with covariance*

$$\mathbb{E}|L(f,t)|^2 = \Gamma_v(f,f,0), \quad \mathbb{E}L(f,t)L(g,s) = \Gamma_v(f,g,t-s), \quad \mathbb{E}|L(g,s)|^2 = \Gamma_v(g,g,0).$$

Additionally, for $L_N(f_{v,a}, t_a)$ we have decomposition and limiting increments similar to (2.4), (2.5).

Note that the covariance of the limiting field on mesoscopic scales in (2.7) does not depend on the fourth cumulant of the entries of X_t, contrary to the macroscopic case (2.2).

Remark 2.4 (Mixing and distance to the edge)**.** *From the prefactor $1 - |v|^2$ in (2.7), relaxation of the dynamics takes longer close to the edge of the spectrum. This is reminiscent of* [22, (1.16)] *which states that at equilibrium individual eigenvalues exhibit diffusive scaling, with quadratic variation proportional to the distance to the edge.*

Remark 2.5 (Averaging by stereographic projection)**.** *The covariance (2.7) can be made explicit:*

$$\Gamma_v(f,g,\tau) = \frac{1}{\pi}\int_{\mathbb{C}^2} \partial_{\overline{z}} f(z) \partial_w g(w) q_r(z-w) \mathrm{d}z \mathrm{d}w, \quad q_r(z) = \frac{r}{\pi(r + |z|^2)^2}, \quad r = \tau(1 - |v|^2).$$

We have $q_r(z)\mathrm{d}z \to \delta_0$ as $r \to 0$, so one recovers (1.4). Moreover $q_r(z)\mathrm{d}z$ is the pushforward, by stereographic projection, of the uniform probability measure on $\mathscr{S}_r$ (the 2-sphere with center 0 and radius r). Denoting $(Q_r f)(z) = \int f(z-w) q_r(w) \mathrm{d}w$ the averaging by this stereographic kernel, Theorem 2.3 means

$$\mathrm{Cov}(L(f,s), L(g,t)) = \frac{1}{\pi^2}\int_{\mathbb{C}} \partial_{\overline{z}} f(z) \partial_z (Q_r g)(z) \mathrm{d}z = \mathrm{Cov}(L_f, L_{Q_r g}), \quad r = |t-s|(1 - |v|^2),$$

relating the covariance of the time-dependent limiting field in Theorem 2.3, to the static field (L_f) from (1.4).

However, $(Q_r)_{r \geqslant 0}$ is not a semigroup ($Q_{r_1 + r_2} \neq Q_{r_1} Q_{r_2}$), so that the limiting field $(L(f,s))_{f,s}$ is not Markovian with respect to time, see Section 2.2.

Remark 2.6 (Existence of Gaussian fields)**.** *The mesoscopic covariance $\Gamma_v(f, g, \tau)$ from Theorem 2.3 can be directly proved to be positive definite, i.e. $\sum_{1 \leqslant i,j \leqslant m} \Gamma_v(f_i, f_j, |t_i - t_j|) \geqslant 0$ for any $m \geqslant 1$, functions f_i and times t_i. For $m = 2$ this follows easily from Schwarz's and then Young's inequality:*

$$\|f\|^2 + \|g\|^2 + 2\mathrm{Re}\langle f, Q_r g\rangle \geqslant \|f\|^2 + \|g\|^2 - 2\|f\| \cdot \|Q_r g\| \geqslant \|f\|^2 + \|g\|^2 - 2\|f\| \cdot \|g\| \geqslant 0$$

for any f, g. For general m a Fourier-based proof will be given in Section 5.5. For (2.2), positive definiteness follows from Theorem 2.2 but a direct proof is unclear.

Remark 2.7 (Restriction to the cylinder)**.** *In Theorem 2.2, dynamical fluctuations at the edge of the spectrum are characterized by a Gaussian field with covariance $\langle f, P_{|t-s|} g\rangle_{H^{1/2}(\partial\mathbb{D})}$. Remarkably, this exact limiting field on the cylinder $\partial\mathbb{D} \times \mathbb{R}$ also describes fluctuations of the unitary Brownian motion* [87, 88, 24]*: at equilibrium when $N \to \infty$,*

$$\mathrm{Cov}(\mathrm{Tr} f(U_N(s)), \mathrm{Tr} g(U_N(t)) \to \langle f, P_{|t-s|} g\rangle_{H^{1/2}(\partial\mathbb{D})}.$$

This dynamically extends the equality between macroscopic fluctuations of eigenvalues for unitary matrices (the right-hand side of (1.1) is also $\langle f, g\rangle_{\mathrm{H}^{1/2}(\partial\mathbb{D})}$) and for the edge of the Ginibre ensemble (when the supports of f, g overlap the boundary, the additional term $\frac{1}{2}\langle f, g\rangle_{H^{1/2}(\partial\mathbb{D})}$ needs to be added to the right-hand side of (1.4), see [81]*).*

Remark 2.8 (Uniformity in the parameters)**.** *We stated Theorem 2.3 only in the bulk regime $|v| < 1$, however a similar statement holds when $|v| = 1$ if $a \leqslant \varepsilon$, for some very small N–independent $\varepsilon > 0$. Additionally, for the sake of clarity we stated Theorems 2.2–2.3 only for two N–independent times s, t, but the proof also gives an analogous result for multiple, N-dependent, times.*

Remark 2.9 (Coupling at equilibrium)**.** *By inspecting the proof of the main result, in the special equilibrium case, it is possible to upgrade the weak convergence in the following sense. There exists a coupling between the fields $(L_N)_{f,t}$ and $(L)_{f,t}$, with $L_N(f,t)$ and $L(f,t)$ being both measurable function of $(B_s)_{-\infty < s \leqslant t}$, such that $(L)_{f,t}$ is a Gaussian field with the covariance (1.7) and $L_N(f,t) - L(f,t)$ converging to 0 in probability as $N \to \infty$, for any fixed f, t.*

2.2 ***Eigenvector overlaps.*** The limiting field from Theorem 2.3 is non–Markovian, as stated below, a fact due to the influence of the eigenbasis on the dynamics (1.5), as the martingales M_i have joint bracket $\mathrm{d}\langle M_i, M_j\rangle_t = \mathscr{O}_{ij}(t)\mathrm{d}t$, where we remind that $\mathscr{O}_{ij} = (R_j^* R_i)(L_j^* L_i)$, with the left and right eigenvectors $\boldsymbol{L}, \boldsymbol{R}$ normalized with $L_i^t R_j = \delta_{ij}$.

Proposition 2.10. *The random field $(L(f,s))$ from Theorem 2.3 is not Markovian with respect to the time variable. More precisely, denoting $\Sigma_{s-} = \sigma(\{L(g,u), u \leqslant s, g \in \mathscr{C}_c^\infty\})$ and $\Sigma_s = \sigma(\{L(g,s), g \in \mathscr{C}_c^\infty\})$, for any $s < t$ there exists $f \in \mathscr{C}_c^\infty$ such that*

$$\mathbb{P}\left(\mathbb{E}\left[L(f,t) \mid \Sigma_{s-}\right] = \mathbb{E}\left[L(f,t) \mid \Sigma_s\right]\right) < 1.$$

This non-Markovianity means that the limiting field keeps memory of some statistics on eigenvectors. This is for example manifested through the following consequence of Theorem 2.3, on space-time correlations between overlaps, at equilibrium: While eigenvalues of Ginibre matrices show exponential decay of correlations in space, the overlaps exhibit polynomial (quartic) decay, even in space-time.

Corollary 2.11. *Consider the dynamics (1.5) at equilibrium, $|v| < 1$ and some $a \in (0, 1/2)$. Then for $|t-s|^{1/2}, |\sigma_i(s) - v|$ and $|\sigma_j(t) - v|$ of order N^{-a}, we have (denoting $c_v = 1 - |v|^2$)*

$$\mathrm{Cov}\Big(\mathscr{O}_{ii}(s), \mathscr{O}_{jj}(t)\Big) \sim \frac{c_v^2}{(c_v|t-s| + |\sigma_i(s) - \sigma_j(t)|^2)^2} \tag{2.8}$$

in the following distributional sense: For any $\tilde{s}_1 < \tilde{s}_2 < \tilde{t}_1 < \tilde{t}_2$ fixed, $s_i = N^{-2a}\tilde{s}_i$, $t_i = N^{-2a}\tilde{t}_i$ and $f_{v,a}, g_{v,a}$ as in (2.6), for some fixed $\varepsilon > 0$ we have

$$\begin{aligned} &\int_{[s_1,s_2]\times[t_1,t_2]} \mathbb{E}\Big[\sum_{i,j} f_{v,a}(\sigma_i(s)) g_{v,a}(\sigma_j(t)) \Big(\frac{\mathscr{O}_{ii}(s)}{N} - c_v\Big)\Big(\frac{\mathscr{O}_{jj}(t)}{N} - c_v\Big)\Big]\mathrm{d}s\mathrm{d}t \\ =& \int_{[s_1,s_2]\times[t_1,t_2]} \int_{\mathbb{C}^2} f_{v,a}(z) g_{v,a}(w) \frac{c_v^2}{(c_v|t-s| + |z-w|^2)^2} \frac{\mathrm{d}z\mathrm{d}w}{\pi^2} \mathrm{d}s\mathrm{d}t + \mathrm{O}(N^{-4a-\varepsilon}). \end{aligned} \tag{2.9}$$

For the relevant application of the above result, f, g are nonnegative with disjoint supports, so that the integral on the right-hand side is $\gtrsim N^{-4a}$: The above corollary identifies the asymptotics of correlations in distributional sense on any mesoscopic scale.

Remark 2.12 (Coherence with static correlations)**.** *For the Ginibre ensemble, pointwise correlations of eigenvector overlaps are known: From* [22, Equation (1.12)]*, when z_1, z_2 are in the bulk of the spectrum at distance of order N^{-a} ($a \in (0, 1/2)$), we have*

$$\mathbb{E}\left(\mathscr{O}_{11}\mathscr{O}_{22} \mid \sigma_1 = z_1, \sigma_2 = z_2\right) = N^2 c_{z_1} c_{z_2} \left(1 + \frac{1}{N^2|z_1 - z_2|^4}\right)\left(1 + \mathrm{O}(N^{-2a+\varepsilon})\right),$$

and by mimicking the proof of [22, Equation (1.9)] *we also have* $\mathbb{E}\left(\mathscr{O}_{11} \mid \sigma_1 = z_1, \sigma_2 = z_2\right) = N c_{z_1}\left(1 + \mathrm{O}(N^{-1+\varepsilon})\right)$*. Both equations together give*

$$\mathbb{E}\left((\mathscr{O}_{11} - N c_{z_1})(\mathscr{O}_{22} - N c_{z_1}) \mid \sigma_1 = z_1, \sigma_2 = z_2\right) = \frac{c_{z_1} c_{z_2}}{|z_1 - z_2|^4} + \mathrm{O}(N^{2-2a}),$$

which agrees with (2.8) when $s = t$ (at least for $a > 1/3$, which implies $|z_1 - z_2|^{-4} \gg N^{2-2a}$).

Remark 2.13 (Microscopic separation of eigenvalues)**.** *In the Ginibre case,* [22] *also provides correlation asymptotics when $\sigma_1 - \sigma_2 \asymp N^{-1/2}$. It remains an interesting problem to understand the joint distribution of overlaps in the more general dynamical situation, when $d((\sigma_1, s), (\sigma_2, t)) \asymp N^{-1/2}$.*

2.3 ***Proof ideas.*** The starting point of the analysis of our non-Hermitian ensembles is Girko's Hermitization method, in a time-dependent setting, which allows to decompose the linear statistics $L_N(f,t)$ as a sum from submicroscopic, microscopic, and mesoscopic scales (see Equation (3.5)) for a family of Hermitian spectra. In the static case, the submicroscopic contributions have been known to be negligible since seminal lower bounds on smallest singular values (see e.g. [89]), and these estimates apply equally to our dynamical setting. In the static case, only recently the fluctuations from the microscopic and mesoscopic scales have been evaluated in [40], which is an important inspiration for our work.

Compared to [40], to treat dynamics the novelties of our proofs are first technical on the microscopic scale, with a very general statement on independence of small singular values (Theorem 4.1), and then conceptual on the mesoscopic scale, with direct emergence of the limiting Gaussian field $L(f,t)$ from the noise in the Dyson Brownian motion (Section 5).

To handle microscopic scales, [40] proved that the smallest singular values of Hermitized matrices corresponding to distant spectral points are independent, by technically difficult variants of the dynamical method [71]. Theorem 4.1 proves that this independence holds in much greater generality, under the assumptions of some weak local law instead of rigidity of the particles, and natural bounds on eigenvector overlaps, which are model-dependent and follow in our case from important estimates in [40]. This theorem could be applied to a wide range of non-Hermitian models with non-trivial mean and a variance profile, or directed graphs. Furthermore, this method has potential to apply to many problems involving eigenvalues statistics close to an hard edge.

Like [40], our proof of this independence of singular values proceeds by dynamics, but it is considerably shorter and more direct. In particular, [40] built on relaxation of the Dyson Brownian motion at the hard edge proved in [36]. With a different method, our main relaxation result (Proposition 4.5) improves on this local ergodicity from [36] in two directions:

(1) It initiates the study of a natural extension of the Dyson Brownian motion, as it covers relaxation for arbitrary, correlated martingales driving individual particles, see (4.5) and the minimal assumption (4.7).

(2) It provides the optimal, submicroscopic error estimates (up to subpolynomial terms). In particular (s_1 and s_1' are the particle closest to the hard edge)

$$|s_1(t) - s_1'(t)| \leqslant \frac{1}{N \cdot \sqrt{Nt}},$$

for any $\boldsymbol{s}(0), \boldsymbol{s}'(0)$ satisfying a weak local law and $\boldsymbol{s}(t), \boldsymbol{s}'(t)$ following the extended Dyson Brownian Motion dynamics.

A key tool for the above relaxation estimate is an observable introduced in [21]. However the proof of (2) is considerably simpler than the optimal relaxation of the eigenvalue gaps in [21], as it bypasses a decomposition into short and long range interactions that has been customary in the dynamics approach [52, 27, 71], and does not require any maximum principle.

Regarding mesoscopic scales, our method is also fully dynamical by generating space-time correlations of linear statistics from a family of coupled dynamics on resolvents, while [40] proceeded by cumulant expansions. In our proof, Gaussianity of the limiting fluctuations easily follows from an explicit writing of the fluctuations as stochastic integrals, with highly concentrated integrands, see Proposition 5.1. In particular, we unveil the (space–time) logarithmic correlation of the linear statistics as the cumulative effect of the Brownian dynamics (1.5). For any $s < t$, this is achieved by splitting the randomness of X_t into two parts: i) the randomness of X_s, and ii) the fresh randomness introduced by $\mathrm{d}B$ to go from X_s to X_t. More precisely, for any $\eta_r > 0$ such that $\eta_t = N^{-1}$ we decompose

$$\begin{aligned}\log\big|\det(X_t - w)\big| &\approx \frac{1}{2}\log\det\left[|X_t - w|^2 + N^{-1}\right]\\ &= \frac{1}{2}\log\det\left[|X_s - w|^2 + \eta_s^2\right] + \frac{1}{2}\int_s^t \mathrm{d}\left(\log\det\left[|X_r - w|^2 + \eta_r^2\right]\right).\end{aligned}$$

For a specific choice $\eta_s \approx N^{-1} + (t-s)$, the above integral becomes purely stochastic (see Proposition 5.1 for more details), so that we reduced the correlations at different times to correlations at the same time but at different spectral parameters:

$$\begin{aligned}\mathrm{Cov}\big[\log\big|\det(X_t - w)\big|, \log\big|\det(X_s - z)\big|\big] &\approx \mathrm{Cov}\left[\frac{1}{2}\log\det\left[|X_s - w|^2 + \eta_s^2\right], \frac{1}{2}\log\det\left[|X_s - z|^2 + N^{-1}\right]\right]\\ &\approx -\log d\big((z,s),(w,t)\big),\end{aligned}$$

with $d\big((z,s),(w,t)\big)$ being defined in (1.6); see (5.10)–(5.12) for detailed calculations giving rise to this parabolic distance.

The method of characteristics is a powerful technique to characterize fluctuations of the spectrum of random matrices. We refer for example to [64, Section 4] for its application to β-ensembles and to [70, Section 7] for Wigner matrices. In this work, we extend its scope to fluctuations in the context of time-dependent and non-Hermitian ensembles, identifying correlation at different times as correlation for the same matrix but at different spectral parameters.

A surprising aspect of Theorems 2.2–2.3 is the remarkably simple limiting space-time logarithmic correlation, despite its emergence from the superposition of the complicated covariances from a family of resolvents (from the Hermitization). This simplicity can be interpreted, in Corollary 2.11, as a statement on space-time correlations between eigenvector overlaps. Indeed Equation (2.9) follows from the combination of Theorem 2.3 and the non-Hermitian dynamics (1.5), at equilibrium. Out of equilibrium, such a quartic decay of correlations seems out of reach: To exhibit overlaps at two distinct times, we need to consider not only forward but also backward dynamics, which are tractable only at equilibrium (see Subsection 5.3).

Finally, we note that in a recent breakthrough [75], universality for non-Hermitian matrices on the finer local scale was obtained through partial Schur transform and supersymmetry. It is unclear whether these methods may apply to multi-time fluctuations as treated in our work.

2.4 Outline. The rest of the paper is organized as follows: In Section 3 we prove our main results Theorems 2.2–2.3. In Section 4 we present a simple, self–contained, decorrelation argument for the small singular values of $X-z_1$, $X-z_2$ when z_1,z_2 are sufficiently far away from each other (i.e. $|z_1-z_2|\gg N^{-1/2}$). Finally, in Section 5 we analyze the resolvent in Girko's formula along the stochastic advection equation, which enables us to identify the Gaussian randomness of the linear statistics as the result of time increments.

2.5 Notations. For two quantities X and Y depending on N, we write that $X=\mathrm{O}(Y)$ or $X\lesssim Y$ if there exists some universal constant $C>0$ such that $|X|\leqslant CY$. We write $X=\mathrm{o}(Y)$, or $X\ll Y$ if the ratio $|X|/Y\to\infty$ as N goes to infinity. We write $X\asymp Y$ if there exists a universal constant $C>0$ such that $Y/C\leqslant|X|\leqslant CY$. We denote $[\![a,b]\!]=[a,b]\cap\mathbb{Z}$ and $[\![n]\!]=[\![1,n]\!]$. Furthermore, for a matrix $\mathbb{C}^{d\times d}$ we denote its normalized trace by $\langle A\rangle:=d^{-1}\mathrm{Tr}[A]$. Because of the Hermitization, in this paper $d=2N$.

Finally, we will write that a sequence of events $(A_N)_{N\geqslant 1}$ holds with very high probability if for any fixed $D>0$ there is a $C>0$ such that $\mathbb{P}(A_N)\geqslant 1-CN^{-D}$ for all $N\geqslant 1$.

3 Time correlations: Proof of Theorems 2.2–2.3

To make our presentation simpler, from now on we only consider random matrices X whose entries χ have a probability density g satisfying

$$g\in L^{1+\alpha}(\mathbb{C}),\qquad \|g\|_{L^{1+\alpha}(\mathbb{C})}\leqslant N^{\beta},\tag{3.1}$$

for some $\alpha,\beta>0$. If X does not satisfy this assumption, then, relying on [90, Theorem 23], we show that the distribution of the linear statistics of X is close to the one of $X+N^{-\gamma}X_{\mathrm{Gin}}$, for any large $\gamma>0$ and X_{Gin} being a complex Ginibre matrix independent of X. Then it is easy to see that the entries of this new matrices satisfy (3.1).

To analyze the linear statistics (2.1) we rely on Girko's formula (cf. [60, 90]):

$$\sum_{\sigma\in\mathrm{Spec}(X)} f(\sigma)=\frac{\mathrm{i}}{4\pi}\int_{\mathbb{C}}\Delta f(z)\int_0^\infty \mathrm{Tr}\big[G^z(\mathrm{i}\eta)\big]\,\mathrm{d}\eta\mathrm{d}z.\tag{3.2}$$

Here G^z denotes the resolvent $G^z(\mathrm{i}\eta):=(W-Z-\mathrm{i}\eta)^{-1}$, with W being the *Hermitization* of X:

$$W:=\begin{pmatrix}0 & X\\ X^* & 0\end{pmatrix},\qquad Z:=\begin{pmatrix}0 & z\\ \overline{z} & 0\end{pmatrix}.\tag{3.3}$$

We will use Girko's formula for the eigenvalues of X_t, with X_t being the solution of the flow (1.5) with initial condition X. By W_t we denote the Hermitization of X_t defined as in (3.3) with X replaced by X_t, and by $G_t^z(\mathrm{i}\eta)$ we denote its resolvent. Note that the spectrum of $W_t - Z$ is symmetric with respect to zero as a consequence of its 2×2 block structure (*chiral symmetry*). We denote the eigenvalues of $W_t - Z$ by $\{\lambda_{\pm i}^z(t)\}_{i\in[n]}$, with $\lambda_{-i}^z(t) = -\lambda_i^z(t)$, and denote the corresponding eigenvectors by $\boldsymbol{w}_{\pm i}^z(t)$. As a consequence of the chiral symmetry, the eigenvectors of $W_t - Z$ are of the form

$$\boldsymbol{w}_{\pm i}^z(t) = \begin{pmatrix} \boldsymbol{u}_i^z(t) \\ \pm \boldsymbol{v}_i^z(t) \end{pmatrix}, \tag{3.4}$$

with $\boldsymbol{u}_i^z(t)$, $\boldsymbol{v}_i^z(t)$ being the left and right singular vectors of $X_t - z$, respectively.

The remainder of this section is divided into several subsections: in Section 3.1 we divide the η–integral in (3.2) into three main regimes, since each one of these regimes will be analyzed using very different techniques, then in Sections 3.2–3.4 we deal with these regimes one by one. Finally, in Section 3.5 we put all these together and conclude the proof of Theorems 2.2–2.3.

3.1 Decomposition in three terms. We split the analysis of Girko's formula into several regimes (cf. [40, Equation (3.10)])

$$\begin{aligned} \sum_{i=1}^N f_{v,a}(\sigma_i(t)) - \mathbb{E}\sum_{i=1}^N f_{v,a}(\sigma_i(t)) &= \frac{1}{4\pi}\int_{\mathbb{C}} \Delta f(z)\big[|\log\det(H_t^z - \mathrm{i}T)| - \mathbb{E}|\log\det(H_t^z - \mathrm{i}T)|\big]\,\mathrm{d}z \\ &\quad - \frac{1}{4\pi\mathrm{i}}\int_{\mathbb{C}} \Delta f_{v,a}(z)\left(\int_0^{\eta_0} + \int_{\eta_0}^{\eta_c} + \int_{\eta_c}^{T}\right)\mathrm{Tr}\big[G_t^z(\mathrm{i}\eta) - \mathbb{E}G_t^z(\mathrm{i}\eta)\big]\,\mathrm{d}\eta\mathrm{d}z \\ &=: J_T(f_{v,a},t) + I_0^{\eta_0}(f_{v,a},t) + I_{\eta_0}^{\eta_c}(f_{v,a},t) + I_{\eta_c}^{T}(f_{v,a},t), \end{aligned} \tag{3.5}$$

with $\eta_0 = N^{-1-\delta_0}$, $\eta_c = N^{-1+\delta_1}$, for some $\delta_0, \delta_1 > 0$. In order to keep the notation for the proof of the macroscopic and mesoscopic CLT unified, with a slight abuse of notation, here we use the convention that $f_{v,a} = f$ when $a = 0$.

The integral J_T can be easily seen to be negligible (see e.g. the proof of [37, Lemma 3] and [3, Theorem 2.3]). Next, we show that the very small η–regime $I_0^{\eta_0}$ is negligible using smoothing inequalities for the smallest singular value of $X - z$ (see Section 3.2). The regime $I_{\eta_0}^{\eta_c}$ will also be negligible (see Section 3.3), for this term we will rely on the asymptotical independence of the small singular values of $X_t - z_1$, $X_t - z_2$ for z_1, z_2 sufficiently away from each other. Finally, we will compute high moments of $I_{\eta_c}^T$ showing that this is the regime where the order one contribution to the lhs. of (3.5) comes from (see Section 3.4).

3.2 Submicroscopic scale. By the lower tail estimate for the smallest singular value of $X - z$ from [89, Theorem 3.2] (see also [38, Equation (4a)]), we readily conclude that the contribution of this regime is negligible. In particular, we have (cf. [40, Lemma 4.4])

$$\mathbb{E}|I_0^{\eta_0}(f_{v,a},t)| \leqslant N^{-c}, \tag{3.6}$$

uniformly in $t \lesssim 1$, for some small fixed constant $c > 0$.

3.3 Microscopic scale. In this section we show that the contribution of the regime $I_{\eta_0}^{\eta_c}$ to the lhs. of (3.5) is also negligible in a second moment sense:

Proposition 3.1. *There exists a small constant $\varepsilon > 0$ such that for δ_0 and δ_1 small enough, and any $t \lesssim 1$, we have*

$$\mathbb{E}|I_{\eta_0}^{\eta_c}(f_{v,a},t)|^2 \leqslant N^{-\varepsilon}. \tag{3.7}$$

Note that (3.7) also implies $\mathbb{E} I_{\eta_0}^{\eta_c}(f_{v,a},t) I_{\eta_0}^{\eta_c}(f_{v,a},s) \leqslant N^{-\varepsilon}$, for any $s \leqslant t$, by a simple Schwarz inequality. The main input to prove Proposition 3.1 is the following independence of resolvents. A similar statement was obtained in [40, Proposition 3.5]. In Section 4 we will give an independent proof, which covers broad classes of initial conditions, and proceeds through a new, simple and quantitative proof of hard edge universality in random matrix theory.

Proposition 3.2. *For fixed $c, \omega_p > 0$, let δ_0, δ_1 be small enough constants. Then uniformly in $|z_l| \leqslant 1-c$, $|z_1 - z_2| \geqslant N^{-1/2+\omega_p}$, and $\eta_1, \eta_2 \in [N^{-1-\delta_0}, N^{-1+\delta_1}]$, we have*[2]

$$\mathbb{E}\langle G_t^{z_1}(\mathrm{i}\eta_1)\rangle\langle G_t^{z_2}(\mathrm{i}\eta_2)\rangle = \mathbb{E}\langle G_t^{z_1}(\mathrm{i}\eta_1)\rangle\mathbb{E}\langle G_t^{z_2}(\mathrm{i}\eta_2)\rangle + \mathrm{O}\left(N^{-\varepsilon}\right). \tag{3.8}$$

Proof of Proposition 3.1. This is straightforward by expanding the square in $\mathbb{E}|I_{\eta_0}^{\eta_c}(f_{v,a},t)|^2$, using (3.8) and $\int |\Delta f| \lesssim 1$ (see the proof of [40, Lemma 4.4]). □

3.4 Mesoscopic scales. In this section we show that the regime $I_{\eta_c}^T$ is the one that gives the order one leading contribution to (3.5).

Proposition 3.3. *Fix $T \geqslant 0$, and denote by Π_p the set of pairings*[3] *on $[p]$. Then, there exists $c = c(p) > 0$ such that, for $t_{i,a} := T + N^{-2a}t_i$, we have*

$$\mathbb{E}\prod_{i\in[p]} I_{\eta_c}^T(f_{v,a}^{(i)}, t_{i,a}) = \sum_{P\in\Pi_p}\prod_{\{i,j\}\in P}\mathbb{E}I_{\eta_c}^T(f_{v,a}^{(i)}, t_{i,a})I_{\eta_c}^T(f_{v,a}^{(j)}, t_{j,a}) + \mathrm{O}(N^{-c}). \tag{3.9}$$

Furthermore, we have

$$\mathbb{E}I_{\eta_c}^T(f_{v,a}^{(i)}, t_{i,a})I_{\eta_c}^T(f_{v,a}^{(j)}, t_{j,a}) = \Gamma(f^{(i)}, f^{(j)}, |t_i - t_j|, \kappa_{4,t_i\wedge t_j}) + \mathrm{O}(N^{-c}), \tag{3.10}$$

with $\Gamma(f,g,\tau,\kappa_{4,t})$ from (2.2) for $a=0$; if $a \in (0,1/2)$ we have the same result with $\Gamma(f,g,\tau,\kappa_{4,t})$ replaced by $\Gamma_v(f,g,\tau)$ from (2.7). Additionally, for any $t_a > 0$ and any $0 \leqslant s_a < t_a$, we have the decomposition

$$I_{\eta_c}^T(f_{v,a}, t_a) =: I_1(f_{v,a}, s_a, t_a) + I_2(f_{v,a}, s_a, t_a), \tag{3.11}$$

with I_1 depending only on X_s, $\mathbb{E}I_1(f,s,t)I_2(f,s,t) = \mathrm{O}(N^{-c})$, and

$$\mathbb{E}|I_1|^2 = \Gamma(f,f,2(t-s),\kappa_{4,t}) + \mathrm{O}(N^{-c}), \qquad \mathbb{E}|I_2|^2 = \Gamma(f,f,0,0) - \Gamma(f,f,2(t-s),\kappa_{4,s}) + \mathrm{O}(N^{-c}), \tag{3.12}$$

for some small fixed $c > 0$. If $a \in (0,1/2)$ we have the same result with $\Gamma(f,g,\tau,\kappa_{4,t})$ replaced by $\Gamma(f,g,\tau)$ from (2.7).

Finally, in the next section we combine (3.6) with Propositions 3.1 and 3.3, to conclude Theorems 2.2–2.3.

3.5 Proof of Theorems 2.2–2.3. Using that J_T from (3.5) is negligible (see e.g. the proof of [37, Lemma 3] and [3, Theorem 2.3]) and the a priori bounds

$$\left|I_0^{\eta_0}(f_{v,a}^{(i)}, t_{i,a})\right| + \left|I_{\eta_0}^{\eta_c}(f_{v,a}^{(i)}, t_{i,a})\right| + \left|I_{\eta_c}^T(f_{v,a}^{(i)}, t_{i,a})\right| \leqslant N^{\xi}$$

with very high probability for any small $\xi > 0$ (see [40, Equation (4.4)] and [41, Equation (4.5)], for the macroscopic and mesoscopic case, respectively), we find that

$$\mathbb{E}\prod_{i\in[p]} L_N(f_{v,a}^{(i)}, t_{i,a}) = \mathbb{E}\prod_{i\in[p]}\left[I_0^{\eta_0}(f_{v,a}^{(i)}, t_{i,a}) + I_{\eta_0}^{\eta_c}(f_{v,a}^{(i)}, t_{i,a}) + I_{\eta_c}^T(f_{v,a}^{(i)}, t_{i,a})\right] + \mathrm{O}(N^{-c}). \tag{3.13}$$

Then by (3.6) and (3.7) we find out that the only order one contribution to the lhs. of (3.13) comes from the large η regime, i.e. we have

$$\mathbb{E}\prod_{i\in[p]} L_N(f_{v,a}^{(i)}, t_{i,a}) = \mathbb{E}\prod_{i\in[p]} I_{\eta_c}^T(f_{v,a}^{(i)}, t_{i,a}) + \mathrm{O}(N^{-c}). \tag{3.14}$$

Finally, Theorem 2.2 readily follows by Proposition 3.3 together with (3.14). □

[2] We state this result for only two different z's for simplicity, but a similar result holds for any finite product of resolvents such that $\min_{i\neq j}|z_i - z_j| \gg N^{-1/2}$.

[3] Note that $\Pi_p = \emptyset$ if p is odd.

4 Independence of singular values

In this section we will make use of the notation $\omega_* \ll \omega^*$ to denote that $\omega_* \leqslant \omega^*/10$, for any two constants ω_*, ω^*. Our main error parameter will be polynomial in N and denoted by

$$\varphi = N^\nu, \tag{4.1}$$

for some fixed (small) constant $\nu > 0$. All other small parameters appearing in this section will satisfy

$$\nu, \omega \ll \omega_t \ll \omega_K \ll \tilde{\omega}.$$

Below is their meaning and where they are introduced:

- ν. The local law holds on scale φ/N (Equation (4.2)).
- ω. The submicroscopic error in independence of singular values is $N^{-1-\omega}$ (Equation (4.4)).
- ω_t. This approximate independence occurs after time at least $N^{-1+\omega_t}$ (Equation (4.4)).
- ω_K. The eigenvector products are small for spectral indices $k \leqslant N^{\omega_K}$ (Equation (4.3)).
- $\tilde{\omega}$. These eigenvector products are of size at most $N^{-\tilde{\omega}}$ (Equation (4.3)).

4.1 Main statement. The main result of this section is the following general statement about asymptotic independence of small singular values. It will imply Proposition 3.2.

Theorem 4.1. *Let X be a $N \times N$ matrix with complex entries, and consider the singular values λ_i^z of $X - z$, for $z \in I := \{z_1, \dots, z_q\}$, q fixed, $z_1, \dots, z_q \in \mathbb{C}$ possibly N-dependent.*

Assume the following local law on scale φ/N holds. There is a $c > 0$ and N_0 such that for any $N \geqslant N_0$ and $z \in I$ there exists $s = s^z$, the Stieltjes transform of a deterministic probability measure $\sigma = \sigma^z$, such that $c \leqslant |\mathrm{Im}\, s(w)| \leqslant c^{-1}$, $|s(w)| + |s'(w)| \leqslant c^{-1}$ for all $|w| \leqslant c$, and such that for any $\mathrm{Im} w \geqslant \varphi/N$, $|w| \leqslant c$ we have

$$\left| \sum_{i=1}^N \frac{1}{\lambda_i^z - w} - s^z(w) \right| \leqslant \frac{1}{\sqrt{N \mathrm{Im} w}}. \tag{4.2}$$

Let X_t be the solution of (1.5), and for $z \in I$ denote the singular values of $X_t - z$ by $\lambda_i^z(t)$. Let $\boldsymbol{u}_i^z(t), \boldsymbol{v}_i^z(t)$ be the left and right singular vectors of $X_t - z$, respectively, and assume that there exists $\omega_K \ll \tilde{\omega}$ such that for any z, w distinct in I and $D > 0$ we have

$$\mathbb{P}\Big(\big|\langle \boldsymbol{u}_i^z(t), \boldsymbol{u}_j^w(t)\rangle\big| + \big|\langle \boldsymbol{v}_i^z(t), \boldsymbol{v}_j^w(t)\rangle\big| \leqslant N^{-\tilde{\omega}}, \quad \forall\, 1 \leqslant i, j \leqslant N^{\omega_K} \text{ and } t \in [0, T] \Big) \geqslant 1 - N^{-D}, \tag{4.3}$$

for a fixed $T > 0$. Let $(X^{(p)})_{1 \leqslant p \leqslant q}$ be independent Ginibre matrices evolving along (1.5), and denote the singular values of $X^{(p)}$ by $\mu_i^{(p)}(t)$. Then there exists a coupling and constants $\omega \ll \omega_t \ll \omega_K$ so that for any $z_p \in I$ we have

$$\mathbb{P}\left(\big|\rho_t(0)\lambda_i^{z_p}(t) - \rho_{\mathrm{sc}}^{(t)}(0)\mu_i^{(p)}(t)\big| \leqslant N^{-1-\omega},\ \forall 1 \leqslant i \leqslant N^\omega \text{ and } t \in [N^{-1+\omega_t}, T] \right) \geqslant 1 - N^{-D}. \tag{4.4}$$

Here ρ_t (resp. $\rho_{\mathrm{sc}}^{(t)}$) is the density of $\sigma^{z_p} \boxplus \mu_{\mathrm{sc}}^{(t)}$ (resp. $\mu_{\mathrm{sc}}^{(1)} \boxplus \mu_{\mathrm{sc}}^{(t)}$) (see the discussion in Section 4.4 for these notations for measures and their free convolutions).

As a consequence of our general Theorem 4.1 we readily conclude the proof of Proposition 3.2.

Proof of Proposition 3.2. By a standard application of a Green's function comparison (GFT) argument, it is enough to prove (3.8) for matrices with an additional Gaussian component of size $\sqrt{t}$, with $t \leqslant N^{-1/2-\varepsilon}$, for any small fixed $\varepsilon > 0$ (see e.g. [40, Lemma 7.5]). We will thus prove (3.8) for matrices having a Gaussian component of size $\sqrt{T}$, with $T = N^{-1+\omega_t}$, for some fixed small $\omega_t > 0$.

For $|z_1 - z_2| \geqslant N^{-1/2+\omega_p}$, by [41, Theorem 3.1], there exist $\tilde{\omega}, \omega_K$ such that (4.3) is satisfied for a fixed time. To show the bound (4.3) uniformly in time it is enough to use a standard grid argument as a consequence of the fact that the bound on eigenvectors is inherited by a bound on product of resolvents (see e.g. the proof of [40, Lemma 7.9]) which is Hölder in time. Additionally, the single resolvent local law [39, Theorem 3.1] implies (4.2). This shows that the assumptions of Theorem 4.1 are satisfied, and (4.4) holds. Given (4.4) as an input (cf. [40, Lemma 7.6–7.7]), using rigidity (cf. [40, Proposition 7.3]), the proof of (3.8) is completely analogous to the proof of [40, Proposition 7.2] and so omitted. □

4.2 ***Reduction to invariance and relaxation.*** We consider coupled processes $s_i(t), r_i(t)$ defined by

$$\mathrm{d}s_i(t) = \frac{1}{\sqrt{2N}}\mathrm{d}b_i^s(t) + \frac{1}{2N}\sum_{j\neq i}\frac{1}{s_i(t)-s_j(t)}\mathrm{d}t, \tag{4.5}$$

$$\mathrm{d}r_i(t) = \frac{1}{\sqrt{2N}}\mathrm{d}b_i^r(t) + \frac{1}{2N}\sum_{j\neq i}\frac{1}{r_i(t)-r_j(t)}\mathrm{d}t, \tag{4.6}$$

where $1 \leqslant |i| \leqslant N$ and $\boldsymbol{s}(0) = (s_i(0))_{1\leqslant|i|\leqslant N}, \mathbf{r}(0) = (r_i(0))_{1\leqslant|i|\leqslant N}$ are such that $s_{-i}(0) = -s_i(0)$, $r_{-i}(0) = r_i(0)$ for $1 \leqslant i \leqslant N$.

In this section, the continuous martingales $b_i^s(t)$ and $b_i^r(t)$ are realized on a common probability space with a common filtration $\mathcal{F}_t$. They do not need to be Brownian motions. They always satisfy the symmetry $b_{-i}^s(t) = -b_i^s(t)$, $b_{-i}^r(t) = -b_i^r(t)$, and the quadratic variation bound

$$\mathrm{d}\langle b_i\rangle_t/\mathrm{d}t \leqslant 1 \text{ a.e.} \tag{4.7}$$

(as a non-decreasing function $\langle b_i\rangle_t$ is differentiable almost everywhere) where $b_i = b_i^s$ or b_i^r. Under these sole assumptions, existence and strong uniqueness hold for (4.5) and (4.6), see Proposition B.1.

We will consider the following additional hypothesis for these martingales.

Assumption 4.2 (Vanishing correlations close to the origin)**.** *For fixed parameters $0 < \omega_K \ll \tilde{\omega}$, $K = N^{\omega_K}$ we have*

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\langle b_i^s - b_i^r\rangle_t\right| \leqslant N^{-\tilde{\omega}}\mathbb{1}_{|i|,|j|\leqslant K} + 4(1-\mathbb{1}_{|i|,|j|\leqslant K}).$$

We denote the empirical particle density of the initial data $\boldsymbol{s}$ as $\mu_0 = \frac{1}{2N}\sum_{1\leqslant|i|\leqslant N}\delta_{s_i(0)}$, and it Stieltjes transform by $m_0(z)$[4]. We assume that μ_0 is regular in the following sense.

Assumption 4.3 ((ν, G)-regularity)**.** *Fix $\nu \in (0,1], G > 0$. Let $\widetilde{\mu}$ be a probability measure with Stieltjes transform $\widetilde{m}$ satisfying, for some fixed $C > 0$,*

$$|\widetilde{m}(z)|, |\widetilde{m}'(z)| \leqslant C, \quad C^{-1} \leqslant \mathrm{Im}[\widetilde{m}(z)] \leqslant C \quad \textit{for all} \quad |z| \leqslant G. \tag{4.8}$$

We say that μ is (ν, G)-regular if, for such a $\tilde{\mu}$, its Stieltjes transform m satisfies

$$|m(z) - \widetilde{m}(z)| \leqslant \frac{\varphi}{\sqrt{N\mathrm{Im}[z]}} \quad \textit{for any } |z| \leqslant G \textit{ with } \mathrm{Im}[z] \geqslant \tfrac{\varphi}{N}. \tag{4.9}$$

Remark 4.4. *For our later use, we may deal with the case that $|\widetilde{m}_0(z)|, |\widetilde{m}_0'(z)|$ may grow with N or $\mathrm{Im}[\widetilde{m}_0(z)]$ may vanishes slowly with N, namely*

$$|\widetilde{m}_0(z)|, |\widetilde{m}_0'(z)| \leqslant N^{\mathrm{o}(1)}, \mathrm{Im}[\widetilde{m}_0(z)] \geqslant N^{-\mathrm{o}(1)}.$$

It is not too hard to see that our argument also extends to this case, as long as we take time t longer enough, to compensate the effect of the error $N^{\mathrm{o}(1)}$. We omit the details to keep the presentation simpler.

Theorem 4.1 will readily follow from the following invariance and relaxation statements.

Proposition 4.5 (Invariance)**.** *Let the processes $\boldsymbol{s}(t), \boldsymbol{r}(t)$ be solutions of the stochastic differential equations (4.5) and (4.6). Assume that the driving continuous martingales satisfy Assumption 4.2. Additionally assume that the initial data $\boldsymbol{s}(0) = \boldsymbol{r}(0)$ satisfies (ν, G)-regularity in the sense of Assumption 4.3.*

Then, for any choice of small constants $\nu, \omega \ll \omega_t \ll \omega_K \ll \tilde{\omega}$ and $t = N^{-1+\omega_t}$, with very high probability

$$|s_i(t_f) - r_i(t_f)| \leqslant N^{-1-\omega}, \textit{ for all } |i| \leqslant N^{\omega}.$$

[4]Within this section we use $m(z)$ to denote the Stieljes transforms of a measure μ. Instead, we point out that in Theorem 4.1 we used $s^z(w)$ to denote the Stieltjes transform of limiting distribution of the singular values of $X - z$.

Proposition 4.6 (Relaxation)**.** *Let the processes* $\boldsymbol{s}(t), \boldsymbol{s}'(t)$ *both satisfy the stochastic differential equation* (4.5)*, with different initial data and the same driving processes* b_i*, which are arbitrary continuous martingales (satisfying (4.7)) and* $\boldsymbol{s}(0), \boldsymbol{s}'(0)$ *are* (ν, G)*-regular around* $\widetilde{\mu}_0, \widetilde{\mu}_0'$ *in the sense of Assumption 4.3. Then for any* $\varepsilon > 0$ *and* $t \geqslant N^{-1+\omega_t}$ *with* $\omega_t \gg \nu$*, for any* $|i| \leqslant N$*, with very high probability one has*

$$|\widetilde{\rho}_t(0) s_i(t) - \widetilde{\rho}_t'(0) s_i'(t)| \leqslant N^{\varepsilon} \cdot \frac{|i|}{N} \cdot \left(\frac{1}{\sqrt{Nt}} + \max(\frac{|i|}{N}, t) \right), \tag{4.10}$$

where $\widetilde{\rho}_t$ *(resp.* $\widetilde{\rho}_t'$*) is the density of* $\widetilde{\mu}_0 \boxplus \mu_{\mathrm{sc}}^{(t)}$ *(resp.* $\widetilde{\mu}_0' \boxplus \mu_{\mathrm{sc}}^{(t)}$*) (see the discussion in Section 4.4).*

Remark 4.7. *When the* b_i*'s are independent, Proposition 4.6 follows by* [36]*. We will give a different, shorter proof, which also covers correlated, general continuous martingales with* $\mathrm{d}\langle b_i \rangle_t / \mathrm{d}t \leqslant 1$*.*

Proof of Theorem 4.1. We first prove that (4.4) holds for any $t \in [N^{-1+\omega_t}, 2N^{-1+\omega_t}]$ for some $\omega, \omega_t > 0$.

Consider the matrix flow

$$\mathrm{d}X_s = \frac{\mathrm{d}B_s}{\sqrt{N}}, \qquad X_0 = Y, \tag{4.11}$$

with $Y = (1-t)^{1/2} X$ and X being an i.i.d. matrix satisfying Assumption 2.1. Fix z_l, for $l = 1, 2$, and denote the Hermitization of $X_s - z_l$ by $H_s^{z_l}$. Then (4.11) induces the following flow on the eigenvalues $x_i^{z_l}(s)$ of $H_s^{z_l}$:

$$\mathrm{d}x_i^{z_l}(s) = \frac{\mathrm{d}b_i^{z_l}(t)}{\sqrt{2N}} + \frac{1}{2N} \sum_{j \neq i} \frac{1}{x_i^{z_l}(s) - x_j^{z_l}(s)} \mathrm{d}s, \tag{4.12}$$

with $b_{-i}^{z_l}(s) = -b_i^{z_l}(s)$. Let $\boldsymbol{w}_i^{z_l}(s)$ be the eigenvectors of $H^{z_l}(s)$ (they have the form (3.4)), then, for $1 \leqslant i, j \leqslant N$, we have

$$\frac{\mathrm{d}}{\mathrm{d}s} \langle b_i^{z_l}, b_j^{z_l} \rangle_s = \delta_{ij}, \qquad \frac{\mathrm{d}}{\mathrm{d}s} \langle b_i^{z_1}, b_j^{z_2} \rangle_s = 4 \mathrm{Re} \big[\langle \boldsymbol{u}_i^{z_1}(s), \boldsymbol{u}_j^{z_2}(s) \rangle \langle \boldsymbol{v}_j^{z_2}(s), \boldsymbol{v}_i^{z_1}(s) \rangle \big].$$

Then, by (4.3), it follows that

$$\frac{\mathrm{d}}{\mathrm{d}s} \langle b_i^{z_1}, b_j^{z_2} \rangle_s \leqslant N^{-2\widetilde{\omega}}, \qquad 1 \leqslant i, j \leqslant N^{\omega_K},$$

with very high probability. This shows that $\{b_i^{z_1}\}$, $\{b_i^{z_2}\}$ satisfy Assumption 4.2. Additionally, by [39, Theorem 3.1] we readily see that the initial condition of (4.11) satisfies Assumption 4.3. We can thus apply Propositions 4.5–4.6 to see that

$$\big| \widetilde{\rho}_t(0) x_i^{z_l}(t) - \rho_{\mathrm{sc}}^{(t)}(0) y_i^{(l)}(t) \big| \leqslant N^{-1-\omega}, \qquad |i| \leqslant N^{\omega},$$

with very high probability. Here $y_i^{(l)}(t)$, with $l = 1, 2$, are two fully independent processes, with $y_i^{(l)}(t)$ the eigenvalues of the Hermitization of a Ginibre matrix, satisfying (4.12) with $b_i^{z_l}(t)$ being replaced by $\beta_i^{(l)}$, where $\beta_i^{(1)}$, $\beta_i^{(2)}$ are a family of $2N$ i.i.d. standard real Brownian motions.

The transfer from this estimate on the (signed) eigenvalues (x_i, y_i) to the singular values (λ_i, μ_i) is elementary.

Finally, for t up to T, the same reasoning applies after changing the initial condition and time to $t_0 = t - N^{-1+\omega_t}$ and running the dynamics on $[t_0, t]$ (the local law (4.2) is satisfied at time t_0 with very high probability, see Proposition 4.13). □

4.3 Proof of Proposition 4.5

We interpolate the two processes in (4.5) and (4.6), by letting $0 \leqslant \alpha \leqslant 1$,

$$\mathrm{d}x_i(t, \alpha) = \frac{\alpha \mathrm{d}b_i^s(t) + (1-\alpha) \mathrm{d}b_i^r(t)}{\sqrt{2N}} + \frac{1}{2N} \sum_{j \neq i} \frac{1}{x_i(t, \alpha) - x_j(t, \alpha)} \mathrm{d}t, \quad 1 \leqslant |i| \leqslant N, \tag{4.13}$$

where $x_i(0, \alpha) = s_i$. We remark that for $\alpha = 1$ we have $x_i(t, \alpha) = s_i(t)$, and for $\alpha = 0$ we have $x_i(t, \alpha) = r_i(t)$.

The following lemma gives some estimates of $\{x_i(t, \alpha)\}$, which will be used repeatedly in the rest of this section. We defer its proof in Section 4.4, after the statement of the local law, Proposition 4.13.

Lemma 4.8. *Adopt the assumptions in Proposition 4.5, and fix a time* $\mathsf{T} = \varphi^{-2}$. *With very high probability there exists a small constant*[5] $c_G > 0$ *such that uniformly for* $0 \leqslant t \leqslant \mathsf{T}$, *we have*

$$c_G \frac{|i-j|}{N} \leqslant |x_i(t,\alpha) - x_j(t,\alpha)| \leqslant \frac{|i-j|}{c_G N}, \quad |i-j| \geqslant \varphi^{10}, \quad |i|,|j| \leqslant c_G N, \tag{4.14}$$

$$|x_i(t,\alpha) - x_i(0,\alpha)| \leqslant \varphi^3 \left(t + \frac{1}{N}\right), \quad |i| \leqslant c_G N. \tag{4.15}$$

If we take derivative with respect to α on both sides of (4.13), we obtain the tangential equation for $u_i(t,\alpha) = \partial_\alpha x_i(t,\alpha)$ (see [71]), which is given by

$$\begin{aligned} \mathrm{d}u_i(t,\alpha) &= \mathrm{d}\xi_i(t) - \sum_{j \neq i} c_{i,j}(t)\,(u_i(t,\alpha) - u_j(t,\alpha))\mathrm{d}t, \\ \xi_i(t) &:= \frac{\mathrm{d}b_i^s(t) - \mathrm{d}b_i^r(t)}{\sqrt{2N}}, \\ c_{ij}(t) &:= \frac{1}{2N(x_i(t,\alpha) - x_j(t,\alpha))^2}, \end{aligned} \tag{4.16}$$

where $u_i(0,\alpha) = 0$ for all $1 \leqslant |i| \leqslant N$. The key for the proof of Proposition 4.5 is the following lemma.

Lemma 4.9. *Under the assumptions from Proposition 4.5, for any* $|i| \leqslant N^{\omega}$ *and* $\alpha \in [0,1]$, *with very high probability we have* $|u_i(t_f,\alpha)| \leqslant N^{-1-\omega}$.

To prove Lemma 4.9, we define the operator $\mathcal{B} = \mathcal{B}(t)$ through

$$(\mathcal{B}v)_i = -\sum_{j \neq i} c_{ij}(t)(v_i - v_j)$$

and we first study the linear differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t} v_i(t) = (\mathcal{B}v(t))_i, \quad 1 \leqslant |i| \leqslant N. \tag{4.17}$$

We need to further introduce the long range and short range operators. Take $c < c_G/2$. Denoting $I_{\mathcal{L}} = \{(i,j) : |i-j| \geqslant \ell, \min\{|i|,|j|\} \leqslant cN\}$, and $I_{\mathcal{S}} = \{(i,j) : i \neq j\} \setminus I_{\mathcal{L}}$, let

$$(\mathcal{S}v)_i = -\sum_{(i,j)\in I_{\mathcal{S}}} c_{ij}(t)(v_i - v_j), \qquad (\mathcal{L}v)_i = -\sum_{(i,j)\in I_{\mathcal{L}}} c_{ij}(t)(v_i - v_j). \tag{4.18}$$

In this definition, we choose the dynamics cutoff parameter

$$\ell = N^{\omega_\ell}, \quad \omega_t \ll \omega_\ell < \omega_K.$$

In particular $\omega_\ell > 10\nu$, so thanks to Lemma 4.8, for any $(i,j) \in I_{\mathcal{L}}$ we have

$$|c_{ij}(t)| \leqslant \frac{N}{c_G |i-j|^2}. \tag{4.19}$$

With $\boldsymbol{\xi}$ defined in (4.16), we consider the following evolution using the short range operator $\mathcal{S}$,

$$\mathrm{d}\boldsymbol{w} = \mathcal{S}\boldsymbol{w} + \mathrm{d}\boldsymbol{\xi}, \quad \boldsymbol{w}(0) = 0. \tag{4.20}$$

Lemma 4.10. *There exists a constant* $c > 0$ *such that for any* $0 \leqslant t \leqslant 1$, *with probability* $1 - e^{-c(\log N)^2}$ *we have*

$$\|\boldsymbol{w}(t)\|_\infty \leqslant (\log N)^2 \sqrt{\frac{t}{N}}.$$

Moreover, this estimate holds for any choice of ℓ, *in particular when* $\mathcal{S}$ *is replaced by* $\mathcal{B}$ *in (4.20).*

[5] In this section the small constant c_G may vary from line to line but only depends on G from the (ν, G)-regularity assumption.

Proof. Let $q = 2p$ with $p \in \mathbb{N}_*$ and $f(t) = \mathbb{E}[\sum_{|k| \leqslant N} w_k(t)^q]$. From Itô's lemma and (4.20) we have

$$f'(t) = -\frac{q}{2} \sum_{(i,j)\in I_{\mathcal{S}}} \mathbb{E}[c_{i,j}(w_i - w_j)(w_i^{q-1} - w_j^{q-1})] + \sum_{|i|\leqslant N} \frac{q(q-1)}{2} \mathbb{E}\big[w_i(t)^{q-2} \frac{\mathrm{d}\langle \xi_i \rangle_t}{\mathrm{d}t}\big].$$

The first term on the right-hand side is non-positive, and the second one is bounded thanks to $\mathrm{d}\langle \xi_i \rangle_t / \mathrm{d}t \lesssim N^{-1}$ and repeated Hölder's inequalities:

$$\mathbb{E}\left[\sum_{|i|\leqslant N} w_i(t)^{q-2}\right] \leqslant 2N\mathbb{E}\left[\Big(\frac{1}{2N}\sum_{|i|\leqslant N} w_i(t)^q\Big)^{\frac{q-2}{q}}\right] \leqslant 2N\mathbb{E}\left[\frac{1}{2N}\sum_{|i|\leqslant N} w_i(t)^q\right]^{\frac{q-2}{q}} = (2N)^{\frac{2}{q}} f(t)^{\frac{q-2}{q}}.$$

We have obtained $f'(t) \leqslant \frac{p(2p-1)(2N)^{1/p}}{N} f(t)^{\frac{p-1}{p}}$ so with Grönwall's inequality

$$\mathbb{E}[\|\mathbf{w}(t)\|_\infty^{2p}] \leqslant f(t) \leqslant 2N \cdot (2p-1)^p \cdot \left(\frac{t}{N}\right)^p.$$

The conclusion follows easily by Markov's inequality. □

Denote the propagator for (4.17) $\mathcal{U} = \mathcal{U}(s,t,\alpha)$ $(s \leqslant t)$, i.e. $v_i(t) = \sum_j \mathcal{U}_{ij}(s,t,\alpha) v_j(s)$. The propagator preserves the sign ($\mathcal{U}_{ij} \geqslant 0$) and the mass ($\sum_j \mathcal{U}_{ij} = 1$). The difference of (4.16) and (4.20) gives $\mathrm{d}(\boldsymbol{u} - \boldsymbol{w}) = \mathcal{B}(\boldsymbol{u} - \boldsymbol{w}) + \boldsymbol{\mathcal{L}} \boldsymbol{w}$ so that, with Duhamel's formula,

$$\boldsymbol{u}(t) = \boldsymbol{w}(t) + \int_0^t \mathcal{U}(s,t,\alpha)\mathcal{L}(s)\boldsymbol{w}(s)\mathrm{d}s. \tag{4.21}$$

For any $|i| \leqslant N$, thanks to Lemma 4.10, we conclude

$$\begin{aligned}\sum_j \mathcal{U}_{ij}(s,t)(\mathcal{L}(s)\boldsymbol{w}(s))_j &\leqslant \sum_j \mathcal{U}_{ij}(s,t) \sum_{|j-k|\geqslant \ell} \frac{CN}{|j-k|^2} |w_k(s)| \leqslant \sum_j \mathcal{U}_{ij}(s,t) \sum_{|j-k|\geqslant \ell} \frac{CN}{|j-k|^2} \|\boldsymbol{w}(s)\|_\infty \\ &\leqslant \sum_j \mathcal{U}_{ij}(s,t) \frac{CN}{\ell} \|\boldsymbol{w}(s)\|_\infty \leqslant \frac{CN}{\ell}\|\boldsymbol{w}(s)\|_\infty \leqslant \frac{CN(\log N)^2\sqrt{Ns}}{\ell},\end{aligned} \tag{4.22}$$

with probability $1 - \mathrm{O}(N^{-D})$ for any fixed s, where C does not depend on s. In the first inequality we used (4.19); in the second line we bound $|w_k(s)|$ by $\|\boldsymbol{w}(s)\|_\infty$; in the last line we first sum over k, then sum over j, and also used Lemma 4.10.

Plugging (4.22) into (4.21) we obtain that with probability $1 - \mathrm{O}(N^{-D})$ we have

$$\boldsymbol{u}(t) = \boldsymbol{w}(t) + \mathrm{O}\left(\int_0^t \frac{C(\log N)^2\sqrt{Ns}}{\ell}\mathrm{d}s\right) = \boldsymbol{w}(t) + \mathrm{O}\left(\frac{(\log N)^2\sqrt{Nt^3}}{\ell}\right) = \boldsymbol{w}(t) + \mathrm{O}\left(N^{-1-\omega}\right), \tag{4.23}$$

provided that $\ell \gg (\log N)^2 N^{3\omega_t/2+\omega}$. Note that the first inequality in (4.23) holding with very high probability requires an intermediate step based on the Markov and Hölder inequalities, because the good sets where (4.22) holds may not have an intersection with large probability as s varies. We refer to [21, proof of Theorem 2.8] for this elementary step.

Lemma 4.9 now follows from combining (4.23) and the following estimates on $\boldsymbol{w}(t)$.

Lemma 4.11. *Let $\boldsymbol{w}$ be the solution of (4.20). Let $\ell = N^{\omega_\ell}$ and $\alpha \in [0,1]$. With very high probability, for any $|i| \leqslant \ell$ and $\nu, \omega \ll \omega_t \ll \omega_\ell < \omega_K \ll \tilde{\omega}$, we have*

$$|w_i(t)| \leqslant \frac{1}{N^{1+\omega}}.$$

Proof. Denote $\zeta = N/\ell$ and $\chi(x)$ be nonnegative, smooth, compactly supported, and such that $\int \chi = 1$. In the rest of this proof, we denote, $b_k(t) = \alpha b_k^s(t) + (1-\alpha) b_k^r(t)$ (from (4.13)). Define $\psi(x) = \int \min\{|x - y|, \frac{c}{2c_G}\}\zeta\chi(\zeta y)\mathrm{d}y$ (where c_G is introduced in (4.14) and c appears in the definition of $I_{\mathcal{L}}$ before (4.18)) and

$$\psi_k(s) = \psi(x_k(s,\alpha)),\ \phi_k(s) = e^{-\zeta\psi_k(s)},\ v_k(s) = \phi_k(s) w_k(s).$$

From the construction $\psi(x) = \frac{c}{2c_G}$ for $|x| \geqslant \frac{c}{c_G}$. The Itô formula gives

$$\mathrm{d}v_k = \sum_{(j,k)\in I_{\mathcal{S}}} c_{jk}\left((v_j - v_k) + \left(\frac{\phi_k}{\phi_j} - 1\right)v_j\right)\mathrm{d}s + (\mathrm{d}\phi_k)\, w_k - \zeta\psi'(x_k)\phi_k(s)\frac{\mathrm{d}\langle b_k, \xi_k\rangle_s}{\sqrt{2N}} + \phi_k \mathrm{d}\xi_k,$$

$$\frac{\mathrm{d}\phi_k}{\phi_k} = -\,\zeta\psi'(x_k)\frac{\mathrm{d}b_k(s)}{\sqrt{2N}} - \zeta\frac{\psi'(x_k)}{2N}\sum_{j\neq k}\frac{\mathrm{d}s}{x_k - x_j} + \frac{1}{2}\left(\frac{\zeta}{2N}\psi''(x_k) + \frac{\zeta^2}{2N}\psi'(x_k)^2\right)\mathrm{d}\langle b_k\rangle_s.$$

Thus if we define $X_s = \sum_{k=1}^N v_k(s)^2$, we obtain

$$\mathrm{d}X_s = \sum_k -2\zeta\psi'(x_k)v_k^2\frac{\mathrm{d}b_k(s)}{\sqrt{2N}} + 2\sum_k v_k\phi_k\frac{\mathrm{d}\xi_k(s)}{\sqrt{2N}} \tag{4.24}$$

$$- \sum_{(j,k)\in I_{\mathcal{S}}} c_{jk}(v_j - v_k)^2\mathrm{d}s \tag{4.25}$$

$$+ \sum_{(j,k)\in I_{\mathcal{S}}} c_{jk}\left(\frac{\phi_k}{\phi_j} + \frac{\phi_j}{\phi_k} - 2\right)v_j v_k \mathrm{d}s \tag{4.26}$$

$$+ \frac{\zeta}{2N}\sum_k \psi''(x_k)v_k^2\mathrm{d}\langle b_k\rangle_s \tag{4.27}$$

$$+ \frac{\zeta^2}{2N}\sum_k \psi'(x_k)^2 v_k^2\mathrm{d}\langle b_k\rangle_s \tag{4.28}$$

$$- \frac{\zeta}{N}\sum_{j<k}\frac{\psi'(x_j)v_j^2 - \psi'(x_k)v_k^2}{x_j - x_k}\mathrm{d}s \tag{4.29}$$

$$- \sum_k 2\zeta\psi'(x_k)\phi_k v_k\frac{\mathrm{d}\langle b_k, \xi_k\rangle_s}{\sqrt{2N}} + \sum_k \mathrm{d}\left\langle \phi_k\xi_k - \zeta\psi'(x_k)v_k\frac{b_k}{\sqrt{2N}}\right\rangle_s \tag{4.30}$$

$$+ \sum_k \phi_k^2\mathrm{d}\langle\xi_k, \xi_k\rangle_s. \tag{4.31}$$

Consider

$$\tau = t \wedge \inf\{s : (4.14) \text{ fails}\} \wedge \inf\left\{s \geqslant 0 : X_s > N^{-2-2\omega}\right\}. \tag{4.32}$$

For $(j,k) \in I_{\mathcal{S}}$ there are two cases: either $\min\{|j|, |k|\} \leqslant cN, |j-k| \leqslant \ell$ or $\min\{|j|, |k|\} \geqslant cN$. From $\|\psi'\|_\infty \leqslant 1$ and the assumption $\zeta = N/\ell$, in the first case we have $\zeta|\psi(x_k) - \psi(x_j)| \leqslant \zeta|x_k - x_j| \leqslant \frac{|k-j|}{c_G\ell} = \mathrm{O}(1)$, where we have used that for $t \leqslant \tau$ from (4.14) we have $|x_k(t) - x_j(t)| \leqslant |k-j|/(c_G N)$ when $|k|, |j| \leqslant c_G N$, and we choose c small enough compared to c_G (say $c = c_G/2$). This implies that $\left|\frac{\phi_k}{\phi_j} + \frac{\phi_j}{\phi_k} - 2\right| = \mathrm{O}(1)\zeta^2|x_k - x_j|^2$. In the second case, $|x_k|, |x_j| \geqslant c/c_G$, and $\zeta|\psi(x_k) - \psi(x_j)| = 0$. One concludes easily that (4.26) is $\mathrm{O}(1)\zeta^2(\ell/N)X_s$. The terms (4.27) and (4.28) are of smaller order by $\|\psi'\|_\infty \leqslant 1$, $\|\psi''\|_\infty \leqslant \zeta$ and $\frac{\mathrm{d}}{\mathrm{d}t}\langle b_k\rangle \leqslant 1$.

For $s \leqslant \tau$, in (4.29) the sums of $\max\{|j|, |k|\} \geqslant cN$ (say $|k| \geqslant cN$) are bounded as

$$2\frac{\zeta}{N}\sum_{|j|\leqslant N^{1-2\nu}, |k|\geqslant N^{1-\nu}}\frac{\psi'(x_j)v_j^2}{x_j - x_k} \leqslant \zeta\sum_j v_j^2 = \zeta X_s \leqslant \frac{\xi}{N^2} = \frac{1}{\ell N}.$$

The sums of $|j|, |k| \leqslant cN$ is of order at most

$$\frac{\zeta}{N}\sum_{(j,k)\in I_{\mathcal{L}}}\frac{v_k^2}{|x_j - x_k|} + \frac{\zeta}{N}\sum_{(j,k)\in I_{\mathcal{S}}}|\psi'(x_j)|\frac{|v_j^2 - v_k^2|}{|x_j - x_k|} + \frac{\zeta}{N}\sum_{(j,k)\in I_{\mathcal{S}}}\|\psi''\|_\infty v_k^2.$$

For $s \leqslant \tau$, the first sum above has order $\zeta(\log N)X_s \leqslant \zeta(\log N)/N^2 = \log N/(N\ell)$. The third sum is at most $(\xi^2\ell/N)X_s \leqslant 1/(N\ell)$. Finally, the second sum is bounded using

$$2\frac{|v_j^2 - v_k^2|}{|x_j - x_k|} \leqslant M^{-1}(v_j + v_k)^2 + M\frac{(v_j - v_k)^2}{(x_j - x_k)^2}.$$

Choosing $M = \varepsilon\zeta^{-1}$ for ε small enough, this proves that (4.29) can be absorbed into the dissipative term (4.25) plus an error of order $(\zeta^2\ell/N)X_s = 1/(N\ell)$.

In (4.30) and (4.31), using $\|\psi'\|_\infty \leqslant 1$, we can bound them by (up to a constant)

$$\sum_k \frac{\zeta\phi_k v_k}{2N} + \frac{\zeta^2 v_k^2}{2N} + \phi_k^2 \frac{\mathrm{d}\langle \xi_k, \xi_k\rangle_s}{\mathrm{d}s} = \sum_k \left(\frac{\zeta\phi_k v_k}{2N} + \phi_k^2 \frac{\mathrm{d}\langle \xi_k, \xi_k\rangle_s}{\mathrm{d}s}\right) + \frac{\zeta^2 X_s}{2N}.$$

Let $K' = \min\{K, \ell^{3/2}\} = N^{\min\{\omega_K, 3\omega_\ell/2\}} \leqslant K$. For $s \leqslant \tau$, if $|k| \geqslant K'$, then $\psi(x_k(s)) \geqslant K'/N = (K'/\ell)\zeta^{-1}$, so that $\phi_k(s) \leqslant e^{-K'/\ell}$ is negligible when $\omega_\ell < \omega_K$. For $|k| \leqslant K'$, we simply bound $|\phi_k| \leqslant 1$, then

$$\begin{aligned}
\sum_k \left(\frac{\zeta\phi_k v_k}{2N} + \phi_k^2 \frac{\mathrm{d}\langle \xi_k, \xi_k\rangle_s}{\mathrm{d}s}\right) &= \sum_{|k|\leqslant K'} \left(\frac{\zeta\phi_k v_k}{2N} + \phi_k^2 \frac{\mathrm{d}\langle \xi_k, \xi_k\rangle_s}{\mathrm{d}s}\right) + \sum_{|k|\geqslant K'} \left(\frac{\zeta\phi_k v_k}{2N} + \phi_k^2 \frac{\mathrm{d}\langle \xi_k, \xi_k\rangle_s}{\mathrm{d}s}\right) \\
&= \sum_{|k|\leqslant K'} \left(\frac{\zeta v_k}{2N} + \frac{\mathrm{d}\langle \xi_k, \xi_k\rangle_s}{\mathrm{d}s}\right) + e^{-K'/\ell} \sum_k \left(\frac{\zeta v_k}{2N} + \frac{\mathrm{d}\langle \xi_k, \xi_k\rangle_s}{\mathrm{d}s}\right) \\
&\lesssim \frac{\zeta K'}{N}\sqrt{\sum_{|k|\leqslant K'} \frac{v_k^2}{2K'}} + \sum_{|k|\leqslant K'} \frac{N^{-\widetilde{\omega}}}{2N} + e^{-K'/\ell}\left(\zeta\sqrt{\frac{\sum_k v_k^2}{N}} + \sum_k \frac{1}{N}\right) \\
&\lesssim \frac{1}{N^{1+\omega_\ell/4}} + \frac{1}{N^{1+\omega_K - \widetilde{\omega}}}
\end{aligned}$$

where in the third line we used that for $|k| \leqslant K$, $\mathrm{d}\langle \xi_k, \xi_k\rangle_s/\mathrm{d}s \lesssim N^{-\widetilde{\omega}}/N$; in the last line we used that for $s \leqslant \tau$, $\sum_k v_k^2(s) = X_s \leqslant 1/N^2$, and $\sqrt{K'}/\ell \leqslant N^{-\omega_\ell/4}$.

We now bound the relevant martingales in (4.24) in terms of the quadratic variation. From [86, Appendix B.6, Equation (18)] with $c = 0$ allowed for continuous martingales, for any continuous martingale M and any $\lambda, \mu > 0$, we have $\mathbb{P}\Big(\sup_{0\leqslant u\leqslant t} |M_u| \geqslant \lambda,\ \langle M\rangle_t \leqslant \mu\Big) \leqslant 2e^{-\frac{\lambda^2}{2\mu}}$. This implies that there exists a c such that for any $N, \varepsilon > 0$ we have

$$\mathbb{P}\left(\sup_{0\leqslant u\leqslant t} |M_u| \geqslant \varphi^\varepsilon \langle M\rangle_t^{1/2}\right) \leqslant c^{-1} e^{-c\varphi^{2\varepsilon}}. \tag{4.33}$$

The stochastic terms in (4.24) have quadratic variations bounded as follows. First, for $s \leqslant \tau$

$$\frac{\mathrm{d}}{\mathrm{d}s}\left\langle \sum_k -\xi\psi'(x_k) v_k^2 \frac{b_k(s)}{\sqrt{2N}}\right\rangle \lesssim \frac{\xi^2}{N}\left(\sum_k v_k^2\right)^2 = \frac{\xi^2 X_s^2}{N} \lesssim \frac{1}{\ell^2 N^{3+4\omega}},$$

Again, for $s \leqslant \tau$ and $k \geqslant K$, $\phi_k(s) \leqslant e^{-K/\ell}$ is negligible. This implies

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s}\left\langle \sum_k v_k\phi_k \frac{\xi_k}{\sqrt{2N}}\right\rangle &\leqslant \sum_{k,j\leqslant K} v_k v_j \phi_k\phi_j \frac{N^{-\tilde{\omega}}}{2N} + \sum_{k\geqslant K \text{ or } j\geqslant K} v_k v_j \phi_k \phi_j \frac{C}{2N} \\
&\leqslant \frac{1}{N^{1+\tilde{\omega}-\omega_K}} \sum_k v_k^2 + e^{-K/\ell}\sum_k v_k^2 \lesssim \frac{1}{N^{1+\tilde{\omega}-\omega_K}} X_s \leqslant \frac{1}{N^{3+\tilde{\omega}+2\omega-\omega_K}}.
\end{aligned}$$

Therefore with very high probability,

$$\sup_{t\leqslant\tau}\left| \int_0^s \sum_k -\xi\psi_k'(u) v_k^2(u) \frac{\mathrm{d}b_k(u)}{\sqrt{2N}}\right| \leqslant \frac{\sqrt{t}}{\ell N^{3/2}}, \quad \sup_{t\leqslant\tau}\left| \int_0^s \sum_k v_k\psi_k \frac{\mathrm{d}\xi_k}{\sqrt{2N}}\right| \leqslant \frac{\sqrt{t}}{N^{3/2} N^{(\tilde{\omega}-\omega_K)/2}}.$$

We have thus proved that with very high probability, for any $0 \leqslant s \leqslant \tau$,

$$\begin{aligned}
X_s &\leqslant C\left(\frac{\log Nt}{N\ell} + \frac{\sqrt{t}}{\ell N^{3/2}} + \frac{\sqrt{t}}{N^{3/2} N^{(\tilde{\omega}-\omega_K)/2}} + \frac{t}{N^{1+\omega_\ell/4}} + \frac{t}{N^{1+\omega_K-\widetilde{\omega}}}\right) \\
&\leqslant \frac{C}{N^2}\left(\frac{1}{N^{\omega_\ell-\omega_t}} + \frac{1}{N^{\omega_\ell-\omega_t/2}} + \frac{1}{N^{(\tilde{\omega}-\omega_K-\omega_t)/2}} + \frac{1}{N^{\omega_\ell/4-\omega_t}} + \frac{1}{N^{\omega_K-\widetilde{\omega}-\omega_t}}\right) \ll \frac{1}{N^{2+2\omega}},
\end{aligned}$$

where we used $\omega \ll \omega_t \ll \omega_\ell < \omega_K \ll \tilde{\omega}$. We conclude that with very high probability $\tau = t$, and $X_t \ll N^{-2-2\omega}$. Thanks to the choice of the stopping time (4.32), $|x_i(t,\alpha) - x_j(t,\alpha)| \leqslant |i-j|\varphi/N$. Thus for any $|i| \leqslant \ell$, $|x_i(t,\alpha)| \lesssim \ell/N$, and $\phi_i(t) \asymp 1$. We conclude that $N^{-2-2\omega} \geqslant X_t \gtrsim w_i(t)^2$, so that $|w_i(t)| \leqslant N^{-1-\omega}$ with very high probability. □

Proof of Proposition 4.5. Proposition 4.5 follows from Lemma 4.9 by integrating from $\alpha = 0$ to $\alpha = 1$:

$$|s_i(t_f) - r_i(t_f)| = |x_i(t_f,1) - x_i(t_f,0)| = \left|\int_0^1 \partial_\alpha x_i(t_f,\alpha)\mathrm{d}\alpha\right| \leqslant \int_0^1 |u_i(t_f,\alpha)|\mathrm{d}\alpha \leqslant N^{-1-\omega}. \tag{4.34}$$

Note that the second inequality holds with very high probability, and requires intermediate steps based on the Markov and Hölder inequalities, similarly to (4.23). □

4.4 ***Local Law.*** In this section, we study a modified version of Dyson Brownian motion where the driving martingales can be essentially arbitrary.

$$\mathrm{d}x_i(t) = \frac{\mathrm{d}B_i}{\sqrt{2N}} + \frac{1}{2N}\sum_{j\neq i}\frac{1}{x_i(t) - x_j(t)}\mathrm{d}t, \quad 1 \leqslant |i| \leqslant N. \tag{4.35}$$

Assumption 4.12. *We assume the martingales satisfy*

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\langle B_i\rangle_t \leqslant 1.$$

Under Assumption 4.12, the existence and strong uniqueness hold for the above stochastic differential equation, see Proposition B.1. We denote the empirical eigenvalue density and its Stieltjes transform as,

$$\mu_t = \frac{1}{2N}\sum_{1\leqslant|i|\leqslant N}\delta_{x_i(t)}, \qquad m_t(z) = \frac{1}{2N}\sum_i \frac{1}{x_i(t) - z}.$$

We assume that the initial data μ_0 is (ν, G)-regular around $\widetilde{\mu}_0$ as in Assumption 4.3.

In the following, we first recall some notations and concepts about the free convolution with the semicircle distribution. The *semicircle distribution* is a measure $\mu_{\mathrm{sc}} \in \mathscr{P}(\mathbb{R})$ whose density $\varrho_{\mathrm{sc}} : \mathbb{R} \to \mathbb{R}_{\geqslant 0}$ with respect to the Lebesgue measure is given by

$$\rho_{\mathrm{sc}}(x) = \frac{(4-x^2)^{1/2}}{2\pi}\cdot \mathbf{1}_{x\in[-2,2]}, \qquad \text{for all } x \in \mathbb{R}.$$

For any real number $t > 0$, we denote the rescaled semicircle probability distribution

$$\mu_{\mathrm{sc}}^{(t)} = t^{-1/2}\rho_{\mathrm{sc}}(t^{-1/2}x)\mathrm{d}x.$$

Let $\widetilde{\mu}_t = \widetilde{\mu}_0 \boxplus \mu_{\mathrm{sc}}^{(t)}$ denote the free convolution between $\widetilde{\mu}_0$ with the rescaled semicircle distribution. By [13, Corollary 2], $\widetilde{\mu}_t$ has a density $\widetilde{\rho}_t : \mathbb{R} \to \mathbb{R}_{\geqslant 0}$ with respect to Lebesgue measure for $t > 0$. Following [13], the Stieltjes transform $\widetilde{m}_t$ of $\widetilde{\mu}_t$ is characterized in the following way. For any $t > 0$, denote the function $\Phi = \Phi^t : \mathbb{H} \to \mathbb{C}$ and the set $\Lambda_t \subseteq \mathbb{H}$ by (see Figure 1)

$$\Phi(z) = z - t\widetilde{m}_0(z); \quad \Lambda_t = \Big\{z \in \mathbb{H} : \mathrm{Im}\big(z - t\widetilde{m}_0(z)\big) > 0\Big\} = \Big\{z \in \mathbb{H} : \int_{-\infty}^{\infty}\frac{\widetilde{\mu}_0(dx)}{|z-x|^2} < \frac{1}{t}\Big\}. \tag{4.36}$$

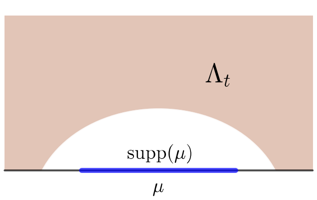

Figure 1: Λ_t as defined in (4.36) is an open subset of the upper half plane $\mathbb{H}$. $\Phi(z)$ is a holomorphic map from Λ_t to $\mathbb{H}$.

From [13, Lemma 4], the function Φ is a homeomorphism from $\overline{\Lambda}_t$ to $\overline{\mathbb{H}}$. Moreover, it is a holomorphic map from Λ_t to $\mathbb{H}$ and a bijection from $\partial\Lambda_t$ to $\mathbb{R}$. For any real number $t \geqslant 0$, $\widetilde{m}_t =: \mathbb{H} \to \mathbb{H}$ is characterized as

$$\widetilde{m}_t\big(u - t\widetilde{m}_0(u)\big) = \widetilde{m}_0(u), \qquad \text{for any } u \in \Lambda_t.$$

In the following we denote the characteristic flow

$$z_t(u) = u - t\widetilde{m}_0(u), \quad \text{for any } u \in \Lambda_t, \quad \widetilde{m}_t(z_t(u)) = \widetilde{m}_0(u). \tag{4.37}$$

If the context is clear, we will simply write $z_t(u)$ as z_t. Note that z satisfies the characteristics equation

$$\partial_t z_t = -\widetilde{m}_t(z_t). \tag{4.38}$$

Lemma 4.8 is an easy consequence of the following proposition.

Proposition 4.13. *Let $\mathsf{T} = \varphi^{-2}$ and recall the definition of φ in (4.1). Assume that μ_0 is (ν, G) regular. Then, we have the following local law with high probability for any $t \in [0, \mathsf{T}]$*

$$|m_t(w) - \widetilde{m}_t(w)| \leqslant \varphi \sqrt{\frac{\mathrm{Im}[\widetilde{m}_0(w)]}{N\mathrm{Im}[w]}}, \quad w \in \{z \in \mathbb{H} : |z| \leqslant G/2, \mathrm{Im}[z] \geqslant \varphi^4 N^{-1+\nu}\}.$$

Proof of Lemma 4.8. The first statement in Lemma 4.8 follows from the lower and upper bound of $m_t(w)$, which follows from Proposition 4.13. We omit its proof and refer to [50, Theorem 1.1] for the proof.

To prove (4.15), we interpolate (4.35) with the standard Brownian motion, for which we know the optimal rigidity, i.e. with very high probability the i-th particle is close to the classical locations with error φ/N. Thanks to Lemma 4.10, the interpolation induces an error bounded by $\varphi\sqrt{t/N}$. It follows that

$$|x_i(t) - \gamma_i(t)| \lesssim \varphi \left(\frac{1}{N} + \sqrt{\frac{t}{N}} + t \right) \lesssim \varphi \left(\frac{1}{N} + t \right),$$

and (4.15) follows, by noticing that the classical locations $\gamma_i(t)$ moves with speed $\mathrm{O}(t)$. □

Recall we denote $\varphi = N^\nu$. For any real number $t \geqslant 0$, we define the spectral domain $\mathcal{D}_t = \mathcal{D}_t(\widetilde{\mu}_t)$ by

$$\mathcal{D}_t := \left\{ z \in \mathbb{H} : |z| \leqslant G - Ct, \mathrm{Im}[z]\,\mathrm{Im}[\widetilde{m}_t(z)] \geqslant \varphi^4/N \right\} \tag{4.39}$$

where the constant C is defined in Equation (4.8). The following lemma collects some properties of the domain $\mathcal{D}_t$.

Lemma 4.14. *Recall the domain $\mathcal{D}_t$ from* (4.39)*, and $\mathsf{T} = \varphi^{-2}$. For N large enough and any time $0 \leqslant t \leqslant \mathsf{T}$, the following holds.*

(i) $\mathcal{D}_t$ is non-empty. For any $z = E_t + \mathrm{i}\eta \in \mathcal{D}_t$, we have

$$\mathrm{Im}[\widetilde{m}_t(z)] \geqslant \varphi^2 \sqrt{\frac{\mathrm{Im}[\widetilde{m}_t(z)]}{N\eta}}. \tag{4.40}$$

(ii) If $z_t(u) = E_t + \mathrm{i}\eta_t \in \mathcal{D}_t$, then $z_s(u) \in \mathcal{D}_s$ for any $0 \leqslant s \leqslant t$.

Proof. From our assumption for $|u| \leqslant G$, $|\widetilde{m}_0(u)| \leqslant C$, and $|z_t(u) - u| = t|\widetilde{m}_0(u)|$. Thus $z_t(u)$ maps $\{u : |u| \leqslant G\}$ surjectively to $\{z : |z| \leqslant G - Ct\}$. We also conclude that for $z = z_t(u)$ with $|z| \leqslant G - Ct$, it holds $\widetilde{m}_t(z) = \widetilde{m}_0(u)$, and (4.40) holds from the definition of $\mathcal{D}_t$. Moreover, for any $z \in \mathcal{D}_t$, we have $\mathrm{Im}[\widetilde{m}_t(z)] \geqslant 1/C$. It is easy to see that $\mathcal{D}_t$ is not empty, and in fact $\{z \in \mathbb{H} : |z| \leqslant G/2, \mathrm{Im}[z] \geqslant C\varphi^4 N^{-1}\} \subset \mathcal{D}_t$. The statement (4.40) follows from the definition of $\mathcal{D}_t$ from(4.39).

For (ii) since $\eta_s \geqslant \eta_t$ from (4.37), if $\eta_t \geqslant \varphi^4/(N\mathrm{Im}[\widetilde{m}_t(z_t)])$, then $\eta_s \geqslant \eta_t \geqslant \varphi^4/(N\mathrm{Im}[\widetilde{m}_t(z_t)]) = \varphi^4/(N\mathrm{Im}[\widetilde{m}_s(z_s)])$. Thus, $z_s(u)$ satisfies the lower bounds required for $z_s(u)$ to be in $\mathcal{D}_s$. □

By the second part of Lemma 4.14, if for any $u = v(u) \in \mathcal{D}_0$ we define

$$\mathfrak{t}(u) := \mathsf{T} \wedge \sup\{t \geqslant 0 : z_t(u) \in \mathcal{D}_t\}, \tag{4.41}$$

then $z_s(u) \in \mathcal{D}_s$ for any $0 \leqslant s \leqslant \mathfrak{t}(u)$. We also define the lattice on the domain $\mathcal{D}_0$ given by

$$\mathcal{L} = \left\{ E + \mathrm{i}\eta \in \mathcal{D}_0 : E \in \mathbb{Z}/N^6, \eta \in \mathbb{Z}/N^6 \right\}. \tag{4.42}$$

Lemma 4.15. *Adopt the assumptions in Proposition 4.13. For any $t \in [0, \mathsf{T}]$ and $w \in \mathcal{D}_t$, there exists some lattice point $u \in \mathcal{L} \cap z_t^{-1}(\mathcal{D}_t)$, such that $|z_t(u) - w| \leqslant N^{-5}$, provided N is large enough.*

Proof. It follows from Lemma 4.14 that if $z_t(u) \in \mathcal{D}_t$, then $u \in \mathcal{D}_0$, and $z_t(u) = u - tm_0(u)$ with $u \in \mathcal{D}_0$. In particular $|\partial_u z_t(u)| = |1 - tm_0'(u)| \leqslant 1 + C\mathsf{T}$. Thus $z_t(u)$ is Lipschitz in u with Lipschitz constant bounded by $(1 + C\mathsf{T})$. Thus for any $w \in \mathcal{D}_t$, there exists some lattice point $u \in \mathcal{L} \cap z_t^{-1}(\mathcal{D}_t)$, such that $|z_t(u) - w| \leqslant (1 + C\mathsf{T})/N^6 \leqslant N^{-5}$, provided N is large enough. □

In the rest of this section, we prove Proposition 4.13 by studying the stochastic differential equation satisfied by $m_s(z)$.

Proof of Proposition 4.13. From Lemma 4.19 in the next subsection applied to the special case $v_k \equiv \frac{1}{N}$, we have

$$\mathrm{d}m_s(z) = m_s(z)\partial_z m_s \mathrm{d}s + \frac{1}{4N^2}\sum_{|k|\leqslant N}\frac{\mathrm{d}\langle B_k\rangle_s - \mathrm{d}s}{(x_k(s) - z)^3} - \frac{1}{2\sqrt{2}N^{3/2}}\sum_{|k|\leqslant N}\frac{\mathrm{d}B_k(s)}{(z - x_k(s))^2}. \tag{4.43}$$

By plugging the characteristic flow (4.38) in (4.43), we get

$$\mathrm{d}m_s(z_s) = -\frac{1}{2\sqrt{2}N^{3/2}}\sum_i \frac{\mathrm{d}B_i(s)}{(x_i(s) - z_s)^2} + (m_s(z_s) - \widetilde{m}_s(z_s))\partial_z m_s(z_s)\mathrm{d}s + \frac{1}{N^2}\sum_i \frac{\mathrm{O}(1)\mathrm{d}s}{(x_i(s) - z_s)^3}. \tag{4.44}$$

To analyze (4.44), we introduce a stopping time

$$\sigma = \mathsf{T} \wedge \inf_{s\geqslant 0}\left\{s : \exists z \in \mathcal{D}_s, |m_s(z) - \widetilde{m}_s(z)| \geqslant \varphi\sqrt{\frac{\mathrm{Im}[\widetilde{m}_s(z_s)]}{N\eta_s}}\right\}.$$

Then for $s \leqslant \sigma$, thanks to (4.40), we have that

$$\mathrm{Im}[m_s(z_s)] \asymp \mathrm{Im}[\widetilde{m}_s(z_s)]. \tag{4.45}$$

Proposition 4.16. *There exists an event Ω, measurable with respect to the paths $\{B_1(s), B_2(s), \cdots, B_N(s)\}_{0\leqslant s\leqslant \mathsf{T}}$, such that $\mathbb{P}[\Omega] \geqslant 1 - Ce^{-(\log N)^2}$ and the following holds. On Ω, for any $u \in \mathcal{L}$ (recall this lattice from (4.42)), denote $z_s(u) = E_s(u) + \mathrm{i}\eta_s(u)$. Then, for any $0 \leqslant s \leqslant \mathfrak{t}(u)$ (recall (4.41)), we have*

$$\int_0^{s\wedge\sigma}\frac{1}{N^{3/2}}\sum_{i=1}^N \frac{\mathrm{d}B_i(\tau)}{|z_\tau(u) - x_i(\tau)|^2} \leqslant \varphi^{1/20}\sqrt{\frac{\mathrm{Im}[\widetilde{m}_{s\wedge\sigma}(z_{s\wedge\sigma})]}{N\eta_{s\wedge\sigma}}}, \tag{4.46}$$

$$\int_0^{s\wedge\sigma}\frac{1}{N^2}\sum_{i=1}^N \frac{1}{|x_i(\tau) - z_\tau(u)|^3}\mathrm{d}\tau \lesssim \frac{1}{N\eta_{s\wedge\sigma}(u)}. \tag{4.47}$$

Proof. For simplicity of notation, we write $\mathfrak{t}(u), E_\tau(u), \eta_\tau(u)$ as $\mathfrak{t}, z_\tau, E_\tau, \eta_\tau$, respectively.

To prove (4.46), we notice by Lemma 4.14 that $z_\tau(u) \in \mathcal{D}_\tau$ for any $0 \leqslant \tau \leqslant \mathfrak{t}$. We define a series of stopping times $0 = t^{(0)} < t^{(1)} < t^{(2)} < \cdots < t^{(m)} = \mathfrak{t}$, as follows:

$$t^{(k)} = \mathfrak{t} \wedge \inf\{\tau > t^{(k-1)} : \eta_\tau < \eta_{t^{(k-1)}}/2\}, \quad k = 1, 2, 3, \cdots, m, \tag{4.48}$$

where $m = m(u)$ might depend on u. From (4.40), η_t cannot be smaller than $1/N^2$, and so any $u \in \mathcal{L}$ must satisfy $m = m(u) \leqslant 10\log N$.

Recall from Assumption 4.12 that $\langle \mathrm{d}B_i, \mathrm{d}B_j\rangle \leqslant 1$. To bound the quadratic variation of the left side of (4.46), for any $s \leqslant t^{(k)} \wedge \sigma$ we have

$$\frac{1}{N^2}\int_0^s \sum_{ij}\frac{\mathrm{d}\tau}{|x_i(\tau) - z_\tau|^2|x_j(\tau) - z_\tau|^2} = \int_0^s \frac{\mathrm{Im}[m_\tau(z_\tau)]^2}{\eta_\tau^2}\mathrm{d}\tau \leqslant \int_0^s \frac{\mathrm{Im}[\widetilde{m}_\tau(z_\tau)]^2}{\eta_\tau^2}\mathrm{d}\tau \lesssim \frac{\mathrm{Im}[\widetilde{m}_s(z_s)]}{\eta_s} \tag{4.49}$$

where we successively used (4.45) and $-\mathrm{Im}[\widetilde{m}_\tau(z_\tau)] = \partial_\tau \eta_\tau$. With (4.33) and (4.49) have proved that for some $c_1 > 0$, with probability $1 - c_1^{-1}e^{-c_1\varphi^{1/10}}$ we have

$$\sup_{0\leqslant\tau\leqslant t^{(k)}}\left|\int_0^{\tau\wedge\sigma}\frac{1}{N^{3/2}}\sum_{i=1}^N \frac{\mathrm{d}B_i(\tau)}{|z_\tau - x_i(\tau)|^2}\right| \leqslant \varphi^{1/20}\sqrt{\frac{\mathrm{Im}[\widetilde{m}_{t^{(k)}\wedge\sigma}(z_{t^{(k)}\wedge\sigma})]}{N\eta_{t^{(k)}\wedge\sigma}}}. \tag{4.50}$$

We define Ω to be the event on which (4.50) holds for all $0 \leqslant k \leqslant m$ and all $u \in \mathcal{L}$. Since $m|\mathcal{L}| \leqslant |\mathcal{L}| \cdot 10 \log N \leqslant N^{20}$, it follows from the discussion above that Ω holds with probability $1 - c_2^{-1} e^{-c_2 \varphi^{1/10}}$ for some $c_2 > 0$. Therefore, for any $s \in [t^{(k-1)}, t^{(k)}]$, the bounds (4.50) and our choice of $t^{(k)}$ (4.48) (with the fact that η_s is non-increasing in s) yield on Ω that, for any $0 \leqslant s \leqslant \mathfrak{t}(u)$, we have

$$\left| \int_0^{s\wedge\sigma} \frac{1}{N^{3/2}} \sum_{i=1}^N \frac{\mathrm{d}B_i(\tau)}{|z_\tau - x_i(\tau)|^2} \right| \leqslant \varphi^{1/20} \sqrt{\frac{\operatorname{Im}[\widetilde{m}_{t^{(k)}\wedge\sigma}(z_{t^{(k)}\wedge\sigma})]}{N\eta_{t^{(k)}\wedge\sigma}}} \leqslant \varphi^{1/20} \sqrt{\frac{\operatorname{Im}[\widetilde{m}_{s\wedge\sigma}(z_{s\wedge\sigma})]}{N\eta_{s\wedge\sigma}}}.$$

This finishes the proof of (4.46).

The error terms in (4.47) can be bounded as

$$\int_0^{t\wedge\sigma} \frac{\mathrm{O}(1)}{N^2} \sum_{i=1}^N \frac{\mathrm{d}s}{|x_i(s) - z_s|^3} \leqslant \int_0^{t\wedge\sigma} \frac{\mathrm{O}(1)}{N} \frac{\operatorname{Im}[m_s(z_s)]\mathrm{d}s}{\eta_s^2} \leqslant \int_0^{t\wedge\sigma} \frac{\mathrm{O}(1)}{N} \frac{\operatorname{Im}[\widetilde{m}_s(z_s)]\mathrm{d}s}{\eta_s^2} \lesssim \frac{1}{N\eta_{t\wedge\sigma}}.$$

The first inequality relies on $|x_i(s) - z_s| \geqslant \eta_s$, and last inequality uses $-\operatorname{Im}[\widetilde{m}_s(z_s)] = \partial_s \eta_s$, see (4.38). □

From the previous proposition, abbreviating $\Delta_s = m_s(z_s) - \widetilde{m}_s(z_s)$, for any $s \leqslant t \wedge \sigma$ we can rewrite Equation (4.44) as

$$\Delta_s = \Delta_0 + \int_0^s \Delta_v \partial_z m_v(z_v)\mathrm{d}v + \mathrm{O}\left(\varphi^{1/20} \sqrt{\frac{\operatorname{Im}[\widetilde{m}_s(z_s)]}{N\eta_s}} \right). \tag{4.51}$$

Thanks to Lemma A.2 (by taking $\phi = \varphi\sqrt{\operatorname{Im}[\widetilde{m}_v(z_v)/N\eta_v]}$), we have

$$|\partial_z m_v(z_v(u))| \leqslant \partial_z \widetilde{m}_v(z_v(u)) + \frac{1}{\eta_v} \sqrt{\varphi \operatorname{Im}[\widetilde{m}_v(z_v)] \sqrt{\frac{\operatorname{Im}[\widetilde{m}_v(z_v)]}{N\eta_v}}} = \partial_z \widetilde{m}_v(z_v(u)) + \varphi^{\frac{1}{2}} \frac{\operatorname{Im}[\widetilde{m}_v(z_v)]^{3/4}}{\eta_v (N\eta_v)^{1/4}}. \tag{4.52}$$

With (4.37), we bound the first term on the right-hand side above with $|\partial_z \widetilde{m}_v(z_v(u))| = \left| \frac{\partial_u \widetilde{m}_v(z_v(u))}{\partial_u z_v(u)} \right| = \left| \frac{\partial_u \widetilde{m}_0(u)}{1 - s\partial_u \widetilde{m}_0(u)} \right| \leqslant 2C$, so that (4.52) gives

$$|\partial_z m_v(z_v(u))| \leqslant \varphi^{\frac{1}{2}} \left(1 + \frac{\operatorname{Im}[\widetilde{m}_v(z_v)]}{\eta_v (N\eta_v)^{1/4}} \right). \tag{4.53}$$

Denoting $\beta_s := \varphi^{\frac{1}{2}} \left(1 + \frac{\operatorname{Im}[\widetilde{m}_s(z_s)]}{\eta_s (N\eta_s)^{1/4}} \right)$, (4.51) and (4.53) imply, for any $s \leqslant t \wedge \sigma$

$$|\Delta_s| \leqslant \int_0^s \beta_v |\Delta_v| \,\mathrm{d}v + \mathrm{O}\left(\varphi^{1/20} \sqrt{\frac{\operatorname{Im}[\widetilde{m}_s(z_s)]}{N\eta_{t\wedge\sigma}}} \right).$$

By Grönwall's inequality, this implies the estimate

$$|\Delta_{t\wedge\sigma}| \leqslant \mathrm{O}\left(\varphi^{1/20} \sqrt{\frac{\operatorname{Im}[\widetilde{m}_{t\wedge\sigma}(z_{t\wedge\sigma})]}{N\eta_{t\wedge\sigma}}} \right) + \varphi^{1/20} \int_0^{t\wedge\sigma} \beta_s \sqrt{\frac{\operatorname{Im}[\widetilde{m}_s(z_s)]}{N\eta_s}} e^{\int_s^{t\wedge\sigma} \beta_\tau \mathrm{d}\tau} \mathrm{d}s. \tag{4.54}$$

For the integral of β_τ, we have

$$\int_s^{t\wedge\sigma} \beta_\tau \mathrm{d}\tau \leqslant \varphi^{\frac{1}{2}} \left((t \wedge \sigma - s) + \frac{1}{(N\eta_{t\wedge\sigma})^{1/4}} \right) \leqslant C.$$

The last term in (4.54) is therefore bounded with

$$\varphi^{\frac{1}{2}+\frac{1}{20}} \int_0^{t\wedge\sigma} \left(1 + \frac{\operatorname{Im}[\widetilde{m}_s(z_s)]}{\eta_s (N\eta_s)^{1/4}} \right) \sqrt{\frac{\operatorname{Im}[\widetilde{m}_s(z_s)]}{N\eta_s}} \mathrm{d}s \lesssim \varphi^{\frac{1}{2}+\frac{1}{20}} \sqrt{\frac{\operatorname{Im}[\widetilde{m}_{t\wedge\sigma}(z_{t\wedge\sigma})]}{N\eta_{t\wedge\sigma}}}. \tag{4.55}$$

It follows by combining (4.54) and (4.55) that

$$|\tilde{m}_{t\wedge\sigma}(z_{t\wedge\sigma}(u)) - m_{t\wedge\sigma}(z_{t\wedge\sigma}(u))| \lesssim \varphi^{\frac{1}{2}+\frac{1}{20}} \sqrt{\frac{\operatorname{Im}[\widetilde{m}_{t\wedge\sigma}(z_{t\wedge\sigma})]}{N\eta_{t\wedge\sigma}}} \ll \varphi \sqrt{\frac{\operatorname{Im}[\widetilde{m}_{t\wedge\sigma}(z_{t\wedge\sigma})]}{N\eta_{t\wedge\sigma}}}.$$

We conclude that with very high probability $t \wedge \sigma = \mathsf{T}$, and thus Proposition 4.13 follows. □

4.5 ***Hard edge universality.*** Before proving Proposition 4.6, we explain that the obtained result is optimal, up to subpolynomial (in N) errors. Without loss of generality we assume that $\widetilde{\rho}_t(0)=\widetilde{\rho}'_t(0)$, and apply the local law estimate (4.9) to obtain the best possible bound on the inital condition of Equation (4.17): $|v_i(0)|=|s_i(0)-s'_i(0)|\lesssim\frac{\sqrt{i}}{N}$ and $v_{-i}(0)=-v_i(0)$. As the expected continuous limit of the propagator $\mathcal{U}$ of (4.17) is $p_{s,t}(x,y)=\frac{t-s}{(t-s)^2+(x-y)^2}$ (see [23]), a perfect homogenization of (4.17) would give

$$|v_i(\alpha,t)|\lesssim\frac{1}{N}\int\frac{t}{t^2+(x-\frac{i}{N})^2}\cdot\mathrm{sgn}(x)(Nx)^{\frac12}\mathrm{d}x\lesssim\frac{i}{N}\cdot\frac{1}{(\max(i,Nt))^{\frac12}}.$$

Integration in $\alpha\in[0,1]$ then gives (4.10) (the necessary additional term $\max(|i|/N,t)$ is irrelevant for small i and is due to the densities difference $\widetilde{\rho}_t(u)-\widetilde{\rho}'_t(u)$ for large u).

Remark 4.17. *If we assume that the initial condition is rigid instead of the weak local law (4.9), i.e. if* $|v_i(0)|=|s_i(0)-s'_i(0)|\lesssim\frac{1}{N}$, *then the above heuristic gives* $|s_i(t)-s'_i(t)|\lesssim\frac{i}{N}\cdot\frac{1}{\max(i,Nt)}$. *The methods developed below likely apply to cover such stronger assumptions and results, but we do not pursue this direction in this article.*

Thanks to Assumption 4.3, the two measures $\widetilde{\mu}_0,\widetilde{\mu}'_0$ have bounded densities $\widetilde{\rho}_0(x),\widetilde{\rho}'_0(x)$, and their densities have bounded derivatives for $|x|\leqslant G$. We can first do a rescaling, to make that the two densities are the same at 0: $\widetilde{\rho}_0(0)=\widetilde{\rho}'_0(0)=A$. Since $t\ll 1$, the free convolution density also satisfies $\widetilde{\rho}_t(0)=\widetilde{\rho}'_t(0)=A+\mathrm{O}(t)$. Thus Proposition 4.6 follows if we can show that

$$|s_i(t)-s'_i(t)|\leqslant N^{\varepsilon}\cdot\frac{|i|}{N}\cdot\left(\frac{1}{\sqrt{Nt}}+\max\left(\frac{|i|}{N},t\right)\right),\quad |i|\leqslant N.$$

We now start the proof of Proposition 4.6. Assumption 4.3 implies that $\widetilde{\rho}_0(x),\widetilde{\rho}'_0(x)$ have bounded derivatives for $|x|\leqslant G$, and it follows

$$\widetilde{\rho}_0(x)=A+\mathrm{O}(|x|),\quad \widetilde{\rho}'_0(x)=A+\mathrm{O}(|x|). \tag{4.56}$$

If we denote the classical locations of $\widetilde{\rho}_0(x)$ as γ_i

$$\frac{i}{N}=\int_0^{\gamma_i}\widetilde{\rho}_0(x)\mathrm{d}x,\quad |i|\geqslant 1,$$

Then for $|i|\leqslant c_G N$, (4.56) implies that

$$\gamma_i=\frac{i}{AN}+\mathrm{O}\left(\frac{i^2}{N^2}\right).$$

By a standard argument (see [51, Corollary 4.2]), (ν,G)-regularity in the sense of Assumption 4.3 implies that

$$\mu_0(I)=\widetilde{\mu}_0(I)+\mathrm{O}\left((N|I|)^{1/2}\right) \tag{4.57}$$

for any interval $I\subset[-c_G,c_G]$ with $|I|\geqslant N^{-1+\nu}$. Since μ_0 and $\widetilde{\mu}_0$ are both symmetric around origin, we conclude that for any $|i|\leqslant cN$

$$|s_i(0)-\gamma_i(0)|\lesssim\frac{i^{1/2}+N^{\nu}}{N}+\frac{i^2}{N^2}.$$

By the same argument we also have that

$$|s'_i(0)-\gamma_i(0)|\lesssim\frac{i^{1/2}+N^{\nu}}{N}+\frac{i^2}{N^2},\qquad |s_i(0)-s'_i(0)|\lesssim\frac{i^{1/2}+N^{\nu}}{N}+\frac{i^2}{N^2}. \tag{4.58}$$

In the rest of this section, we analyze the coupling from (4.13):

$$\mathrm{d}x_i(t,\alpha)=\frac{\mathrm{d}B_i}{\sqrt{2N}}+\frac{1}{2N}\sum_{j\neq i}\frac{1}{x_i(t,\alpha)-x_j(t,\alpha)}\mathrm{d}t,\quad 1\leqslant|i|\leqslant N, \tag{4.59}$$

where $x_i(0,\alpha)=\alpha s_i+(1-\alpha)s'_i$, $\quad 0\leqslant\alpha\leqslant 1$, and the Brownian motions satisfies $\mathrm{d}\langle B_i\rangle/\mathrm{d}t\leqslant 1$. We consider the corresponding tangential equation (4.17) satisfied by $v_i(t)=v_i(\alpha,t)=\frac{\mathrm{d}}{\mathrm{d}\alpha}x_i(\alpha,t)$

We first state qualitative properties of (4.17). The elementary proof is left to the reader.

Lemma 4.18. *Consider (4.17) with initial condition* $\mathbf{v}$.

(i) If $\mathbf{v}$ *is antisymmetric*($v_i = -v_{-i}$) *and* $v_i \geqslant 0$ *for all* $i \geqslant 1$*, then* $v_i(t) \geqslant 0$ *for all* $i \geqslant 1$*, and* $t \geqslant 0$*.*

(ii) If $v_{-N} \leqslant \ldots \leqslant v_{-1} \leqslant v_1 \leqslant \ldots \leqslant v_N$*, then for any* $t \geqslant 0$ *we have* $v_{-N}(t) \leqslant \ldots \leqslant v_{-1}(t) \leqslant v_1(t) \leqslant \ldots \leqslant v_N(t)$*.*

We now define the key observable from [21],

$$f_t(z) = \sum_{1\leqslant |i|\leqslant N} \frac{v_i(t)}{x_i(t)-z}, \tag{4.60}$$

which satisfies the following stochastic advection equation.

Lemma 4.19. *For any* $\operatorname{Im} z \neq 0$*, we have*

$$\mathrm{d}f_t = m_t(z)\partial_z f_t \mathrm{d}t + \frac{1}{2N}\sum_{|k|\leqslant N} \frac{v_k(t)(\mathrm{d}\langle B_k\rangle_t - \mathrm{d}t)}{(x_k(t)-z)^3} - \frac{1}{\sqrt{2N}}\sum_{|k|\leqslant N}\frac{v_k(t)}{(z-x_k(t))^2}\mathrm{d}B_k(t).$$

Proof. It is a simple application of Itô's formula. We omit the time index. First,

$$\mathrm{d}f = \sum_{|k|\leqslant N}\frac{\mathrm{d}v_k}{x_k-z} + \sum_{|k|\leqslant N} v_k \mathrm{d}\frac{1}{x_k-z}. \tag{4.61}$$

Applying again the Itô formula $\mathrm{d}(x_k-z)^{-1} = -(x_k-z)^{-2}\mathrm{d}x_k + \frac{\mathrm{d}\langle B_k\rangle}{2N}(x_k-z)^{-3}\mathrm{d}t$, with (4.59) we naturally decompose the second sum above as (I)+[(II)+(III)]$\mathrm{d}t$ where

$$\text{(I)} = -\frac{1}{\sqrt{2N}}\sum_{|k|\leqslant N}\frac{v_k}{(z-x_k)^2}\mathrm{d}B_k,$$

$$\text{(II)} = \frac{1}{2N}\sum_{\ell\neq k}\frac{v_k}{x_\ell - x_k}\frac{1}{(x_k-z)^2},$$

$$\text{(III)} = \frac{1}{2N}\sum_{|k|\leqslant N}\frac{v_k \mathrm{d}\langle B_k\rangle}{(x_k-z)^3}.$$

Concerning the first sum in (4.61), by (4.17) we have

$$\begin{aligned}\sum_{|k|\leqslant N}\frac{\partial_t v_k}{x_k-z} &= \sum_{\ell\neq k}\frac{v_\ell - v_k}{2N(x_\ell-x_k)^2(x_k-z)} = \frac{1}{2}\sum_{\ell\neq k}\frac{v_\ell-v_k}{2N(x_\ell-x_k)^2}\left(\frac{1}{x_k-z}-\frac{1}{x_\ell-z}\right)\\ &= \frac{1}{4N}\sum_{\ell\neq k}\frac{v_\ell-v_k}{x_\ell-x_k}\frac{1}{(x_k-z)(x_\ell-z)} = -\frac{1}{2N}\sum_{\ell\neq k}\frac{v_k}{x_\ell-x_k}\frac{1}{(x_k-z)(x_\ell-z)}.\end{aligned}$$

Combining with (II), we obtain

$$\text{(II)} + \sum_{|k|\leqslant N}\frac{\partial_t v_k}{x_k-z} = \frac{1}{2N}\sum_{\ell\neq k}\frac{v_k}{x_\ell-x_k}\frac{1}{x_k-z}\left(\frac{1}{x_k-z}-\frac{1}{x_\ell-z}\right) = \frac{1}{2N}\sum_{\ell\neq k}\frac{v_k}{(x_k-z)^2}\frac{1}{x_\ell-z}.$$

All singularities have disappeared. We obtained (II)+(III)+$\sum_{|k|\leqslant N}\frac{\partial_t v_k}{x_k-z} = m(z)\partial_z f + \frac{1}{2N}\sum_{|k|\leqslant N}\frac{v_k(t)(\mathrm{d}\langle B_k\rangle_t - \mathrm{d}t)}{(x_k(t)-z)^3}$. Summation of the remaining term (I) concludes the proof. □

Instead of considering the stochastic advection equation directly for (4.60) we will exploit positivity of the v_i's for $i \geqslant 1$ by defining

$$f_s(z) = f_s^{(1)}(z) + f_s^{(2)}(z)$$

where

$$f_s^{(1)}(z) = \sum_{1\leqslant|i|\leqslant N}\frac{v_i^{(1)}(s)}{x_i(s)-z}, \qquad v_i^{(1)}(0) = \operatorname{sgn}(i)\cdot\frac{i^{1/2}}{N},$$

and

$$f_s^{(2)}(z) = \sum_{1\leqslant|i|\leqslant N}\frac{v_i^{(2)}(s)}{x_i(s)-z}, \qquad v_i^{(2)}(0) = \operatorname{sgn}(i)\cdot\frac{i^2}{N^2}.$$

Lemma 4.20. *For any $|z| \leqslant G/4$ with $E \geqslant 0$, $\eta := \operatorname{Im} z \geqslant N^{-1+\nu}$, we have*

$$|\operatorname{Im} f_0^{(1)}(z)| + |\operatorname{Im} f_0^{(2)}(z)| \leqslant E\sqrt{\frac{N}{\max(E,\eta)}} + N\log N\ E\max(\eta, E).$$

Proof. We start with the contribution from $f_0^{(2)}$. By symmetry of the x_k and v_k's, so we can write

$$\begin{aligned}|\operatorname{Im} f_0^{(2)}(z)| &= \left|\operatorname{Im}\sum_{k\geqslant 1}\left(\frac{v_k^{(2)}(0)}{x_k(0)-z} - \frac{v_k^{(2)}(0)}{-x_k(0)-z}\right)\right| \leqslant \eta\sum_{k\geqslant 1}|v_k^{(2)}(0)|\left|\frac{1}{|x_k - E|^2+\eta^2} - \frac{1}{|x_k(0)+E|^2+\eta^2}\right| \\ &= 4\eta E\sum_{k\geqslant 1}|v_k^{(2)}(0)|\frac{x_k(0)}{|x_k(0)-z|^2|x_k(0)+z|^2} \leqslant 4\eta E\sum_{k\geqslant 1}\frac{k^2}{N^2}\frac{x_k(0)}{|x_k(0)-z|^2|x_k(0)+z|^2}.\end{aligned} \tag{4.62}$$

From (4.57) and the assumption $\operatorname{Im} z \geqslant N^{-1+\nu}$ for eigenvalues $x_k \leqslant G/2$, and the assumption $x_k \leqslant C$ for the complementary regime, we have

$$\text{right-hand side of (4.62)} \lesssim \eta EN\int_0^c \frac{x^3}{|z-x|^2|z+x|^2}\mathrm{d}x + \eta EN.$$

If $\eta > E$ we have $|z-x| \asymp |z+x|$, and we obtain

$$\int_0^c \frac{x^3}{|z-x|^2|z+x|^2}\mathrm{d}x \lesssim \int_0^c \frac{x^3}{|z-x|^4}\mathrm{d}x \lesssim \log N.$$

If $\eta < E$, we have $|z+x| \asymp \max(x, E)$ so that

$$\begin{aligned}\int_0^c \frac{x^3}{|z-x|^2|z+x|^2}\mathrm{d}x &\lesssim \int_0^c \frac{x^3}{|z-x|^2\max(x,E)^2}\mathrm{d}x \lesssim E^{-2}\int_0^E \frac{x^3}{|z-x|^2}\mathrm{d}x + \int_E^c \frac{x}{|z-x|^2}\mathrm{d}x \\ &\lesssim \frac{E}{\eta} + \int_0^{c-E}\frac{(E+x)\mathrm{d}x}{\eta^2+x^2} \lesssim \frac{E}{\eta} + \log\frac{c-E}{\eta} \lesssim \frac{E}{\eta} + \log N.\end{aligned}$$

We have therefore obtained $|\operatorname{Im} f_0^{(2)}(z)| \lesssim N\log N\, E\max(\eta, E)$.

One can treat $f_0^{(1)}$ similarly, obtaining

$$|\operatorname{Im} f_0^{(1)}(z)| \leqslant 4\eta E\sum_{k\geqslant 1}\frac{\sqrt{k}}{N}\frac{x_k(0)}{|x_k(0)-z|^2|x_k(0)+z|^2} \lesssim \sqrt{N}E\int_0^c \frac{x^{3/2}\eta}{|z-x|^2|z+x|^2}\mathrm{d}x \lesssim E\sqrt{\frac{N}{\max(E,\eta)}},$$

concluding the proof. □

We define $y = \frac{\varphi^5}{N}$ and

$$\mathscr{S} = \left\{z = E + \mathrm{i}y : y < E < \frac{G}{10}\right\}.$$

In the following lemma and its proof, for a fixed $t > 0$ we use the convention

$$\partial_s z_s = \widetilde{m}_{t-s}(z_s),\ 0 \leqslant s \leqslant t \tag{4.63}$$

for the characteristics (note the sign change compared to (4.37): the characteristics now move upwards), for coherence with the notation from [21].

Lemma 4.21. *Let $\mathsf{T} = N^{-2\nu}$. With very high probability, for any $z = E + \mathrm{i}y \in \mathscr{S}$ and $0 < t < \mathsf{T}$ we have*

$$|\operatorname{Im} f_0^{(1)}(z_t) - \operatorname{Im} f_t^{(1)}(z)| \leqslant E\sqrt{\frac{N}{\max(E,t)}} + NE\max(E,t),$$

and the same estimate holds for $\operatorname{Im} f_0^{(2)}(z_t) - \operatorname{Im} f_t^{(2)}(z)$.

Proof. In the following, we abbreviate f_s for either $f_s^{(1)}$ or $f_s^{(2)}$, as the proof is the same in both cases (and accordingly we denote $v_k = v_k^{(1)}$ or $v_k^{(2)}$).

For any $1 \leqslant \ell, m \leqslant N^{12}\mathsf{T}$, we define $t_\ell = \ell N^{-12}$ and $z^{(m)} = E_m + \mathrm{i}y$ where $E_m = mN^{-12}$. We also define the stopping times (with respect to $\mathcal{F}_t = \sigma(B_k(s), 0 \leqslant s \leqslant t, 1 \leqslant k \leqslant N)$)

$$\tau_{\ell,m} = \inf\left\{0 \leqslant s \leqslant t_\ell : |\operatorname{Im} f_s(z_{t_\ell - s}^{(m)}) - \operatorname{Im} f_0(z_{t_\ell}^{(m)})| > \frac{1}{2}\left(E_m\sqrt{\frac{N}{\max(E_m,t_\ell)}} + NE_m \max(E_m, t_\ell)\right)\right\},$$

$$\tau_0 = \inf\left\{0 \leqslant t \leqslant 1 \mid \exists z \in [0, G/2] \times [\frac{\varphi^5}{N}, 1] \text{ s.t. } |m_t(w) - \widetilde{m}_t(w)| \geqslant \varphi\sqrt{\frac{\operatorname{Im}[\widetilde{m}_0(w)]}{N\operatorname{Im}[w]}}\right\},$$

$$\tau = \min\{\tau_0, \tau_{\ell,m} : 0 \leqslant \ell, m \leqslant N^{12}\mathsf{T}, \varphi^5 N^{-1} < E_m < G/10\},$$

with the convention $\inf \varnothing = \mathsf{T}$. We will prove that for any $D > 0$ there exists $\widetilde{N}_0(D)$ such that for any $N \geqslant \widetilde{N}_0(D)$, we have

$$\mathbb{P}(\tau = \mathsf{T}) > 1 - N^{-D}. \tag{4.64}$$

We first explain why the above equation implies the expected result by a grid argument in t and z.

On the one hand, we have the sets inclusion

$$\{\tau = \mathsf{T}\} \bigcap_{1 \leqslant \ell \leqslant N^{12}\mathsf{T}, 1 \leqslant m \leqslant N^{12}, 1 \leqslant k \leqslant N} A_{\ell,m,k} \subset \bigcap_{z \in \mathscr{S}, 0 < t < \mathsf{T}} \left\{|\operatorname{Im} f_s(z_{t-s}) - \operatorname{Im} f_0(z_t)| < E\sqrt{\frac{N}{\max\{E,t\}}} + NE\max(E,t)\right\} \tag{4.65}$$

where

$$A_{\ell,m,k} = \left\{\sup_{t_\ell \leqslant u \leqslant t_{\ell+1}} \left|\int_{t_\ell}^u \frac{v_k(s)\mathrm{d}B_k(s)}{(z^{(m)} - x_k(s))^2}\right| < N^{-4}\right\}.$$

Indeed, for any given z and t, chose $t_\ell, z^{(m)}$ such that $t_\ell \leqslant t < t_{\ell+1}$ and $|z - z_m| < N^{-5}$. Then $|f_t(z) - f_t(z^{(m)})| < N^{-2}$, say, as follows directly from the definition of f_t and the crude estimate $|v_k(t)| < 1$ (obtained by maximum principle). Moreover, we can bound the time increments using Lemma 4.19:

$$\mathrm{d}f_s = m_s(z)\partial_z f_s \mathrm{d}s + \frac{1}{2N}\sum_{|k| \leqslant N} \frac{v_k(s)(\mathrm{d}\langle B_k\rangle_s - \mathrm{d}s)}{(x_k(s) - z)^3} - \frac{1}{\sqrt{2N}}\sum_{|k| \leqslant N} \frac{v_k(s)}{(z - x_k(s))^2}\mathrm{d}B_k(s). \tag{4.66}$$

Thanks to the trivial estimates $|m_t(E + \mathrm{i}\eta)| \leqslant \eta^{-1}$, $|\partial_z f_t(E + \mathrm{i}\eta)| \leqslant N\|v(0)\|_\infty \eta^{-2} \leqslant N\eta^{-2}$ and $|\partial_{zz} f_t(E + \mathrm{i}\eta)| \leqslant N\eta^{-3}$, under the event $\cap_k A_{\ell,m,k}$ (to bound the martingale term) we have $|f_t(z) - f_{t_\ell}(z^{(m)})| < N^{-2}$.

For $M_u = \int_{t_\ell}^u \frac{v_k(s)\mathrm{d}B_k(s)}{(z^{(m)} - x_k(s))^2}$, we have the deterministic estimate $\langle M\rangle_{t_{\ell+1}} \leqslant N^{-12}(\varphi^2/N)^{-4}\|v(0)\|_\infty^2 \leqslant \varphi^{-8}N^{-8}$, so that (4.33) with $\mu = \varphi^{-8}N^{-8}$ gives $\mathbb{P}(A_{\ell,m,k}) \geqslant 1 - e^{-c\varphi^{1/5}}$ and therefore, for any $D > 0$, for large enough N we have

$$\mathbb{P}\Big(\bigcap_{1 \leqslant \ell \leqslant N^{12}\mathsf{T}, 1 \leqslant m \leqslant N^{12}, 1 \leqslant k \leqslant N} A_{\ell,m,k}\Big) \geqslant 1 - N^{-D}. \tag{4.67}$$

Equations (4.64), (4.65), (4.67) conclude the proof of the proposition.

We now prove (4.64). We abbreviate $t = t_\ell$, $z = z^{(m)}$ for some $1 \leqslant \ell, m \leqslant N^{12}\mathsf{T}$. Let $g_u(z) = f_u(z_{t-u})$. Composing (4.66) with the characteristics (4.63) we obtain [6]

$$\begin{aligned}\mathrm{d}g_{u\wedge\tau}(z) = (m_u(z_{t-u}) - \widetilde{m}_u(z_{t-u}))\partial_z f_u(z_{t-u})\mathrm{d}(u \wedge \tau) + \frac{1}{2N}\sum_i \frac{(\mathrm{d}\langle B_i\rangle_u/\mathrm{d}u - 1)v_i(u)}{(x_i(u) - z_{t-u})^3}\mathrm{d}(u \wedge \tau) \\ - \frac{\mathrm{O}(1)}{\sqrt{N}}\sum_k \frac{v_k(u)}{(z_{t-u} - x_k(u))^2}\mathrm{d}B_k(u \wedge \tau).\end{aligned} \tag{4.68}$$

We first bound

$$\int_0^t |(m_u(z_{t-u}) - \widetilde{m}_u(z_{t-u}))\partial_z f_u(z_{t-u})|\,\mathrm{d}(u \wedge \tau) \leqslant \int_0^t \frac{\varphi}{\sqrt{N\eta_{t-u}}}\,|\partial_z f_u(z_{t-u})|\,\mathrm{d}(u \wedge \tau).$$

[6] In this paper, we abbreviate $u \wedge t = \min(u, t)$ when u and t are time variables.

To bound $m_u - \widetilde{m}_u$ above, we have used $0 \leqslant u \leqslant \tau$. Moreover,

$$f_s(z) = \sum_{i=1}^{N} \frac{x_i(t)v_i(t)}{x_i(t)^2 - z^2}$$

so that (from Lemma 4.18 we know that $v_i(u) \geqslant 0$ for all $u \geqslant 0, i \geqslant 1$)

$$|\partial_z f_u(z_{t-u})| \leqslant |z_{t-u}| \sum_{i=1}^{N} \frac{x_i(u)v_i(u)}{|x_i(u) - z_{t-u}|^2 \cdot |x_i(u) + z_{t-u}|^2}. \tag{4.69}$$

Let $I_j = I_j(u) = \{i \in [\![1, N]\!] : x_i(u) \in \frac{\varphi^5}{N}[j, j+1]\}$, $0 \leqslant j \leqslant cN/\varphi^5$. Then

$$|\partial_z f_u(z_{t-u})| \leqslant |z_{t-u}| \sum_{0 \leqslant j \leqslant cN/\varphi^5} \left(\max_{i \in I_j} \frac{x_i(u)}{|x_i(u) - z_{t-u}|^2 \cdot |x_i(u) + z_{t-u}|^2} \right) \cdot \sum_{i \in I_j} v_i(u) + N|z_{t-u}| \int_{G/10}^{C} |x - z_{t-u}|^{-2} \mathrm{d}x, \tag{4.70}$$

where the above integral is due to points $x_i(u) \geqslant G/10$, and use the local law in the domain $G/10 < x_i < G/2$, the upper bound $x_i < C$, and the maximum principle bound $|v_i| \leqslant 1$. The above integral contributes to at most

$$N|z_{t-u}| \int_{G/10}^{C} |x - z_{t-u}|^{-2} \mathrm{d}x \lesssim \frac{N|z_{t-u}|}{|z_{t-u} - \frac{G}{10}|}.$$

The above error term is $\lesssim N|z_{t-u}| \frac{\max(E, \eta_{t-u})}{\eta_{t-u}}$ (here $E = \mathrm{Im} z$), which we will now show to be the contribution from the sum in (4.70): the integral term in (4.70) can be ignored from now.

For each $1 \leqslant j \leqslant cN/\varphi^5$ (note that we omit the case $j = 0$ which will be treated separately), pick a $n = n_j$ such that $|\mathrm{Re} z^{(n)} - \varphi^5 \frac{j}{N}| < N^{-9}$ and denote $e_j = e_{j,N} = \mathrm{Re} z^{(n)}$. First, as $v_k(u) \geqslant 0$ for any $k \geqslant 1$ and u, we have

$$\mathrm{Im}\, f_u(z^{(n)}) = y e_j \sum_{i=1}^{N} \frac{x_i(u)v_i(u)}{|x_i(u) - z^{(n)}|^2 |x_i(u) + z^{(n)}|^2} \gtrsim y \sum_{i \in I_j} \frac{v_i(u)}{|x_i(u) - z^{(n)}|^2} \gtrsim y^{-1} \sum_{i \in I_j} v_i(u), \tag{4.71}$$

where in the first inequality we used that for $i \in I_j$ ($j \geqslant 1$) we have $x_i \asymp e_j \asymp |x_i + z^{(n)}|$. To estimate $\mathrm{Im}\, f_u(z^{(n)})$, introduce ℓ such that $t_\ell \leqslant u < t_{\ell+1}$. On the event $\cap_k A_{\ell,m,k}$ and $u \leqslant \tau$, we have $|f_u(z^{(n)}) - f_{t_\ell}(z^{(n)})| < N^{-2}$ as seen easily from (4.66). We therefore proved

$$\sum_{k \in I_j} v_k(u) \leqslant y\, \mathrm{Im}\, f_{t_\ell}(z^{(n)}) + N^{-2} \leqslant \frac{\varphi^5}{N} \left(e_j \sqrt{\frac{N}{\max(e_j, u)}} + N e_j \max(e_j, u) \right). \tag{4.72}$$

We used $t_\ell \leqslant u \leqslant \tau$ for the second inequality. For $j = 0$ we simply bound $\sum_{k \in I_0} v_k(u) \leqslant \sum_{k \in I_1} v_k(u)$ because the evolution preserves monotonicity, as proved in Lemma 4.18 (ii).

We have therefore obtained (using the local law on scale φ^5/N, i.e. $\tau \leqslant \tau_0$)

$$|\partial_z f_u(z_{t-u})| \leqslant \frac{\varphi^5}{N} |z_{t-u}| \sum_{0 \leqslant j \leqslant N/\varphi^5} \frac{e_j}{|e_j - z_{t-u}|^2 \cdot |e_j + z_{t-u}|^2} \cdot \left(e_j \sqrt{\frac{N}{\max(e_j, u)}} + N e_j \max(e_j, u) \right).$$

The contribution from the first term ($e_j \sqrt{\frac{N}{\max(e_j, u)}}$) is

$$N^{1/2} |z_{t-u}| \int_0^1 \frac{x^2}{|x - z_{t-u}|^2 \cdot |x + z_{t-u}|^2} \frac{\mathrm{d}x}{\sqrt{\max(x, u)}}.$$

If $t - u \leqslant E = \mathrm{Re} z$, the above integral is

$$\lesssim \frac{1}{\eta_{t-u}} \int_0^1 \frac{\eta_{t-u}}{|x - z_{t-u}|^2} \cdot \frac{x^2}{x^2 + E^2} \frac{\mathrm{d}x}{\sqrt{\max(x, u)}} \lesssim \frac{1}{\eta_{t-u} \sqrt{\max(E, u)}}.$$

If $t-u \geqslant E$, it is

$$\lesssim \int_0^{t-u} \frac{x^2}{|t-u|^4}\frac{\mathrm{d}x}{\sqrt{\max(x,u)}} + \int_{t-u}^1 \frac{x^2}{x^4}\frac{\mathrm{d}x}{\sqrt{\max(x,u)}} \lesssim \frac{1}{\eta_{t-u}\sqrt{\max(u,\eta_{t-u})}}.$$

Similarly, the contribution from the second term ($Ne_j \max(e_j,u)$) is

$$N|z_{t-u}|\int_0^1 \frac{x^2}{|x-z_{t-u}|^2\cdot|x+z_{t-u}|^2}\max(x,u)\mathrm{d}x \lesssim N|z_{t-u}|\frac{\max(E,u,\eta_{t-u})}{\eta_{t-u}}.$$

All together, we have proved

$$|\partial_z f_u(z_{t-u})| \lesssim N^{1/2}\frac{|z_{t-u}|}{\eta_{t-u}\sqrt{\max(E,u,\eta_{t-u})}} + N|z_{t-u}|\frac{\max(E,u,\eta_{t-u})}{\eta_{t-u}},$$

so that

$$\begin{aligned}\int_0^t \frac{\varphi}{\sqrt{N\eta_{t-u}}}|\partial_z f_u(z_{t-u})|\,\mathrm{d}(u\wedge\tau) &\lesssim \varphi\int_0^t \frac{\max(E,\eta_{t-u})}{\eta_{t-u}^{3/2}\sqrt{\max(E,u,\eta_{t-u})}}\mathrm{d}u + \varphi\sqrt{N}\int_0^t \frac{\max(E,\eta_{t-u})\max(E,u,\eta_{t-u})}{\eta_{t-u}^{3/2}}\mathrm{d}u\\ &\lesssim \frac{\varphi}{\sqrt{y}}\frac{E}{\sqrt{\max(E,t)}} + \frac{\varphi^2\sqrt{N}}{\sqrt{y}}E\max(E,t) \lesssim \varphi^{-1/2}E\sqrt{\frac{N}{\max(E,t)}} + \varphi^{-1/2}NE\max(E,t).\end{aligned}$$

With the same reasoning, we now bound the contribution in (4.68) from

$$\frac{1}{N}\sum_i \frac{(\mathrm{d}\langle B_i\rangle_u/\mathrm{d}u-1)v_i(u)}{(x_i(u)-z_{t-u})^3} = \partial_{zz}\frac{1}{2N}\sum_i \frac{(\mathrm{d}\langle B_i\rangle_u/\mathrm{d}u-1)v_i(u)}{x_i(u)-z_{t-u}} = \frac{1}{N}\sum_{i\geqslant 1}\partial_{zz}\frac{\mathrm{O}(1)x_i v_i(u)}{x_i(u)^2-z_{t-u}^2}. \qquad (4.73)$$

Note that

$$|\frac{1}{N}\partial_{zz}\frac{xv}{x^2-z^2}| \lesssim \frac{xv}{N}\left(\frac{1}{|x-z|^2|x+z|^2} + \frac{|z|^2}{|x-z|^3|x+z|^3}\right) \lesssim \frac{xv|z|}{|x-z|^2|x+z|^2}$$

where we used $1/N \leqslant |z|$, $|z| \leqslant |x+z|$ and $1/N \leqslant |x-z|$. This proves that the contribution from (4.73) in (4.68) can be bounded by the right-hand side in (4.69) and therefore has a smaller contribution.

We finally want to bound $\sup_{0\leqslant s\leqslant t}|M_s|$, let $E = \mathrm{Re}[z_{t-u}]$, then

$$\begin{aligned}M_s &:= \mathrm{Im}\int_0^s \frac{1}{\sqrt{N}}\sum_k \frac{v_k(u)}{(z_{t-u}-x_k(u))^2}\mathrm{d}B_k(u\wedge\tau)\\ &= \int_0^s \frac{1}{\sqrt{N}}\sum_{k=1}^N \mathrm{Im}\left(\frac{1}{(z_{t-u}-x_k(u))^2} + \frac{1}{(z_{t-u}+x_k(u))^2}\right)v_k(u)\mathrm{d}B_k(u\wedge\tau)\\ &= \int_0^s \eta_{t-u}\frac{1}{\sqrt{N}}\sum_{k=1}^N\left(\frac{E-x_k(u)}{|z_{t-u}-x_k(u)|^4} + \frac{E+x_k(u)}{|z_{t-u}+x_k(u)|^4}\right)v_k(u)\mathrm{d}B_k(u\wedge\tau).\end{aligned}$$

With (4.33) with very high probability we have

$$\sup_{0\leqslant s\leqslant t}|M_s|^2 \leqslant \varphi^{1/10}\int_0^t \frac{\eta_{t-u}^2}{N}\left(\sum_k\left|\frac{E-x_k(u)}{|z_{t-u}-x_k(u)|^4} + \frac{E+x_k(u)}{|z_{t-u}+x_k(u)|^4}\right|v_k(u)\right)^2\mathrm{d}(u\wedge\tau). \qquad (4.74)$$

With (4.72) and the local law ($\tau \leqslant \tau_0$) the above sum is bounded with (we omit the terms corresponding to $x_i > G/10$ as they are shown to be negligible thanks to the local law and the simple bound $|v_i| \leqslant 1$)

$$\begin{aligned}&N^{1/2}\int_0^c \left|\frac{E-x}{|z_{t-u}-x|^4} + \frac{E+x}{|z_{t-u}+x|^4}\right|\frac{x}{\sqrt{\max(x,u)}}\mathrm{d}x + \varphi N\int_0^c\left|\frac{E-x}{|z_{t-u}-x|^4} + \frac{E+x}{|z_{t-u}+x|^4}\right|x\max(x,u)\mathrm{d}x\\ &\lesssim N^{1/2}\left(\frac{E}{\eta_{t-u}^2\sqrt{\max(E,u)}}\mathbb{1}_{t-u\leqslant E} + \frac{1}{\eta_{t-u}\sqrt{\max(\eta_{t-u},u)}}\mathbb{1}_{t-u\geqslant E}\right)\\ &+ N\left(\frac{E}{\eta_{t-u}^2}\max(E,u)\mathbb{1}_{t-u\leqslant E} + \frac{\max(\eta_{t-u},u)}{\eta_{t-u}}\mathbb{1}_{t-u\geqslant E}\right).\end{aligned}$$

Hence the bracket in (4.74) is upper bounded with the sum of these two terms:

$$\varphi^{1/10}\int_0^t\left(\frac{E}{\eta_{t-u}\sqrt{\max(E,u)}}\mathbb{1}_{t-u\leqslant E}+\frac{1}{\sqrt{\max(\eta_{t-u},u)}}\mathbb{1}_{t-u\geqslant E}\right)^2\mathrm{d}u\lesssim\frac{\varphi^{1/10}}{y}\frac{E^2}{\max(E,t)},$$

$$\varphi^{1/10}N\int_0^t\left(\frac{E\max(E,u)}{\eta_{t-u}}\mathbb{1}_{t-u\leqslant E}+\max(\eta_{t-u},u)\mathbb{1}_{t-u\geqslant E}\right)^2\mathrm{d}u\lesssim\frac{\varphi^{1/10}}{y}N(E\max(E,t))^2.$$

This concludes the proof. □

Corollary 4.22. *For any $t\in[\varphi^{100}/N,1]$ and $|i|<N$, uniformly in α we have, with overwhelming probability,*

$$|v_i(t,\alpha)|\leqslant\varphi^8\frac{i}{N}\left(\frac{1}{\sqrt{Nt}}+\max(\frac{i}{N},t)\right).$$

Proof. We have $|v_i(0)|\leqslant\varphi(v_i^{(1)}(0)+v_i^{(2)}(0))$ from (4.58); from Lemma 4.18 (i), this implies that for any $t\geqslant 0, i\geqslant 1$ we have $|v_i(t)|\leqslant\varphi(v_i^{(1)}(t)+v_i^{(2)}(t))$. From Equation (4.71) and Lemmas 4.20 and 4.21, we have

$$v_i^{(1)}(t)+v_i^{(2)}(t)\lesssim\frac{\varphi^6}{N}\left(\gamma_i\sqrt{\frac{N}{\max(\gamma_i,t)}}+N\gamma_i\max(\gamma_i,t)\right)\lesssim\varphi^6\frac{i}{N}\left(\frac{1}{\sqrt{Nt}}+\max(\frac{i}{N},t)\right)$$

with very high probability. This concludes the proof. □

Proof of Proposition 4.6. Based on Corollary 4.22, the proof of Proposition 4.6 proceeds similarly to (4.34), details are left to the reader. □

5 Resolvent fluctuations from the dynamics noise

It is well known [2] that for any $t\geqslant 0$ the resolvents G_t^z becomes approximately deterministic as $n\to\infty$. Its deterministic approximation is given by

$$M^z=M^z(\mathrm{i}\eta):=\begin{pmatrix}m^z & -zu^z\\ -\overline{z}u^z & m^z\end{pmatrix},\qquad u^z=u^z(\mathrm{i}\eta):=\frac{m^z(\mathrm{i}\eta)}{\mathrm{i}\eta+m^z(\mathrm{i}\eta)},$$

with $m^z=m^z(\mathrm{i}\eta)$ being the unique solution of the cubic equation

$$-\frac{1}{m^z}=\mathrm{i}\eta+m^z-\frac{|z|^2}{\mathrm{i}\eta+m^z},\qquad \eta\mathrm{Im}[m^z]>0.$$

Note that on the imaginary axis m^z is purely imaginary and that u^z is real. Furthermore, note that the deterministic approximation of G_t does not depend on time since the first two moments of X_t are preserved along the flow (1.5) and M^z is determined only by those moments.

5.1 Stochastic advection equation and consequences. In this section we analyze the evolution of the resolvent $G_t^z(\mathrm{i}\eta):=(W_t-Z-\mathrm{i}\eta)^{-1}$ along the flow (1.5). Here we perform the analysis in the regime $|z|^2\leqslant 1-c$, for some small fixed $c>0$, since this is the regime of interest of Proposition 3.3. The main result of this section is a decomposition theorem for $I_{\eta_c}^T(f,t)$ into the sum of two terms: one that depends only on the initial condition and one that is a martingale (see Proposition 5.1). Before stating this result we introduce the notation

$$E_1:=\begin{pmatrix}1&0\\0&0\end{pmatrix},\qquad E_2:=\begin{pmatrix}0&0\\0&1\end{pmatrix}. \tag{5.1}$$

Proposition 5.1. *For any $s<t$, we have*

$$\begin{aligned}I_{\eta_c}^T(f,t)=&-\frac{1}{4\pi\mathrm{i}}\int_{\mathbb{C}}\Delta f(z)\int_{\eta_s(\eta_c,t)}^{\eta_s(T,t)}\mathrm{Tr}\big[G_s^{z_s(z)}(\eta')-\mathbb{E}G_s^{z_s(z)}(\eta')\big]\,\mathrm{d}\eta'\mathrm{d}z\\&+\frac{1}{4\pi\mathrm{i}\sqrt{N}}\sum_{i\neq j}\int_{\mathbb{C}}\Delta f(z)\int_s^t\int_{\eta_\tau(\eta_c,t)}^{\eta_\tau(T,t)}\mathrm{Tr}\big[G_\tau^{z_\tau(z)}(\mathrm{i}\eta')^2E_i\mathrm{d}B_\tau E_j\big]\Bigg]\mathrm{d}\eta\mathrm{d}z+\mathrm{O}\left(\frac{N^\xi}{N\eta_c}\right),\end{aligned} \tag{5.2}$$

with very high probability for any $\xi > 0$. Here $\sum_{i\neq j}$ denotes a summation over the indices $(i,j) \in \{(1,2),(2,1)\}$.

This immediately shows the decomposition in (3.11) with the integrals I_1, I_2 being defined as

$$I_1 = I_1(f,s,t) := -\frac{1}{4\pi \mathrm{i}} \int_{\mathbb{C}} \Delta f(z) \int_{\eta_s(\eta_c,t)}^{\eta_s(T,t)} \mathrm{Tr}\big[G_s^{z_s(z)}(\eta') - \mathbb{E}G_s^{z_s(z)}(\eta')\big] \,\mathrm{d}\eta' \mathrm{d}z,$$
$$I_2 = I_2(f,s,t) := \frac{1}{4\pi \mathrm{i}\sqrt{N}} \sum_{i\neq j} \int_{\mathbb{C}} \Delta f(z) \int_s^t \int_{\eta_\tau(\eta_c,t)}^{\eta_\tau(T,t)} \mathrm{Tr}\big[G_\tau^{z_\tau(z)}(\mathrm{i}\eta')^2 E_i \mathrm{d}B_\tau E_j\big]\Bigg] \,\mathrm{d}\eta \mathrm{d}\tau \mathrm{d}z.$$

Proof of Proposition 5.1. By Itô's formula (using the short–hand notations $G = G_t^z(\mathrm{i}\eta)$) we obtain

$$\begin{aligned} \mathrm{d}\langle G(\mathrm{i}\eta)\rangle &= -\frac{1}{\sqrt{N}}\sum_{i\neq j}\langle G^2 E_i \mathrm{d}B_t E_j\rangle + \frac{1}{2}\langle WG^2\rangle \mathrm{d}t + 2\sum_{i\neq j}\langle GE_i\rangle\langle G^2E_j\rangle \mathrm{d}t \\ &= -\frac{1}{\sqrt{N}}\sum_{i\neq j}\langle G^2 E_i \mathrm{d}B_t E_j\rangle + \frac{1}{2}\langle G\rangle \mathrm{d}t + \frac{1}{2}\langle (Z+\mathrm{i}\eta)G^2\rangle \mathrm{d}t + 2\sum_{i\neq j}\langle GE_i\rangle\langle G^2E_j\rangle \mathrm{d}t. \end{aligned} \tag{5.3}$$

In order to control the evolution of the resolvent along the flow (5.3) we consider the characteristics[7] (cf. [41, Eq. (5.4)], [42, Eq. (5.4)]):

$$\partial_t \eta_t = -\mathrm{Im} m^{z_t}(\mathrm{i}\eta_t) - \frac{\eta_t}{2}, \qquad \partial_t z_t = -\frac{z_t}{2}. \tag{5.4}$$

Note that along the characteristics we have

$$\eta_0 = e^{t/2}\eta_t + (e^{t/2} - e^{-t/2})\mathrm{Im} m^{z_t}(\mathrm{i}\eta_t), \quad z_0 = e^{t/2} z_t, \quad m^{z_0}(\mathrm{i}\eta_0) = e^{-t/2} m^{z_t}(\mathrm{i}\eta_t), \quad u^{z_0}(\mathrm{i}\eta_0) = e^{-t} u^{z_t}(\mathrm{i}\eta_t). \tag{5.5}$$

Then, by plugging (5.4) into (5.3), we get (using the notations $G_t := G_t^{z_t}(\mathrm{i}\eta_t)$ and $M_t := M^{z_t}(\mathrm{i}\eta_t)$)

$$\mathrm{d}\langle G_t\rangle = -\frac{1}{\sqrt{N}}\sum_{i\neq j}\langle G_t^2 E_i \mathrm{d}B_t E_j\rangle + \frac{1}{2}\langle G_t\rangle \mathrm{d}t + \langle G_t - M_t\rangle\langle G^2\rangle \mathrm{d}t, \tag{5.6}$$

where we used that by the chiral symmetry of $W_t - Z$ we have $2\langle G_t E_i\rangle = \langle G_t\rangle$. Subtracting the expectation in (5.6), we obtain

$$\mathrm{d}\langle G_t - \mathbb{E}G_t\rangle = -\frac{1}{\sqrt{N}}\sum_{i\neq j}\langle G_t^2 E_i \mathrm{d}B_t E_j\rangle + \frac{1}{2}\langle G_t - \mathbb{E}G_t\rangle \mathrm{d}t + \langle G_t - M_t\rangle\langle G_t^2\rangle \mathrm{d}t - \mathbb{E}\langle G_t - M_t\rangle\langle G_t^2\rangle \mathrm{d}t.$$

Then, writing $G_t^2 = M_t' + (G_t^2 - M_t')$ (here prime denotes the derivative $\partial_{\mathrm{i}\eta}$), we get

$$\mathrm{d}\langle G_t - \mathbb{E}G_t\rangle = -\frac{1}{\sqrt{N}}\sum_{i\neq j}\langle G_t^2 E_i \mathrm{d}B_t E_j\rangle + \left(\frac{1}{2} + \langle M_t'\rangle\right)\langle G_t - \mathbb{E}G_t\rangle \mathrm{d}t + \langle G_t - M_t\rangle\langle G_t^2 - M_t'\rangle \mathrm{d}t - \mathbb{E}\langle G_t - M_t\rangle\langle G_t^2 - M_t'\rangle \mathrm{d}t,$$

so that

$$\begin{aligned} \langle G_t - \mathbb{E}G_t\rangle &= \exp\left(\int_0^t \left[\frac{1}{2} + \partial_{\eta_s} m^{z_s}(\mathrm{i}\eta_s)\right] \mathrm{d}s\right)\langle G_0 - \mathbb{E}G_0\rangle \\ &\quad - \frac{1}{\sqrt{N}}\sum_{i\neq j}\int_0^t \exp\left(\int_0^s \left[\frac{1}{2} + \partial_{\eta_\tau} m^{z_\tau}(\mathrm{i}\eta_\tau)\right] \mathrm{d}\tau\right)\langle G_s^2 E_i \mathrm{d}B_s E_j\rangle \,\mathrm{d}s + O\left(\frac{N^\xi}{(N\eta_t)^2}\right) \end{aligned} \tag{5.7}$$

with very high probability, where we used that $|\langle G_t^2 - M_t'\rangle| \lesssim N^\xi (N\eta_t^2)^{-1}$ for any arbitrary small $\xi > 0$ by [41, Theorem 3.3] for $z_1 = z_2$ and $\eta_1 = \eta_2$.

We now plug the expression (5.7) into Girko's formula (3.2); for simplicity of notation within this proof we choose $s = 0$. Choose $\eta_0 = \eta_0(\eta) = \eta_0(\eta, t)$ and $z_0 = z_0(z) = z_0(z,t)$ such that $z_t = z$, $\eta_t = \eta$, with η_t, z_t

[7] Our convention here is that the characteristics move upwards as in (4.37), contrary to (4.63).

being the solutions of (5.4) with initial conditions η_0, z_0. Then plugging (5.7) into Girko's formula (3.2) we find that

$$I_{\eta_c}^T(f,t) = -\frac{1}{4\pi\mathrm{i}}\int_{\mathbb{C}} \Delta f(z) \int_{\eta_c}^T \exp\left(\int_0^t \left[\frac{1}{2}+\partial_{\eta_s} m^{z_s}(\mathrm{i}\eta_s)\right] \mathrm{d}s\right) \mathrm{Tr}\big[G_0^{z_0(z)}(\mathrm{i}\eta_0(\eta)) - \mathbb{E}G_0^{z_0(z)}(\mathrm{i}\eta_0(\eta))\big]$$
$$+ \frac{1}{4\pi\mathrm{i}\sqrt{N}}\sum_{i\neq j}\int_{\mathbb{C}} \Delta f(z) \int_{\eta_c}^T \int_0^t \exp\left(\int_0^s \left[\frac{1}{2}+\partial_{\eta_\tau} m^{z_\tau}(\mathrm{i}\eta_\tau)\right] \mathrm{d}\tau\right) \mathrm{Tr}\big[G_s^{z_s(z)}(\mathrm{i}\eta_s(\eta))^2 E_i \mathrm{d}B_s E_j\big]\,\mathrm{d}\eta \mathrm{d}z^2$$
$$+ \mathrm{O}\left(\frac{N^\xi}{N\eta_c}\right), \tag{5.8}$$

with $z_s(z) = e^{(t-s)/2}z$. Performing the change of variables $\eta_0(\eta,t)\to\eta'$ and $\eta_s(\eta,t)\to\eta'$, in the first and second line of (5.8) respectively, and using

$$\partial_\eta \eta_0(\eta) = \partial_{\eta_t}\eta_0(\eta_t) = \exp\left(\int_0^t \left[\frac{1}{2}+\partial_{\eta_s} m^{z_s}(\mathrm{i}\eta_s)\right]\mathrm{d}s\right),$$

we obtain (5.2). □

5.2 Proof of Proposition 3.3. Proposition 3.3 consists of two main statements: i) the integrals $I_{\eta_c}^T(f_{v,a}^{(i)}, t_i)$ obey a Wick theorem and we have an explicit formula for their covariance, ii) each one of those integrals can be decomposed as the sum of two terms (see (3.12)). The main input for both these results is the decomposition of $I_{\eta_c}^T(f,t)$ from Proposition 5.1.

The Wick theorem factorization in (3.9) immediately follows from (5.2) together with [40, Proposition 3.3]. We present its proof in Section 5.2.3.

We now turn to the proof of ii). We recall that, by Proposition 5.1, we immediately obtain (3.11) and that I_1, I_2 are (approximately) uncorrelated. Furthermore, the first relation in (3.12) immediately follows by [40, Proposition 3.3] as it consists of the variance of the product of two resolvents evaluated at the same time; hence to conclude the proof of ii) we only need to compute the variance of the martingale term I_2.

In the following, to keep the presentation simple, we present the proof of Proposition 3.3 only in the macroscopic case $a = 0$. The proof in the mesoscopic case is completely analogous after replacing any reference to [40] with the corresponding one in [41]. Additionally, for simplicity of notation we also assume that $s = 0$; the general case can be achieved by a simple time–shift.

We divide this section into three subsections: in Section 5.2.1 we compute $\mathbb{E}I_{\eta_c}^T(f,t), I_{\eta_c}^T(g,0)$, in Section 5.2.2 we compute the variance of the martingale term in (5.2), concluding the proof of ii), in Section 5.2.3 we prove a Wick theorem, concluding the proof of i).

Before presenting these proof we comment on their relation with the proof of [40, Proposition 3.3]. In the reminder of this section the local law [41, Theorem 5.2] is a fundamental input to compute the deterministic approximation of product of resolvents of the form $G_s^{z_1}G_s^{z_2}$, i.e. this local law is used for any fixed time $0 \leqslant s \leqslant t$, but only for resolvents evaluated at the same time. On the other hand, the CLT for resolvents [40, Proposition 3.3] is used only at time $s = 0$ (see (5.9) below) to compute the correlation of the initial condition.

5.2.1 Correlation between $I_{\eta_c}^T(f,t), I_{\eta_c}^T(g,0)$. From now on $c > 0$ is a small fixed constant that may change from line to line. By [40, Proposition 3.3], for $\eta_i \geqslant n^{-1+\delta_1}$, it follows that

$$\mathbf{E}\mathrm{Tr}\big[G_0^{v(z_1)}(\mathrm{i}\eta_1) - \mathbb{E}G_0^{v(z_1)}(\mathrm{i}\eta_1)\big]\mathrm{Tr}\big[G_0^{z_2}(\mathrm{i}\eta_2) - \mathbb{E}G_0^{z_2}(\mathrm{i}\eta_2)\big]$$
$$= \partial_{\eta_1}\partial_{\eta_2}\log\Big[1 + e^{-t}(|v(z_1)z_2|^2u_1^2u_2^2 - m_1^2m_2^2) - 2e^{-t/2}u_1u_2\mathrm{Re}[v(z_1)\overline{z_2}]\Big] - \kappa_4\partial_{\eta_1}\partial_{\eta_2}m_1^2m_2^2 + \mathrm{O}(N^{-c}), \tag{5.9}$$

where we used the notations $m_1 := m^{v(z_1)}(\mathrm{i}\eta_1)$, $u_1 := u^{v(z_1)}(\mathrm{i}\eta_1)$, and similar notations for m_2, u_2.

Adding back the small η_1 and η_2 regimes (see the proof of [40, Lemma 4.7]), and performing the (η_1,η_2)-integrals in Girko's formula we conclude that (note that the expectation of the martingale term in (5.8) vanishes)

$$
\begin{aligned}
\mathbb{E}I^T_{\eta_c}(f,t)I^T_{\eta_c}(g,0) = \frac{1}{8\pi^2}\int\int \Delta f(z_1)\Delta g(z_2)\,\mathrm{d}z_1\mathrm{d}z_2 \\
\times\Bigg[-\frac{1}{2}\log\Big[1+|z_2v(z_1)|^2(u^{v(z_1)}(\mathrm{i}\eta_0(\eta_1)))^2(u^{z_2}(\mathrm{i}\eta_2))^2 \\
-(m^{v(z_1)}(\mathrm{i}\eta_0(\eta_1)))^2(m^{z_2}(\mathrm{i}\eta_2))^2-2u^{v(z_1)}(\mathrm{i}\eta_0(\eta_1))u^{z_2}(\mathrm{i}\eta_2)\mathrm{Re}[v(z_1)\overline{z_2}]\Big] \\
+\frac{\kappa_4}{2}m^{v(z_1)}(\mathrm{i}\eta_0(\eta_1))^2m^{z_2}(\mathrm{i}\eta_2)^2\Bigg]\Bigg|_{\substack{\eta_1=0,\\ \eta_2=0}} +\mathrm{O}(N^{-c}).
\end{aligned}
\tag{5.10}
$$

Then, using (5.5), for the log–term in (5.10), we conclude

$$
\begin{aligned}
&\log\big[1+|z_2v(z_1)|^2(u^{v(z_1)}(\mathrm{i}\eta_0(\eta_1)))^2(u^{z_2}(\mathrm{i}\eta_2))^2-(m^{v(z_1)}(\mathrm{i}\eta_0(\eta_1)))^2(m^{z_2}(\mathrm{i}\eta_2))^2-2u^{v(z_1)}(\mathrm{i}\eta_0(\eta_1))u^{z_2}(\mathrm{i}\eta_2)\mathrm{Re}[v(z_1)\overline{z_2}]\big] \\
&=\log\big[1+e^{-t}(|z_2z_1|^2u^{z_1}(i\eta_1)^2(u^{z_2}(\mathrm{i}\eta_2))^2-m^{z_1}(i\eta_1)^2(m^{z_2}(\mathrm{i}\eta_2))^2)-2e^{-t/2}u^{z_1}(i\eta_1)u^{z_2}(\mathrm{i}\eta_2)\mathrm{Re}[z_1\overline{z_2}]\big].
\end{aligned}
\tag{5.11}
$$

Evaluating (5.11) at $\eta_1=\eta_2=0$ we obtain

$$
\mathrm{rhs.}(5.11)=\Theta(z_1,z_2,t):=-\frac{1}{2}\begin{cases}\log\big[|(1-e^{-t})(1-|z_1|^2)+|z_1-e^{-t/2}z_2|^2\big] & |z_1|,|z_2|\leqslant 1\\ \log|z_l-e^{-t/2}z_m|^2-\log|z_l|^2 & |z_m|\leqslant 1,\,|z_l|>1\\ \log|e^{-t/2}-z_1\overline{z_2}|^2-\log|z_1z_2|^2 & |z_1|,|z_2|>1.\end{cases}
\tag{5.12}
$$

Furthermore, for $\eta_1=\eta_2=0$ we also compute the term in the last line of (5.10):

$$
m^{v(z_1)}(\mathrm{i}\eta_0(\eta_1))^2=e^{-t}m^{z_1}(\mathrm{i}\eta_1)=e^{-t}(1-|z_1|^2),\qquad m^{z_2}(\mathrm{i}\eta_2)^2=(1-|z_2|^2).
\tag{5.13}
$$

Plugging (5.12)–(5.13) into (5.10) we thus obtain

$$
\mathbb{E}I^T_{\eta_c}(f,t)I^T_{\eta_c}(g,0)\approx\frac{1}{8\pi^2}\int\int\Delta f(z_1)\Delta g(z_2)\,\mathrm{d}z_1\mathrm{d}z_2\Big[\Theta(z_1,z_2,t)+\frac{\kappa_4}{2}e^{-t}(1-|z_1|^2)(1-|z_2|^2)\Big].
\tag{5.14}
$$

We now conclude the proof of (3.9) performing integration by parts in z_1 and $\overline{z_2}$ in (5.14). Here we omit the details of the tedious explicit computations to compute the second line of (3.9) given (5.12), since they are analogous to [40]; we only point out the very minor differences. The computations of the term with the fourth cumulant κ_4 are exactly the same as in [40, Lemma 4.10], in fact the only difference compared to [40] is the scaling factor e^{-t} in (5.14). For the term with $\Theta(z_1,z_2,t)$, using that

$$
\partial_{z_1}\partial_{\overline{z_2}}\log\big[|z_l-e^{-t/2}z_m|^2\big]=0,
$$

when $|z_m|\leqslant 1$, $|z_l|>1$ and proceeding exactly as in [40, Appendix A], we conclude (cf. [40, Lemma 4.9])

$$
\frac{1}{8\pi^2}\int_{\mathbb{C}^2}\Delta f(z_1)\Delta g(z_2)\Theta(z_1,z_2,t)\,\mathrm{d}z_1\mathrm{d}z_2=-\int_{\mathbb{D}^2}\partial_{\overline{z_1}}f\partial_{z_2}g\partial_{z_1}\partial_{\overline{z_2}}K(z_1,z_2,t)\,\mathrm{d}z_1\mathrm{d}z_2+\lim_{\varepsilon\to 0}\mathcal{I}_\varepsilon,
\tag{5.15}
$$

with $K(z_1,z_2,t)$ from (1.8), and

$$
\begin{aligned}
\mathcal{I}_\varepsilon:=&\frac{1}{2\pi^2}\int_{|z_1|\geqslant 1}\mathrm{d}z_1\int_{\substack{|e^{-t/2}-z_1\overline{z}_2|\geqslant\varepsilon,\\ |z_2|\geqslant 1}}\mathrm{d}z_2\,\partial_{z_1}f(z_1)\partial_{\overline{z_2}}g(z_2)\frac{1}{(e^{-t/2}-\overline{z}_1z_2)^2}\\
&+\frac{1}{2\pi^2}\int_{|z_1|\geqslant 1}\mathrm{d}z_1\int_{\substack{|e^{-t/2}-z_1\overline{z}_2|\geqslant\varepsilon,\\ |z_2|\geqslant 1}}\mathrm{d}z_2\,\partial_{\overline{z_1}}f(z_1)\partial_{z_2}\overline{g(z_2)}\frac{1}{(e^{-t/2}-z_1\overline{z}_2)^2}.
\end{aligned}
$$

To compute the second term in the rhs. of (5.15) we proceed similarly to [40, Eqs. (4.33)–(4.35)]. Using the change of variables $\overline{z_1}\to 1/\overline{z_1}$ and $z_2\to 1/z_2$ the integrals in $\mathcal{I}_\varepsilon$ are equal to the integral of $1/(e^{-t/2}\overline{z_1}z_2-1)^2$ over the domain $|z_i|\leqslant 1$ and $|e^{-t/2}\overline{z_1}z_2-1|\geqslant\varepsilon$. By a density argument it is enough to compute the second term in the rhs. of (5.15) for polynomials

$$
f(z_1)=\sum_{k,l\geqslant 0}a_{kl}z_1^k\overline{z_1}^l,\qquad g(z_2)=\sum_{k,l\geqslant 0}b_{kl}z_2^k\overline{z_2}^l.
$$

Using that

$$\lim_{\varepsilon\to 0}\int_{|z_1|\leqslant 1}\int_{\substack{|e^{-t/2}\overline{z_1}z_2-1|\geqslant\varepsilon,\\ |z_2|\leqslant 1}} z_1^{\alpha}\overline{z_1}^{\beta}z_2^{\alpha'}\overline{z_2}^{\beta'}\,\mathrm{d}z_1\mathrm{d}z_2=\frac{\pi^2}{(\alpha+1)(\alpha'+1)}\delta_{\alpha\beta}\delta_{\alpha'\beta'}$$

and proceeding exactly as in [40, Eqs. (4.34)] we conclude

$$\lim_{\varepsilon\to 0}\mathcal{I}_\varepsilon=\frac{1}{2}\sum_{\substack{k,k',l,l'\geqslant 0\\ m\in\mathbf{Z}}}|m|e^{-\frac{t}{2}|m|}a_{k,l}\overline{b_{k',l'}}\delta_{k,l+m}\delta_{k',l'+m}=\frac{1}{2}\langle f,P_t g\rangle_{H^{1/2}(\partial\mathbb{D})},\tag{5.16}$$

with P_t being the Poisson kernel defined as $P_t f(z):=\sum_{\mathbb{Z}}e^{-|m|t/2}\widehat{f}_m z^m$. Combining (5.14)–(5.15) and (5.16) we conclude the proof of (3.10), and so of part i).

5.2.2 Second moment of the martingale term. We now compute the second moment of the martingale term in the second line of (5.2). For fix $(i,j)\in\{(1,2),(2,1)\}$, we compute

$$\mathbb{E}\int_0^t\mathrm{Tr}\big[[G_s^{z_{1,s}}(\mathrm{i}\eta_1)]^2E_i\mathrm{d}B_sE_j\big]\int_0^t\mathrm{Tr}\big[[G_s^{z_{2,s}}(\mathrm{i}\eta_2)]^2E_i\mathrm{d}B_sE_j\big]=\mathbb{E}\int_0^t\mathrm{Tr}\big[G_s^{z_{1,s}}(\mathrm{i}\eta_1)^2E_iG_s^{z_{2,s}}(\mathrm{i}\eta_2)^2E_j\big]\,\mathrm{d}s.$$

Furthermore, we note that

$$\mathrm{Tr}\big[G_s^{z_{1,s}}(\mathrm{i}\eta_1)^2E_iG_s^{z_{2,s}}(\mathrm{i}\eta_2)^2E_j\big]=-\partial_{\eta_1}\partial_{\eta_2}\mathrm{Tr}\big[G_s^{z_{1,s}}(\mathrm{i}\eta_1)E_iG_s^{z_{2,s}}(\mathrm{i}\eta_2)E_j\big].$$

Plugging this into the computation of the second moment of the martingale term I_2 in (5.2), we obtain

$$\frac{1}{16\pi^2 n}\iint_{\mathbb{C}}\Delta f(z_1)\Delta f(z_2)\int_0^t\iint_{\eta_s(\eta_c,t)}^{\eta_s(T,t)}\partial_{\eta_1}\partial_{\eta_2}\mathrm{Tr}\big[G_s^{z_{1,s}}(\mathrm{i}\eta_1)E_iG_s^{z_{2,s}}(\mathrm{i}\eta_2)E_j\big]\,\mathrm{d}\eta_1\mathrm{d}\eta_2\mathrm{d}z_1^2\mathrm{d}z_2^2.\tag{5.17}$$

We can thus perform the (η_1,η_2)–integrations in (5.17) and obtain

$$\frac{1}{16\pi^2 n}\iint_{\mathbb{C}}\Delta f(z_1)\Delta f(z_2)\int_0^t\mathrm{Tr}\big[G_s^{z_{1,s}}(\mathrm{i}\eta_{1,s}(\eta_c))E_iG_s^{z_{2,s}}(\mathrm{i}\eta_{2,s}(\eta_c))E_j\big]\,\mathrm{d}z_1^2\mathrm{d}z_2^2.\tag{5.18}$$

Here we omitted a negligible error smaller than n^{-10} coming from the upper extreme of integration in the (η_1,η_2)–integrals.

We now compute the leading deterministic term in (5.18). For this purpose we recall the multi–resolvent local law from [41, Theorem 3.3]. Define $G_i:=(W-Z_i-\mathrm{i}\eta_i)^{-1}$, with Z_i as in (3.3), and denote by M_i its deterministic approximation. Then, for a deterministic matrix $A\in\mathbb{C}^{2N\times 2N}$, the deterministic approximation of G_1AG_2 is given by

$$M_A^{z_1,z_2}:=\big(1-M_1\mathcal{S}[\cdot]M_2\big)^{-1}[M_1AM_2].$$

Here $\mathcal{S}[\cdot]$ denotes the *covariance operator*, which is defined by (recall the definition of E_1,E_2 from (5.1))

$$\mathcal{S}[\cdot]:=2\langle\cdot E_1\rangle E_2+2\langle\cdot E_2\rangle E_1.$$

By the local law from [41, Theorem 3.3], we then have

$$\big|\big\langle\big[G_s^{z_{1,s}}(\mathrm{i}\eta_{1,s}(\eta_c))E_iG_s^{z_{2,s}}(\mathrm{i}\eta_{2,s}(\eta_c))-M_{E_i}^{z_{1,s},z_{2,s}}(\mathrm{i}\eta_{1,s}(\eta_c),\mathrm{i}\eta_{2,s}(\eta_c))\big]E_j\big\rangle\big|\lesssim\frac{N^\xi}{N\eta_{*,s}^2},\tag{5.19}$$

with very high probability for $\xi>0$ arbitrary small and $\eta_{*,s}:=\eta_{1,s}(\eta_c)\wedge\eta_{2,s}(\eta_c)$. We are thus left to compute

$$\int_0^t\langle M_{E_i}^{z_{1,s},z_{2,s}}(\mathrm{i}\eta_{1,s}(\eta_c),\mathrm{i}\eta_{2,s}(\eta_c))E_j\rangle\,\mathrm{d}s,$$

and then plug the answer into (5.18). In fact the error term in (5.19), after the time integration, can by estimated by N^{-c} for some small fixed $c>0$ and so shown to be negligible.

Using the notations $m_{i,s} := m^{z_{i,s}}(\mathrm{i}\eta_{i,s}(\eta_c))$, $u_{i,s} := u^{z_{i,s}}(\mathrm{i}\eta_{i,s}(\eta_c))$, we then compute

$$\begin{aligned}
&\sum_{(i,j)\in\{(1,2),(2,1)\}} \langle M_{E_i}^{z_{1,s},z_{2,s}}(\mathrm{i}\eta_{1,s}(\eta_c),\mathrm{i}\eta_{2,s}(\eta_c))E_j\rangle \\
&\qquad= \frac{u_{1,s}u_{2,s}\mathrm{Re}[z_{1,s}\overline{z_{2,s}}] - |z_{1,s}z_{2,s}|^2u_{1,s}^2u_{2,s}^2 + m_{1,s}^2m_{2,s}^2}{1+|z_{1,s}z_{2,s}|^2u_{1,s}^2u_{2,s}^2 - m_{1,s}^2m_{2,s}^2 - 2u_{1,s}u_{2,s}\mathrm{Re}[z_{1,s}\overline{z_{2,s}}]} \\
&\qquad= -\frac{1}{2}\partial_s \log\big[1+e^{-2(t-s)}|z_1z_2|^2u_1^2u_2^2 - e^{-2(t-s)}m_1^2m_2^2 - 2e^{-(t-s)}u_1u_2\mathrm{Re}[z_{1,s}\overline{z_{2,s}}]\big],
\end{aligned} \tag{5.20}$$

where we used that $z_{i,s} = e^{(t-s)/2}z_i$, $u_{i,s} = e^{-(t-s)}u_i$, and $m_{i,s} = e^{-(t-s)/2}m_i$. Plugging (5.20) evaluated at $\eta_1 = \eta_2 = 0$ into (5.18), we thus conclude (neglecting negligible errors of size N^{-c})

$$\begin{aligned}
\mathbb{E}|I_2|^2 &= \mathbb{E}\left|\frac{1}{4\pi\mathrm{i}\sqrt{N}}\int_{\mathbb{C}}\Delta f(z)\int_0^t\int_{\eta_s(\eta_c,t)}^{\eta_s(T,t)}\mathrm{Tr}\big[G_s^{z_s(z)}(\mathrm{i}\eta')^2E_i\mathrm{d}B_sE_j\big]\right]\mathrm{d}\eta'\mathrm{d}z^2\Bigg|^2 \\
&= -\frac{1}{16\pi^2}\int_{\mathbb{C}}\int_{\mathbb{C}}\Delta f(z_1)\Delta f(z_2)\times\int_0^t\partial s\begin{cases}\log\big[|(1-e^{-2(t-s)})(1-|z_1|^2)+|z_1-e^{-(t-s)}z_2|^2\big] & |z_1|,|z_2|\leqslant 1\\ \log|z_l-e^{-(t-s)}z_m|^2-\log|z_l|^2 & |z_m|\leqslant 1,\, |z_l|>1\\ \log|e^{-(t-s)}-z_1\overline{z_2}|^2-\log|z_1z_2|^2 & |z_1|,|z_2|>1\end{cases} \\
&= -\frac{1}{16\pi^2}\int_{\mathbb{C}}\int_{\mathbb{C}}\Delta f(z_1)\Delta f(z_2)\times\begin{cases}\log|z_1-z_2|^2-\log\big[|(1-e^{-2t})(1-|z_1|^2)+|z_1-e^{-t}z_2|^2\big] & |z_1|,|z_2|\leqslant 1\\ \log|z_1-z_2|^2-\log|z_l-e^{-t}z_m|^2 & |z_m|\leqslant 1,\, |z_l|>1\\ \log|1-z_1\overline{z_2}|^2-\log|e^{-t}-z_1\overline{z_2}|^2 & |z_1|,|z_2|>1\end{cases}
\end{aligned} \tag{5.21}$$

Finally, performing integration by parts in (5.21) as explained at the end of Section 5.2.1, but for $t=0$ (e.g. we are in the same setting of [40]), we conclude the second equality (3.12), and so the proof of part ii) as well.

5.2.3 Wick Theorem Consider a product of the form (recall that for notational simplicity we only consider the case $a=0$)

$$\mathbb{E}\prod_{i\in[p]} I_{\eta_c}^T(f^{(i)},t_i). \tag{5.22}$$

By Proposition 5.1, each of these integrals can be decomposed as

$$I_{\eta_c}^T(f^{(i)},t_i) = I_1(f^{(i)},0,t_i) + I_2(f^{(i)},0,t_i) + \mathcal{O}(N^{-c}).$$

By (5.21) and the martingale representation theorem, we can write

$$I_2(f^{(i)},0,t_i) = \int_0^t \mathcal{C}_s^{1/2}\mathrm{d}\mathfrak{b}_s + \mathcal{O}(N^{-c}), \tag{5.23}$$

for some small fixed $c>0$, where $\mathfrak{b}_s\in\mathbb{R}^p$ is a standard Brownian motion, independent of the $I_1(f^{(i)},0,t_i)$'s, and $\mathcal{C}_s$ is a $p\times p$ matrix such that

$$(\mathcal{C}_s)_{ij} = \frac{V_s\big(f^{(i)}+f^{(j)}\big)-V_s\big(f^{(i)}-f^{(j)}\big)}{4}, \qquad V_s(f) := \text{integrand in the second line of rhs. (5.21).}$$

This shows that, modulo a negligible error N^{-c}, the $I_2(f^{(i)},0,t_i)$ are jointly a multivariate Gaussian random variable with covariance $\int_0^t(\mathcal{C}_s)_{ij}\mathrm{d}s$.

Now a Wick theorem as in (3.9) for the product (5.22) easily follows. In fact, (5.23) readily implies a Wick theorem for $I_2(f^{(i)},0,t_i)$; we are thus left only with products of the $I_1(f^{(i)},0,t_i)$'s . Then, by [40, Proposition 3.3], we also conclude that the $I_1(f^{(i)},0,t_i)$'s satisfy a Wick theorem as they consists of product of resolvents evaluated at the same time. This concludes the proof of of Proposition 3.3.

5.3 *Proof of Corollary 2.11.* From Theorem 2.3, for any compactly supported F, G we expect

$$\mathbb{E}\big[L_N(F_{v,a}, s_a)L_N(G_{v,a}, t_a)\big] = \Gamma_v(F, G, t-s)\cdot(1+\mathrm{o}(1)) \tag{5.24}$$

as $N\to\infty$. Strictly speaking this convergence of the covariance does not follow from Theorem 2.3, which states convergence in distribution. However, an inspection of the proof (which proceeds by asymptotics of the moments) in the special equilibrium case immediately shows that (5.24) holds, notably because (3.6) is correct at equilibrium, the matrix entries having a smooth density and Wegner estimates applying easily.

Second, we will need a slight generalization of (5.24) to F, G non-necessarily with compact support, as it will be applied to $F = \frac{1}{2\pi} f_{v,a} * \log, G = \frac{1}{2\pi} g_{v,a} * \log$, with f, g with compact support. Again, this generalization poses no problem: The starting point of the proof, Equation (3.2), can be directly replaced with

$$\sum_i F(\sigma_i) = \frac{\mathrm{i}}{4\pi}\int_{\mathbb{C}} f_{v,a}(z)\int_0^\infty \mathrm{Tr}\big[G^z(\mathrm{i}\eta)\big]\,\mathrm{d}\eta\mathrm{d}z$$

and the remainder of the proof is then unchanged.

In sum, our starting point will be that for any $a>0$, $f,g\in H_0^2(\mathbb{C})$, there exists a $c>0$ such that uniformly in s,t in compact sets we have (remember $c_v = 1-|v|^2$)

$$\mathbb{E}\big[L_N(F, s_a)L_N(G, t_a)\big] = -\frac{1}{16\pi^2}\int_{\mathbb{C}^2} f_{v,a}(z)g_{v,a}(w)\log\left(c_v|t_a-s_a|+|z-w|^2\right)\mathrm{d}z\mathrm{d}w\cdot(1+\mathrm{O}(N^{-c})), \tag{5.25}$$

where $F(z) = \frac{1}{2\pi}\int f_{v,a}(z-w)\log|w|\mathrm{d}w$ and $G(z)$ is defined similarly.

On the other hand, from [22, Proposition A.1] we have $\langle M_k, M_k\rangle = 0$ so that $\langle \mathrm{Re}M_k, \mathrm{Im}M_k\rangle = 0$ and $\langle \mathrm{Re}M_k\rangle = \langle\mathrm{Im}M_k\rangle$. The Itô formula therefore gives

$$\mathrm{d}F(\sigma_k(t)) = \nabla F\cdot\mathrm{d}M_k - \frac{1}{2}\nabla F\cdot\sigma_k\mathrm{d}t + \frac{1}{4}\Delta F\,\mathrm{d}\langle M_k\rangle = \nabla F\cdot\mathrm{d}M_k - \frac{1}{2}\nabla F\cdot\sigma_k\mathrm{d}t + \frac{f}{4}\frac{\mathscr{O}_{kk}}{N}\mathrm{d}t. \tag{5.26}$$

Let $t' = t_1+s_2-s_1$. The difference $\mathbb{E}\big[L_N(F,s_2)L_N(G,t')\big] - \mathbb{E}\big[L_N(F,s_2)L_N(G,t_1)\big]$ can be written in two manners from (5.25) and (5.26), which gives

$$\begin{aligned}&\int_{t_1}^{t'}\mathbb{E}\left[L_N(F,s_2)\sum_k\Big(-\frac{1}{2}\nabla G(\sigma_k(t))\cdot\sigma_k(t) + \frac{g(\sigma_k(t))}{4}\frac{\mathscr{O}_{kk}(t)}{N}\Big)\right]\mathrm{d}t\\ &= -\int_{t_1}^{t'}\frac{1}{16\pi^2}\int_{\mathbb{C}^2} f_{v,a}(z)g_{v,a}(w)\frac{c_v}{c_v|t-s_2|+|z-w|^2}\,\mathrm{d}z\mathrm{d}w\mathrm{d}t + \mathrm{O}(N^{-4a-c+\varepsilon})\end{aligned} \tag{5.27}$$

where we have used that $\int_{\mathbb{C}^2} f_{v,a}(z)g_{v,a}(w)\log\left(c_v|t_a-s_a|+|z-w|^2\right)\mathrm{d}z\mathrm{d}w = \mathrm{O}(N^{-4a+\varepsilon})$ for any $\varepsilon>0$, and that the expectation of the term containing $\mathrm{d}M_k$ vanishes.

We now wish to remove the contribution from the gradient term in (5.27). For this we rely on the following rigidity estimates: With very high probability we have

$$|L_N(F,s_2)| \leqslant N^{-2a+\varepsilon}, \qquad \left|\sum_k \nabla G(\sigma_k)\cdot\sigma_k - \mathbb{E}\sum_k\nabla G(\sigma_k)\cdot\sigma_k\right| \leqslant N^{-a+\varepsilon}. \tag{5.28}$$

Indeed, consider a partition of the the constant on $\mathbb{C}$: $1 = \sum_{n\geqslant 1}\chi_n$, where χ_n is supported on $\{|z|\leqslant 2^i\}$ and $\|\chi_n^{(k)}\|_\infty \leqslant C_k$ for all n. We now define $F_n(z) = F(z)\chi_n(N^a(z-v))$. On the support of f we have $F = \mathrm{O}(N^{-2a+\varepsilon})$, and more generally $\|F_n^{(k)}\|_\infty \leqslant \widetilde{C}_k N^{-2a+\varepsilon}N^{ka}$. Together with [28, Theorem 1.2], this implies $\mathbb{P}(|L_N(F_n,s_2)|\geqslant N^{-2a+\varepsilon}) = \mathrm{O}(N^{-D})$ for any fixed $\varepsilon, D>0$, and $n = \mathrm{O}(\log N)$. With a union bound we conclude $\mathbb{P}(|L_N(F,s_2)|\geqslant N^{-2a+\varepsilon}) = \mathrm{O}(N^{-D})$, and finaly by taking large moments and Hölder's inequality this implies the bound on the left-hand side of (5.28). The right-hand side of (5.28) follows from a similar dyadic decomposition, as we have $\nabla G = \mathrm{O}(N^{-a+\varepsilon})$ on the support of g, which is the most singular part of the test function.

In sum, by rigidity of the spectrum we have proved

$$\int_{t_1}^{t'}\mathbb{E}\left[L_N(F,s_2)\sum_k\nabla G(\sigma_k(t))\cdot\sigma_k(t)\right]\mathrm{d}t = \mathrm{O}(N^{-5a+\varepsilon}),$$

which together with 5.27 gives

$$\int_{t_1}^{t'} \mathbb{E}\left[L_N(F,s_2)\sum_k \frac{g(\sigma_k(t))}{4}\frac{\mathscr{O}_{kk}(t)}{N}\right]\mathrm{d}t = -\int_{t_1}^{t'}\frac{1}{16\pi^2}\int_{\mathbb{C}^2} f_{v,a}(z)g_{v,a}(w)\frac{c_v}{c_v|t-s_2|+|z-w|^2}\,\mathrm{d}z\mathrm{d}w\mathrm{d}t+\mathrm{O}(N^{-4a-c+\varepsilon}).$$

By reversibility at equilibrium, we have

$$\mathbb{E}\left[L_N(F,s_2)\sum_k \frac{g(\sigma_k(t))}{4}\frac{\mathscr{O}_{kk}(t)}{N}\right] = \mathbb{E}\left[L_N(F,t_1)\sum_k \frac{g(\sigma_k(s_2+t_1-t))}{4}\frac{\mathscr{O}_{kk}(s_2+t_1-t)}{N}\right],$$

so that we have proved

$$\int_{s_1}^{s_2} \mathbb{E}\left[L_N(F,t_1)\sum_k \frac{g(\sigma_k(s))}{4}\frac{\mathscr{O}_{kk}(s)}{N}\right]\mathrm{d}s = -\int_{s_1}^{s_2}\frac{1}{16\pi^2}\int_{\mathbb{C}^2} f_{v,a}(z)g_{v,a}(w)\frac{c_v}{c_v|t_1-s|+|z-w|^2}\,\mathrm{d}z\mathrm{d}w\mathrm{d}s+\mathrm{O}(N^{-4a-c+\varepsilon}).$$

By subtracting the analogous formula with t_1 replaced with t_2 we have

$$\begin{aligned}\int_{s_1}^{s_2} \mathbb{E}&\left[(L_N(F,t_2)-L_N(F,t_1))\sum_k \frac{g(\sigma_k(s))}{4}\frac{\mathscr{O}_{kk}(s)}{N}\right]\mathrm{d}s\\ &= -\int_{s_1}^{s_2}\frac{1}{16\pi^2}\int_{\mathbb{C}^2} f_{v,a}(z)g_{v,a}(w)\int_{t_1}^{t_2}\partial_t\frac{c_v}{c_v|t-s|+|z-w|^2}\,\mathrm{d}z\mathrm{d}w\mathrm{d}s\mathrm{d}t+\mathrm{O}(N^{-4a-c+\varepsilon}).\end{aligned}$$

With (5.26) to evaluate $L_N(F,t_2)-L_N(F,t_1)$ we obtain (2.9). □

5.4 Proof of Proposition 2.10. Note first that from Remark 2.5 we have (remember $s<t$ here) $\mathbb{E}[L(g,s)L(f,t)] = \mathbb{E}[L(g,s)L(Q_{t-s}f,s)]$ for any $g\in\mathscr{C}^\infty$, so that

$$\mathbb{E}[L(f,t)\mid\Sigma_s] = L(Q_{t-s}f,s)$$

almost surely. In a proof by contradiction, assume that $\mathbb{P}\left(\mathbb{E}\left[L(f,t)\mid\Sigma_{s-}\right]=\mathbb{E}\left[L(f,t)\mid\Sigma_s\right]\right)=1$ for any $f\in\mathscr{C}^\infty$, and let $u<s$. Together with $\Sigma_u\subset\Sigma_{s-}$ and the above equation thus gives

$$\begin{aligned}L(Q_{t-u}f,u) = \mathbb{E}[L(f,t)\mid\Sigma_u] = \mathbb{E}[\mathbb{E}[L(f,t)\mid\Sigma_{s-}]\mid\Sigma_u] &= \mathbb{E}[\mathbb{E}[L(f,t)\mid\Sigma_s]\mid\Sigma_u]\\ &= \mathbb{E}[L(Q_{t-s}f,s)\mid\Sigma_u] = L(Q_{s-u}Q_{t-s}f,u)\text{ a.s.}\end{aligned}$$

so that $L((Q_{t-u}-Q_{s-u}Q_{t-s})f,u)=0$ almost surely. If $L(h,u)=0$ a.s. then its variance vanishes, so $\int|\nabla h|^2=0$ and h is constant. Hence $(Q_{t-u}-Q_{s-u}Q_{t-s})f=0$ (Q_r preserves the mean). As this holds for any $f\in\mathscr{C}^\infty$, we have

$$q_{t-u}(z) = \int q_{s-u}(w)q_{t-s}(z-w)\mathrm{d}w. \tag{5.29}$$

A calculation gives, for any $\xi=(\xi_1,\xi_2)\in\mathbb{R}^2$ and defining $z\cdot\xi=\xi_1\mathrm{Re}z+\xi_2\mathrm{Im}z$,

$$\begin{aligned}\hat{q}_r(\xi) = \int q_r(z)e^{-\mathrm{i}z\cdot\xi}\mathrm{d}z &= \int_0^\infty\int_0^{2\pi} e^{\mathrm{i}x|\xi|\cos\theta}\frac{r}{\pi(r+x^2)^2}\mathrm{d}\theta x\mathrm{d}x\\ &= \int_0^\infty J_0(x|\xi|)\frac{2r}{(r+x^2)^2}\mathrm{d}\theta x\mathrm{d}x = |\xi|\sqrt{r}K_1(|\xi|\sqrt{r}),\end{aligned} \tag{5.30}$$

where J_α (resp. K_α) is the Bessel function of the first kind (resp. modified Bessel function of the second kind) with index α. Equation (5.29) therefore gives, for any $r_1,r_2,|\xi|>0$,

$$|\xi|\sqrt{r_1+r_2}K_1(|\xi|\sqrt{r_1+r_2}) = |\xi|^2\sqrt{r_1r_2}K_1(|\xi|\sqrt{r_1})K_1(|\xi|\sqrt{r_2}),$$

which can be easily proved to be wrong, e.g. by choosing $r_2=|\xi|=1$ and comparing the Taylor series for small r_1. □

5.5 Existence of the limiting Gaussian field. As explained in Remark 2.6, the existence of the limiting Gaussian field on mesoscopic scales is equivalent to $\sum_{1\leqslant i,\,j,\leqslant m}\Gamma(f_i,f_j,|t_i-t_j|)\geqslant 0$, i.e.

$$\sum_{1\leqslant i,\,j,\leqslant m}\int \partial_{\overline{z}}f_i(z)(q_{|t_i-t_j|}*\partial_z f_j)(z)\mathrm{d}z\geqslant 0.$$

In Fourier space we obtain the condition

$$\sum_{1\leqslant i,\,j,\leqslant m}\int |\xi|^2\hat{f}_i(\xi)\overline{\hat{f}_j}(\xi)\hat{q}_{|t_i-t_j|}(\xi)\mathrm{d}\xi\geqslant 0.$$

For the above we clearly only need to prove $\sum_{1\leqslant i,\,j,\leqslant m}v_i\overline{v_j}\hat{q}_{|t_i-t_j|}(\xi)\mathrm{d}\xi\geqslant 0$ for any complex ξ, v_i's and real t_i's. By (5.30) we have $\hat{q}_{|t_i-t_j|}(\xi)=|\xi|\cdot|t_i-t_j|^{1/2}K_1(|\xi|\cdot|t_i-t_j|^{1/2})$, so it is sufficient to show

$$\sum_{1\leqslant i,\,j,\leqslant m}z_i\overline{z_j}|s_i-s_j|^{1/2}K_1(|s_i-s_j|^{1/2})\geqslant 0 \tag{5.31}$$

for any complex z_i's and real s_i's. The function $g(t)=|t|^{1/2}K_1(|t|^{1/2})$ has non-negative Fourier transform:

$$\hat{g}(u)=\int_{\mathbb{R}}|t|^{1/2}K_1(|t|^{1/2})e^{-\mathrm{i}ut}\mathrm{d}t=\frac{1}{4u^2}\int_0^\infty\frac{v}{v^2+1}e^{-\frac{v}{4u}}\mathrm{d}v\geqslant 0,$$

which implies (5.31) and concludes the proof. □

A Stieltjes transform

In this section we recall various results concerning Stieltjes transforms. To that end, fix a probability measure μ. We define the *Stieltjes transform* of μ to be the function $m=m_\mu:\mathbb{H}\to\mathbb{H}$ for any complex number $z\in\mathbb{H}$ setting

$$m(z)=\int_{-\infty}^{\infty}\frac{\mu(\mathrm{d}x)}{x-z}.$$

Here $\mathbb{H}$ denotes the upper half complex plane. We have the following estimates on the Stieltjes transform and its derivatives.

Lemma A.1. *Let $m(z)=m_\mu(z)$ be the Stieltjes transform of a probability measure μ. For any integer $p\geqslant 1$, we denote its p-th derivative by $m^{(p)}(z)$. Then*

$$|m(z)|\leqslant\frac{1}{\mathrm{dist}(z,\mathrm{supp}(\mu))},\qquad |m'(z)|\leqslant\frac{\mathrm{Im}[m(z)]}{\mathrm{Im}[z]},\quad |m^{(p)}(z)|\leqslant\frac{p!\mathrm{Im}[m(z)]}{\mathrm{dist}(z,\mathrm{supp}(\mu))^{p-1}\mathrm{Im}[z]}.$$

Proof. The Stieltjes transform is given by

$$|m(z)|=\left|\int_{\mathbb{R}}\frac{\mathrm{d}\mu(x)}{x-z}\right|\leqslant\int_{\mathbb{R}}\frac{\mathrm{d}\mu(x)}{|z-x|}\leqslant\frac{1}{\mathrm{dist}(z,\mathrm{supp}(\mu))}.$$

For the second statement of (A.2), we have

$$|m'(z)|=\left|\int_{\mathbb{R}}\frac{\mathrm{d}\mu(x)}{(x-z)^2}\right|\leqslant\int_{\mathbb{R}}\frac{\mathrm{d}\mu(x)}{|z-x|^2}=\frac{1}{\mathrm{Im}[z]}\int_{\mathbb{R}}\frac{\mathrm{Im}[z]\mathrm{d}\mu(x)}{|z-x|^2}=\frac{\mathrm{Im}[m(z)]}{\mathrm{Im}[z]}.$$

For the last statement of (A.2), we have

$$|m^{(p)}(z)|=\left|\int_{\mathbb{R}}\frac{p!\mathrm{d}\mu(x)}{(x-z)^{p+1}}\right|\leqslant\int_{\mathbb{R}}\frac{p!\mathrm{d}\mu(x)}{|z-x|^{p+1}}\leqslant\int_{\mathbb{R}}\frac{p!\mathrm{d}\mu(x)}{|z-x|^2\mathrm{dist}(z,\mathrm{supp}(\mu))^{p-1}}=\frac{p!\mathrm{Im}[m(z)]}{\mathrm{dist}(z,\mathrm{supp}(\mu))^{p-1}\mathrm{Im}[z]},$$

concluding the proof. □

Lemma A.2. *Let $m(z) = m_\mu(z)$ be the Stieltjes transform of a probability measure μ. For any small $\eta > 0$, and two complex numbers w, z with $\eta/C \leqslant \mathrm{Im}[w], \mathrm{Im}[z] \leqslant C\eta$, and $|w - z| \leqslant C\eta$, we have*

$$\mathrm{Im}[m(w)] \asymp \mathrm{Im}[m(z)]. \tag{A.1}$$

If we further assume that there exists a control parameter $0 < \phi \leqslant \mathrm{Im}[m(z)]$, such that

$$|m(w) - \widetilde{m}(w)|, |m(z) - \widetilde{m}(z)| \leqslant \phi,$$

and $|w - z| \asymp \eta\sqrt{\phi/\mathrm{Im}[m(z)]}$, then

$$|m'(z)| \lesssim \max |\widetilde{m}'(u)| + \frac{\sqrt{\phi \mathrm{Im}[m(z)]}}{\eta}. \tag{A.2}$$

Proof. For the first statement (A.1), (A.2) gives that

$$|\partial_u \mathrm{Im}[m(u)]| \leqslant |m'(u)| \leqslant |\frac{\mathrm{Im}[m(u)]}{\mathrm{Im}[u]},$$

which gives that $|\partial_u \log \mathrm{Im}[m(u)]| \leqslant 1/\mathrm{Im}[u]$. By integrating it from z to w and from w to z, we conclude that

$$\log \frac{\mathrm{Im}[m(w)]}{\mathrm{Im}[m(z)]} \lesssim 1, \quad \log \frac{\mathrm{Im}[m(z)]}{\mathrm{Im}[m(w)]} \lesssim 1.$$

Then (A.1) follows.

For the second statement (A.2), by Taylor expansion

$$|m(w) - m(z) - (w - z)m'(z)| \leqslant \frac{1}{2}|w - z|^2 \sup |m''(u)|$$

where the supremum is on $[z, w]$. Then we get

$$\begin{aligned}|m'(z)| &\leqslant \frac{|m(w) - m(z)|}{|w - z|} + \frac{1}{2}|w - z| \sup |m''(u)| \lesssim \max |\widetilde{m}'(u)| + \frac{\phi}{|w - z|} + |w - z|\frac{\mathrm{Im}[m(z)]}{\mathrm{Im}[z]^2} \\ &\lesssim \max |\widetilde{m}'(u)| + \frac{\sqrt{\phi \mathrm{Im}[m(z)]}}{\mathrm{Im}[z]},\end{aligned}$$

where we used that $|w - z| \asymp \mathrm{Im}[z]\sqrt{\phi/\mathrm{Im}[m(z)]}$. ☐

B Well-posedness

The purpose of this short appendix is to prove that the Equation (4.13) is well-posed. More generally, we consider the stochastic differential equation

$$\mathrm{d}x_i(t) = \frac{\mathrm{d}b_i(t)}{\sqrt{2N}} + \frac{1}{2N} \sum_{j \neq i} \frac{1}{x_i(t) - x_j(t)} \mathrm{d}t, \quad 1 \leqslant |i| \leqslant N, \tag{B.1}$$

where $0 < x_1(0) < \cdots < x_N(0)$, $x_{-i}(0) = -x_i(0)$, $(i \geqslant 1)$, $(b_i)_{1 \leqslant i \leqslant N}$ is a collection of continuous martingales, and $b_{-i}(t) = -b_i(t)$, $(i \geqslant 1)$. Note that the drift in (B.1) includes a repulsive term between x_i and x_{-i}, equal to $\frac{1}{2N}\frac{1}{2x_i}$.

Proposition B.1. *If $\frac{\mathrm{d}\langle b_i \rangle_t}{\mathrm{d}t} \leqslant 1$ for any $i \geqslant 1, t \geqslant 0$, then existence and strong uniqueness hold for the stochastic differential equation (B.1).*

Note that this proposition does not require indepedence of the b_i's, or bounds on the off-diagonal brackets.

Proof. For any $\varepsilon > 0$, let $\tau_\varepsilon = \inf\{t \geqslant 0 : |x_i(t) - x_j(t)| = \varepsilon \text{ for some } i \neq j\}$ and $\mu_C = \inf\{t \geqslant 0 : |x_i(t)| = C \text{ for some } i\}$. It is well-known that existence and strong uniqueness hold if one proves $\tau_\varepsilon \to \infty$ a.s. as $\varepsilon \to 0$, and $\mu_C \to \infty$ as $C \to \infty$, see e.g. [83].

Still following [83], consider $\phi(\mathbf{x}) = -\sum^*_{i,j} \log|x_i - x_j|$, where here and in the following $\sum^*_{i,j}$ (resp. $\sum^*_i$) means summation over couples $-N \leqslant i \neq j \leqslant N$ such that $i, j \neq 0$ (resp. $-N \leqslant i \leqslant N$ such that $i \neq 0$). Let $\mathbf{y}_t := \mathbf{x}_{t\wedge\tau_\varepsilon}$, then Itô's formula gives

$$\begin{aligned}
\mathrm{d}\phi(\mathbf{y}_t) &= \sum_i^* \partial_i\phi(\mathbf{y}_t)\mathrm{d}y_i(t) + \frac{1}{2}\sum_{1\leqslant|i|,|j|\leqslant N} \partial_{i,j}\phi(\mathbf{y}_t)\mathrm{d}\langle y_i, y_j\rangle_t \\
&= -\sum_i^* \left(\sum_{j:j\neq i}^* \frac{1}{y_i - y_j}\right)\cdot\left(\frac{\mathrm{d}b_i}{\sqrt{2N}} + \frac{1}{2N}\sum_{j:j\neq i}^* \frac{\mathrm{d}t}{y_i - y_j}\right) - \frac{1}{2}\frac{1}{2N}\sum_{i,j}^* \frac{\mathrm{d}\langle b_i, b_j\rangle}{(y_i - y_j)^2} + \frac{1}{2}\frac{1}{2N}\sum_i^*\sum_{j:j\neq i}^* \frac{\mathrm{d}\langle b_i\rangle}{(y_i - y_j)^2} \\
&= \mathrm{d}M_t + \frac{1}{2N}\left(-\sum_i^*\left(\sum_{j:j\neq i}^* \frac{1}{y_i - y_j}\right)^2 \mathrm{d}t - \frac{1}{2}\sum_{i,j}^* \frac{\mathrm{d}\langle b_i, b_j\rangle}{(y_i - y_j)^2} + \frac{1}{2}\sum_{i,j}^* \frac{\mathrm{d}\langle b_i\rangle}{(y_i - y_j)^2}\right),
\end{aligned}$$

where M is a local martingale. Note that by Cauchy-Schwarz we have

$$|\mathrm{d}\langle b_i, b_j\rangle/\mathrm{d}t| \leqslant (\mathrm{d}\langle b_i\rangle/\mathrm{d}t)^{1/2}(\mathrm{d}\langle b_j\rangle/\mathrm{d}t)^{1/2} \leqslant 1,$$

so that the above parenthesis is smaller than ($\sum^*_{i,j,k}$ below means summation over all $-N \leqslant i, j, k \leqslant N$ such that i, j, k are all distinct and non-zero)

$$-\sum_i^*\left(\sum_{j:j\neq i}^* \frac{1}{y_i - y_j}\right)^2 + \sum_{i,j}^* \frac{1}{(y_i - y_j)^2} = -\sum_{i,j,k}^* \frac{1}{(y_i - y_j)(y_i - y_k)} = 0,$$

where the last equality relies on $\frac{1}{(a-b)(a-c)} + \frac{1}{(b-a)(b-c)} + \frac{1}{(c-a)(c-b)} = 0$. This shows that $(\phi(\mathbf{y}_t))_{t\geqslant 0}$ is a supermartingale.

Let $A = \{\tau_\varepsilon \leqslant \mu_C, \tau_\varepsilon \leqslant T\}$ where ε, C are chosen so that $\min_{i\neq j}|x_i(0) - x_j(0)| \geqslant \varepsilon$, and $\max_i |x_i(0)| \leqslant C$. Then, using the supermartingale property for the first inequality below, we have

$$\begin{aligned}
\mathbb{E}[\phi(\mathbf{y}_0)] &\geqslant \mathbb{E}[\phi(\mathbf{y}_{T\wedge\mu_C})] = \mathbb{E}[\phi(\mathbf{y}_{T\wedge\mu_C})\mathbb{1}_A] + \mathbb{E}[\phi(\mathbf{y}_{T\wedge\mu_C})\mathbb{1}_{A^c}] \\
&\geqslant (2(-\log\varepsilon) + (2N(2N-1) - 2)(-\log(2C)))\mathbb{P}(A) + 2N(2N-1)(-\log(2C))(1 - \mathbb{P}(A)).
\end{aligned}$$

This implies $\mathbb{P}(A) \to 0$ as $\varepsilon \to 0$, so $\mathbb{P}(\tau_0 \leqslant \mu_C, \tau_0 \leqslant T) = 0$. The proof will therefore be complete if $\mu_C \to \infty$ a.s. as $C \to \infty$. Let $Z_t = \sum_i z_i(t)^2$ where $z_i(t) = x_i(t \wedge \mu_C)$, then Itô's formula gives

$$\mathrm{d}Z_t = 2\sum_i z_i \mathrm{d}z_i + \sum_i \mathrm{d}\langle z_i\rangle = 2\sum_i z_i \frac{\mathrm{d}b_i}{\sqrt{2N}} + N(2N-1) + \frac{1}{2N}\sum_i \mathrm{d}\langle b_i\rangle,$$

where we have used $\sum^*_{i,j} \frac{z_i}{z_i - z_j} = N(2N-1)$. With our assumption on $\frac{\mathrm{d}\langle b_i\rangle_t}{\mathrm{d}t}$, this implies $\mathbb{E}(Z_t) \leqslant Z_0 + at$ for any t and some constant a depending on N. We now define the martingale $M_t = \int_0^t 2\sum_i z_i \frac{\mathrm{d}b_i}{\sqrt{2N}}$. With the Burkholder-Davis-Gundy inequality, we have

$$\mathbb{E}[\sup_{0\leqslant u\leqslant t}|M_u|^2] \leqslant C_1\mathbb{E}[\langle M\rangle_t] \leqslant C_2\int_0^t \sum_{i,j}\mathbb{E}[|z_i z_j \mathrm{d}\langle b_i, b_j\rangle|] \leqslant C_3\int_0^t \sum_{i,j}\mathbb{E}[|z_i|^2 + |z_j|^2]\mathrm{d}t \leqslant C_4\int_0^t \mathbb{E}[Z_t]\mathrm{d}t \leqslant C_5 + C_6 t$$

where C_1 is universal and the other constants C_i above and below depend only on n. Note that $|Z_u| \leqslant |Z_0| + |M_u| + C_7 u$, so from the above we obtain $\mathbb{E}[\sup_{0\leqslant u\leqslant t}|Z_u|^2] \leqslant C_8 + C_9 t$, and in particular

$$\mathbb{P}(\mu_C \leqslant t) = \mathbb{P}(\max_{0\leqslant u\leqslant t}|x_i(u)| \geqslant C) = \mathbb{P}(\max_{0\leqslant u\leqslant t}|z_i(u)| \geqslant C) \leqslant \frac{C_8 + C_9 t}{C^2}.$$

Taking $C \to \infty$ we obtain $\mathbb{P}(\mu_\infty \leqslant t) = 0$ for any fixed $t \geqslant 0$, so that $\mathbb{P}(\mu_\infty = \infty) = 1$. This concludes the proof. $\square$

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