The Eigenvector Moment Flow and local Quantum Unique Ergodicity

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We prove that the distribution of eigenvectors of generalized Wigner matrices is universal both in the bulk and at the edge. This includes a probabilistic version of local quantum unique ergodicity and asymptotic normality of the eigenvector entries. The proof relies on analyzing the eigenvector flow under the Dyson Brownian motion. The key new ideas are: (1) the introduction of the eigenvector moment flow, a multi-particle random walk in a random environment, (2) an effective estimate on the regularity of this flow based on maximum principle and (3) optimal finite speed of propagation holds for the eigenvector moment flow with very high probability.

Keywords: Universality, Quantum unique ergodicity, Eigenvector moment flow.

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1 Introduction

Wigner envisioned that the laws of the eigenvalues of large random matrices are new paradigms for universal statistics of large correlated quantum systems. Although this vision has not been proved for any truly interacting quantum system, it is generally considered to be valid for a wide range of models. For example, the quantum chaos conjecture by Bohigas-Giannoni-Schmit [6] asserts that the eigenvalue statistics of the Laplace operator on a domain or manifold are given by the random matrix statistics, provided that the corresponding classical dynamics are chaotic. Similarly, one expects that the eigenvalue statistics of random Schrödinger operators (Anderson tight binding models) are given by the random matrix statistics in the delocalization regime. Unfortunately, both conjectures are far beyond the reach of the current mathematical technology.

In Wigner's original theory, the eigenvector behaviour plays no role. As suggested by the Anderson model, random matrix statistics coincide with delocalization of eigenvectors. A strong notion of delocalization, at least in terms of "flatness of the eigenfunctions", is the quantum ergodicity. For the Laplacian on a negative curved compact Riemannian manifold, Shnirel'man [31], Colin de Verdière [11] and Zelditch [36] proved that quantum ergodicity holds. More precisely, let $(\psi_k)_{k\geqslant 1}$ denote an orthonormal basis of eigenfunctions of the Laplace-Beltrami operator, associated with increasing eigenvalues, on a negative curved manifold $\mathcal{M}$ (or more generally, assume only that the geodesic flow of $\mathcal{M}$ is ergodic) with volume measure μ. Then, for any open set $A\subset\mathcal{M}$, one has

$$\lim_{\lambda\to\infty}\frac{1}{N(\lambda)}\sum_{j:\lambda_j\leqslant\lambda}\Big|\int_A|\psi_j(x)|^2\mu(\mathrm{d}x)-\int_A\mu(\mathrm{d}x)\Big|^2=0,$$

where $N(\lambda)=|\{j:\lambda_j\leqslant\lambda\}|$. Quantum ergodicity was also proved for d-regular graphs under certain assumptions on the injectivity radius and spectral gap of the adjacency matrices [3]. Random graphs are considered a good paradigm for many ideas related to quantum chaos [24].

An even stronger notion of delocalization is the quantum unique ergodicity conjecture (QUE) proposed by Rudnick-Sarnak [30], i.e., for any negatively curved compact Riemannian manifold $\mathcal{M}$, the eigenstates become equidistributed with respect to the volume measure μ: for any open $A\subset\mathcal{M}$ we have

$$\int_A|\psi_k(x)|^2\mu(\mathrm{d}x)\underset{k\to\infty}{\longrightarrow}\int_A\mu(\mathrm{d}x). \tag{1.1}$$

Some numerical evidence exists for both eigenvalue statistics and the QUE, but a proper understanding of the semiclassical limit of chaotic systems is still missing. One case for which QUE was rigorously proved concerns arithmetic surfaces, thanks to tools from number theory and ergodic theory on homogeneous spaces [20, 21, 26]. For results in the case of general compact Riemannian manifolds whose geodesic flow is Anosov, see [2].

A major class of matrices for which one expects that Wigner's vision holds is the Wigner matrices, i.e., random matrices with matrix elements distributed by identical mean-zero random variables. For this class of matrices, the Wigner-Dyson-Mehta conjecture states that the local statistics are independent of the laws of the matrix elements and depend only on the symmetry class. This conjecture was recently solved for an even more general class: the generalized Wigner matrices for which the distributions of matrix entries can vary and have different variances. (See [17, 18] and [16] for a review. For earlier results on this conjecture for Wigner matrices, see [14, 34] for the bulk of the spectrum and [19, 32, 33] for the edge). One key ingredient of the method initiated in [14] proceeds by interpolation between Wigner and Gaussian ensembles through Dyson Brownian motion, a matrix process that induces an autonomous evolution of eigenvalues. The fundamental conjecture for Dyson Brownian motion, the Dyson conjecture, states that the time to local equilibrium is of order $t \gtrsim 1/N$, where N is the size of the matrix. This conjecture was resolved in [19] (see [14] for the earlier results) and is the underlying reason for the universality.

Concerning the eigenvectors distribution, complete delocalization was proved in [19] for generalized Wigner matrices in the following sense : with very high probability

$$\max |u_i(\alpha)| \leqslant \frac{(\log N)^{C \log \log N}}{\sqrt{N}},$$

where C is a fixed constant and the maximum ranges over all coordinates α of the L^2-normalized eigenvectors, $u_1, \dots, u_N$ (a stronger estimate was obtained for Wigner matrices in [13], see also [8] for a delocalization bound for the Laplacian on deterministic regular graphs). Although this bound prevents concentration of eigenstates onto a set of size less than $N(\log N)^{-C \log \log N}$, it does not imply the "complete flatness" of type (1.1). In fact, if the eigenvectors are distributed by the Haar measure on the orthogonal group, the weak convergence

$$\sqrt{N} u_i(\alpha) \to \mathscr{N} \tag{1.2}$$

holds, where $\mathscr{N}$ is a standard Gaussian random variable and the eigenvector components are asymptotically independent. Since the eigenvectors of GOE are distributed by the Haar measure on the orthogonal group, this asymptotic normality (1.2) holds for GOE (and a similar statement holds for GUE). For Wigner ensembles, by comparing with GOE, this property was proved for eigenvectors in the bulk by Knowles-Yin and Tao-Vu [22, 35] under the condition that the first four moments of the matrix elements of the Wigner ensembles match those of the standard normal distribution. For eigenvectors near the edges, the matching condition can be reduced to only the first two moments [22].

In this paper, we develop a completely new method to show that this asymptotic normality (1.2) and independence of eigenvector components hold for generalized Wigner matrices without any moment matching condition. In particular, even the second moments are allowed to vary as long as the matrix stays inside the generalized Wigner class. From the law of large numbers of independent random variables, this implies the *local quantum unique ergodicity*, to be specified below, with high probability. In fact, we will prove a stronger form of asymptotic normality in the sense that any projection of the eigenvector is asymptotically normal, see Theorem 1.2. This can be viewed as the eigenvector universality for the generalized Wigner ensembles.

The key idea in this new approach is to analyze the "Dyson eigenvector flow". More precisely, the Dyson Brownian motion is induced by the dynamics in which matrix elements undergo independent Brownian motions. The same dynamics on matrix elements yield a flow on the eigenvectors. This eigenvector flow, which we will call the *Dyson eigenvector flow*, was computed in the context of Brownian motion on ellipsoids [28], real Wishart processes [9], and for GOE/GUE in [4] (see also [1]). This flow is a diffusion process on a compact Lie group ($O(N)$ or $U(N)$) endowed with a Riemannian metric. This diffusion process roughly speaking can be described as follows. We first randomly choose two eigenvectors, u_i and u_j. Then we randomly rotate these two vectors on the circle spanned by them with a rate $(\lambda_i - \lambda_j)^{-2}$ depending on the eigenvalues. Thus the eigenvector flow depends on the eigenvalue dynamics. If we freeze the eigenvalue flow, the eigenvector flow is a diffusion with time dependent singular coefficients depending on the eigenvalues.

Due to its complicated structure, the Dyson eigenvector flow has never been analyzed. Our key observation is that the dynamics of the moments of the eigenvector entries can be viewed as a multi-particle random walk in a random environment. The number of particles of this flow is one half of the degree of polynomials in the eigenvector entries, and the (dynamic) random environment is given by jump rates depending on the eigenvalues. We shall call this flow the *eigenvector moment flow*. If there is only one particle, this flow is the random walk with the random jump rate $(\lambda_i - \lambda_j)^{-2}$ between two integer locations i and j. This one dimensional random walk process was analyzed locally in [15] for the purpose of the single gap universality between eigenvalues. An important result of [15] is the Hölder regularity of the solutions. In higher dimensions, the jump rates depend on the locations of nearby particles and the flow is not a simple tensor product of the one dimensional process. Fortunately, we find that this flow is reversible with respect to an explicit equilibrium measure. The Hölder regularity argument in [15] can be extended to any dimension to prove that the solutions of the moment flow are locally Hölder continuous. From this result and the local semicircle law (more precisely, the isotropic local semicircle law proved in [23] and [5]), one can obtain that the bulk eigenvectors generated by a Dyson eigenvector flow satisfy local quantum unique ergodicity, and the law of the entries of the eigenvectors are Gaussian.

Instead of showing the Hölder regularity, we will directly prove that the solution to the eigenvector moment flow converges to a constant. This proof is based on a maximum principle for parabolic differential equations and the local isotropic law [5] previously mentioned. It yields the convergence of the eigenvector moment flow to a constant for $t \gtrsim N^{-1/4}$ with explicit error bound. This immediately implies that all eigenvectors (in the bulk and at the edge) generated by a Dyson eigenvector flow satisfy local quantum unique ergodicity, and the law of the entries of the eigenvectors are Gaussian.

The time to equilibrium $t \gtrsim N^{-1/4}$ mentioned above is not optimal and the correct scaling of relaxation to equilibrium is $t \sim N^{-1}$ in the bulk, similar to Dyson's conjecture for relaxation of bulk eigenvalues to local equilibrium. In other words, we expect that Dyson's conjecture can be extended to the eigenvector flow bulk as well. We will give a positive answer to this question in Theorem 7.1. A key tool in proving this theorem is a finite speed of propagation estimate for the eigenvector moment flow. An estimate of this type was first proved in [15, Section 9.6], but it requires a difficult level repulsion estimate. In Section 6, we will prove an optimal finite speed of propagation estimate without using any level repulsion estimate.

In order to prove that the eigenvectors of the original matrix ensemble satisfy quantum ergodicity, it remains to approximate the Wigner matrices by Gaussian convoluted ones, i.e., matrices that are a small time solution to the Dyson Brownian motion. We invoke the Green function comparison theorem in a version similar to the one stated in [22]. For bulk eigenvectors, we can remove this small Gaussian component by a continuity principle instead of the Green function comparison theorem: we will show that the Dyson Brownian motion preserves the detailed behavior of eigenvalues and eigenvectors up to time $N^{-1/2}$ directly by using the Itô formula. This approach is much more direct and there is no need to construct moment matching matrices.

The eigenvector moment flow developed in this paper can be applied to other random matrix models. For example, the local quantum unique ergodicity holds for covariance matrices (for the associated flow and results, see Appendix C) and a certain class of Erdős-Rényi graphs. To avoid other technical issues, in this paper we only consider generalized Wigner matrices. Before stating the results and giving more details about the proof, we recall the definition of the considered ensemble.

Definition 1.1. *A generalized Wigner matrix H_N is an Hermitian or symmetric $N \times N$ matrix whose upper-triangular matrix elements $h_{ij} = \overline{h_{ji}}$, $i \leqslant j$, are independent random variables with mean zero and variance $\sigma_{ij}^2 = \mathbb{E}(|h_{ij}|^2)$ satisfying the following additional two conditions:*

(i) Normalization: for any $j \in [\![1, N]\!]$, $\sum_{i=1}^N \sigma_{ij}^2 = 1$.

(ii) Non-degeneracy: there exists a constant C, independent of N, such that $C^{-1}N^{-1} \leqslant \sigma_{ij}^2 \leqslant CN^{-1}$ for all $i, j \in [\![1, N]\!]$. In the Hermitian case, we furthermore assume that, for any $i < j$, $\mathbb{E}((\mathbf{h}_{ij})^* \mathbf{h}_{ij}) \geqslant cN^{-1}$ in the sense of inequality between 2×2 positive matrices, where $\mathbf{h}_{ij} = (\Re(h_{ij}), \Im(h_{ij}))$.

Moreover, we assume that all moments of the entries are finite: for any $p \in \mathbb{N}$ there exists a constant C_p such that for any i, j, N we have

$$\mathbb{E}(|\sqrt{N} h_{ij}|^p) < C_p. \tag{1.3}$$

In the following, $(u_i)_{i=1}^N$ denotes an orthonormal eigenbasis for H_N, a matrix from the (real or complex) generalized Wigner ensemble. The eigenvector u_i is associated with the eigenvalue λ_i, where $\lambda_1 \leqslant \ldots \leqslant \lambda_N$.

Theorem 1.2. *Let $(H_N)_{N \geqslant 1}$ be a sequence of generalized Wigner matrices. Then there is a $\delta > 0$ such that for any $m \in \mathbb{N}$, $I \subset \mathbb{T}_N := [\![1, N^{1/4}]\!] \cup [\![N^{1-\delta}, N - N^{1-\delta}]\!] \cup [\![N - N^{1/4}, N]\!]$ with $|I| = m$ and for any unit vector $\mathbf{q}$ in $\mathbb{R}^N$, we have*

$$\begin{aligned} &\sqrt{N}(|\langle \mathbf{q}, u_k \rangle|)_{k \in I} \to (|\mathscr{N}_j|)_{j=1}^m && \textit{in the symmetric case,} \\ &\sqrt{2N}(|\langle \mathbf{q}, u_k \rangle|)_{k \in I} \to (|\mathscr{N}_j^{(1)} + \mathrm{i}\mathscr{N}_j^{(2)}|)_{j=1}^m && \textit{in the Hermitian case,} \end{aligned} \tag{1.4}$$

in the sense of convergence of moments, where all $\mathscr{N}_j, \mathscr{N}_j^{(1)}, \mathscr{N}_j^{(2)}$, are independent standard Gaussian random variables. This convergence holds uniformly in I and $|\mathbf{q}| = 1$. More precisely, for any polynomial P in m variables, there exists $\varepsilon = \varepsilon(P) > 0$ such that for large enough N we have

$$\begin{aligned} &\sup_{I \subset \mathbb{T}_N, |I| = m, |\mathbf{q}| = 1} \left| \mathbb{E}\left(P\left(\left(N|\langle \mathbf{q}, u_k \rangle|^2 \right)_{k \in I} \right) \right) - \mathbb{E}\left(P\left((|\mathscr{N}_j|^2)_{j=1}^m \right) \right) \right| \leqslant N^{-\varepsilon}, \\ &\sup_{I \subset \mathbb{T}_N, |I| = m, |\mathbf{q}| = 1} \left| \mathbb{E}\left(P\left(\left(2N|\langle \mathbf{q}, u_k \rangle|^2 \right)_{k \in I} \right) \right) - \mathbb{E} P\left((|\mathscr{N}_j^{(1)}|^2 + |\mathscr{N}_j^{(2)}|^2)_{j=1}^m \right) \right| \leqslant N^{-\varepsilon}, \end{aligned} \tag{1.5}$$

respectively for the real and complex generalized Wigner ensembles.

The restriction on the eigenvector in the immediate regime $[\![N^{1/4}, N^{1-\delta}]\!]$ (and similarly for its reflection) was due to that near the edges the level repulsion estimate, Definition 5.1, (or the gap universality) was only written in the region $[\![1, N^{1/4}]\!]$ (see the discussion after Definition 5.1 for references regarding this matter). There is no doubt that these results can be extended to the the immediate regime with only minor modifications in the proofs. Here we state our theorem based on existing written results.

The normal convergence (1.4) was proved in [35] under the assumption that the entries of H_N have moments matching the standard Gaussian distribution up to order four, and if their distribution is symmetric (in particular the fifth moment vanishes).

This convergence of moments implies in particular joint weak convergence. Choosing $\mathbf{q}$ to be an element of the canonical basis, Theorem 1.2 implies in particular that any entry of an eigenvector is asymptotically normally distributed, modulo the (arbitrary) phase choice. Because the above convergence holds for any $|\mathbf{q}| = 1$, asymptotic joint normality of the eigenvector entries also holds. Since eigenvectors are defined only up to a phase, we define the equivalence relation $u \sim v$ if $u = \pm v$ in the symmetric case and $u = \lambda v$ for some $|\lambda| = 1$ in the Hermitian case.

Corollary 1.3 (Asymptotic normality of eigenvectors for generalized Wigner matrices)**.** *Let $(H_N)_{N \geqslant 1}$ be a sequence of generalized Wigner matrices, $\ell \in \mathbb{N}$. Then for any $k \in \mathbb{T}_N$ and $J \subset [\![1, N]\!]$ with $|J| = \ell$, we have*

$$\begin{aligned} &\sqrt{N}(u_k(\alpha))_{\alpha \in J} \to (\mathscr{N}_j)_{j=1}^\ell && \textit{for the real generalized Wigner ensemble,} \\ &\sqrt{2N}(u_k(\alpha))_{\alpha \in J} \to (\mathscr{N}_j^{(1)} + \mathrm{i}\mathscr{N}_j^{(2)})_{j=1}^\ell && \textit{for the complex generalized Wigner ensemble,} \end{aligned}$$

in the sense of convergence of moments modulo $\sim$, where all $\mathscr{N}_j, \mathscr{N}_j^{(1)}, \mathscr{N}_j^{(2)}$, are independent standard Gaussian variables. More precisely, for any polynomial P in ℓ variables (resp. Q in 2ℓ variables) there exists

ε depending on P (resp. Q) such that, for large enough N,

$$\sup_{\substack{J\subset [\![1,N]\!], |J|=\ell,\\ k\in\mathbb{T}_N}} \left|\mathbb{E}\left(P\left(\sqrt{N}(e^{\mathrm{i}\omega}u_k(\alpha))_{\alpha\in J}\right)\right) - \mathbb{E}P\left((\mathscr{N}_j)_{j=1}^{\ell}\right)\right| \leqslant N^{-\varepsilon}, \tag{1.6}$$

$$\sup_{\substack{J\subset [\![1,N]\!], |J|=\ell,\\ k\in\mathbb{T}_N}} \left|\mathbb{E}\left(Q\left(\sqrt{2N}(e^{\mathrm{i}\omega}u_k(\alpha), e^{-\mathrm{i}\omega}\overline{u_k(\alpha)})_{\alpha\in J}\right)\right) - \mathbb{E}Q\left((\mathscr{N}_j^{(1)}+\mathrm{i}\mathscr{N}_j^{(2)}, \mathscr{N}_j^{(1)}-\mathrm{i}\mathscr{N}_j^{(2)})_{j=1}^{\ell}\right)\right| \leqslant N^{-\varepsilon},$$

for the symmetric (resp. Hermitian) generalized Wigner ensembles. Here ω is independent of H_N and uniform on the binary set $\{0,\pi\}$ (resp. $(0,2\pi)$).

By characterizing the joint distribution of the entries of the eigenvectors, Theorem 1.2 and Corollary 1.3 imply that for any eigenvector a probabilistic equivalent of (1.1) holds. For $a_N : [\![1,N]\!] \to [-1,1]$ we denote $|a_N| = |\{1 \leqslant \alpha \leqslant N : a_N(\alpha) \neq 0\}|$ the size of the support of a_N, and $\langle u_k, a_N u_k\rangle = \sum |u_k(\alpha)|^2 a_N(\alpha)$.

Corollary 1.4 (Local quantum unique ergodicity for generalized Wigner matrices)**.** *Let $(H_N)_{N\geqslant 1}$ be a sequence of generalized (real or complex) Wigner matrices. Then there exists $\varepsilon > 0$ such that for any $\delta > 0$, there exists $C > 0$ such that the following holds: for any $(a_N)_{N\geqslant 1}$, $a_N : [\![1,N]\!] \to [-1,1]$ with $\sum_{\alpha=1}^N a_N(\alpha) = 0$ and $k \in \mathbb{T}_N$, we have*

$$\mathbb{P}\left(\left|\frac{N}{|a_N|}\langle u_k, a_N u_k\rangle\right| > \delta\right) \leqslant C\left(N^{-\varepsilon} + |a_N|^{-1}\right). \tag{1.7}$$

Under the condition that the first four moments of the matrix elements of the Wigner ensembles match those of the standard normal distribution, (1.7) can also be proved from the results in [22, 35]; the four moment matching were reduced to two moments for eigenvectors near the edges [22].

The quantum ergodicity for a class of sparse regular graphs was proved by Anantharaman-Le Masson [3], partly based on pseudo-differential calculus on graphs from [25]. The main result in [3] is for deterministic graphs, but for the purpose of this paper we only state its application to random graphs (see [3] for details and more general statements). If $u_1, \dots, u_N$ are the (L^2-normalized) eigenvectors of the discrete Laplacian of a uniformly chosen $(q+1)$-regular graph with N vertices, then for any fixed $\delta > 0$ we have, for any $q \geqslant 1$ fixed,

$$\mathbb{P}\left(\sharp\{k : |\langle u_k, a_N u_k\rangle| > \delta\} > \delta N\right) \underset{N\to\infty}{\longrightarrow} 0,$$

where a_N may be random (for instance, it may depend on the graph). The results in [3] were focused on very sparse deterministic regular graphs and are very different from our setting for generalized Wigner matrices.

Notice that our result (1.7) allows the test function to have a very small support and it is valid for any k. This means that eigenvectors are flat even in "microscopic scales". However, the equation (1.7) does not imply that all eigenvectors are completely flat simultaneously with high probability, i.e., we have not proved the following statement:

$$\mathbb{P}\left(\sup_{1\leqslant k\leqslant N} |\langle u_k, a_N u_k\rangle| > \delta\right) \to 0$$

for a_N with support of order N. This strong form of QUE, however, holds for the Gaussian ensembles.

In the following section, we will define the Dyson vector flow and, for the sake of completeness, prove the well-posedness of the eigenvector stochastic evolution. In Section 3 we will introduce the eigenvector moment flow and prove the existence of an explicit reversible measure. In Section 4, we will prove Theorem 1.2 under the additional assumption that H_N is the sum of a generalized Wigner matrix and a Gaussian matrix with small variance. The proof in this section relies on a maximum principle for the eigenvector moment flow. We will prove Theorem 1.2 by using a Green function comparison theorem in Section 5. In Section 6, we will prove that the speed of propagation for the eigenvector moment flow is finite with very high probability. This estimate will enable us to prove in Section 7 that the relaxation to equilibrium for the eigenvector moment flow in the bulk is of order $t \gtrsim N^{-1}$. The appendices contain a continuity estimate for the Dyson Brownian motion up to time $N^{-1/2}$, and some basic results concerning the generator of the Dyson vector flow as well as analogue results for covariance matrices.

2 Dyson Vector Flow

In this section, we first state the stochastic differential equation for the eigenvectors under the Dyson Brownian motion. This evolution is given by (2.3) and (2.5). We then give a concise form of the generator for this Dyson vector flow. We will follow the usual slight ambiguity of terminology by naming both the matrix flow and the eigenvalue flow a Dyson Brownian motion. In case we wish to distinguish them, we will use matrix Dyson Brownian motion for the matrix flow.

Definition 2.1. *Hereafter is our choice of normalization for the Dyson Brownian motion.*

(i) Let $B^{(s)}$ be a $N \times N$ matrix such that $B^{(s)}_{ij} (i < j)$ and $B^{(s)}_{ii}/\sqrt{2}$ are independent standard Brownian motions, and $B^{(s)}_{ij} = B^{(s)}_{ji}$. The $N \times N$ symmetric Dyson Brownian motion $H^{(s)}$ with initial value $H^{(s)}_0$ is defined as

$$H^{(s)}_t = H^{(s)}_0 + \frac{1}{\sqrt{N}} B^{(s)}_t, \tag{2.1}$$

(ii) Let $B^{(h)}$ be a $N \times N$ matrix such that $\Re(B^{(h)}_{ij}), \Im(B^{(h)}_{ij}) (i < j)$ and $B^{(h)}_{ii}/\sqrt{2}$ are independent standard Brownian motions, and $B^{(h)}_{ji} = (B^{(h)}_{ij})^*$. The $N \times N$ Hermitian Dyson Brownian motion $H^{(h)}$ with initial value $H^{(t)}_0$ is

$$H^{(h)}_t = H^{(h)}_0 + \frac{1}{\sqrt{2N}} B^{(h)}_t,$$

Definition 2.2. *We refer to the following stochastic differential equations as the Dyson Brownian motion for (2.2) and (2.4) and the Dyson vector flow for (2.3) and (2.5).*

(i) Let $\boldsymbol{\lambda}_0 \in \Sigma_N = \{\lambda_1 < \cdots < \lambda_N\}$, $\boldsymbol{u}_0 \in \mathrm{O}(N)$, and $B^{(s)}$ be as in Definition 2.1. The symmetric Dyson Brownian motion/vector flow with initial condition $(\lambda_1, \ldots, \lambda_N) = \boldsymbol{\lambda}_0$, $(u_1, \ldots, u_N) = \boldsymbol{u}_0$, is

$$\mathrm{d}\lambda_k = \frac{\mathrm{d}B^{(s)}_{kk}}{\sqrt{N}} + \left(\frac{1}{N} \sum_{\ell \neq k} \frac{1}{\lambda_k - \lambda_\ell} \right) \mathrm{d}t, \tag{2.2}$$

$$\mathrm{d}u_k = \frac{1}{\sqrt{N}} \sum_{\ell \neq k} \frac{\mathrm{d}B^{(s)}_{k\ell}}{\lambda_k - \lambda_\ell} u_\ell - \frac{1}{2N} \sum_{\ell \neq k} \frac{\mathrm{d}t}{(\lambda_k - \lambda_\ell)^2} u_k. \tag{2.3}$$

(ii) Let $\boldsymbol{\lambda}_0 \in \Sigma_N$, $\boldsymbol{u}_0 \in \mathrm{U}(N)$, and $B^{(h)}$ be as in Definition 2.1. The Hermitian Dyson Brownian motion/vector flow with initial condition $(\lambda_1, \dots, \lambda_N) = \boldsymbol{\lambda}_0$, $(u_1, \dots, u_N) = \boldsymbol{u}_0$, is

$$\mathrm{d}\lambda_k = \frac{\mathrm{d}B^{(h)}_{kk}}{\sqrt{2N}} + \left(\frac{1}{N}\sum_{\ell \neq k} \frac{1}{\lambda_k - \lambda_\ell}\right)\mathrm{d}t, \tag{2.4}$$

$$\mathrm{d}u_k = \frac{1}{\sqrt{2N}}\sum_{\ell \neq k} \frac{\mathrm{d}B^{(h)}_{k\ell}}{\lambda_k - \lambda_\ell} u_\ell - \frac{1}{2N}\sum_{\ell \neq k} \frac{\mathrm{d}t}{(\lambda_k - \lambda_\ell)^2} u_k. \tag{2.5}$$

The theorem below contains the following results. (a) The above stochastic differential equations admit a unique strong solution, this relies on classical techniques and an argument originally by McKean [27]. (b) The matrix Dyson Brownian motion induces the standard Dyson Brownian motion (for the eigenvalues) and Dyson eigenvector flow. This statement was already proved in [4]. (c) For calculation purpose, one can condition on the trajectory of the eigenvalues to study the eigenvectors evolution. For the sake of completeness, this theorem is proved in the appendix.

With a slight abuse of notation, we will write $\boldsymbol{\lambda}_t$ either for $(\lambda_1(t), \dots, \lambda_N(t))$ or for the $N \times N$ diagonal matrix with entries $\lambda_1(t), \dots, \lambda_N(t)$.

Theorem 2.3. *The following statements about the Dyson Brownian motion and eigenvalue/vector flow hold.*

(a) Existence and strong uniqueness hold for the system of stochastic differential equations (2.2), (2.3). Let $(\boldsymbol{\lambda}_t, \boldsymbol{u}_t)_{t \geqslant 0}$ be the solution. Almost surely, for any $t \geqslant 0$ we have $\boldsymbol{\lambda}_t \in \Sigma_N$ and $\boldsymbol{u}_t \in \mathrm{O}(N)$.

(b) Let $(H_t)_{t \geqslant 0}$ be a symmetric Dyson Brownian motion with initial condition $H_0 = \boldsymbol{u}_0 \boldsymbol{\lambda}_0 \boldsymbol{u}_0^*$, $\boldsymbol{\lambda}_0 \in \Sigma_N$. Then the processes $(H_t)_{t \geqslant 0}$ and $(\boldsymbol{u}_t \boldsymbol{\lambda}_t \boldsymbol{u}_t^*)_{t \geqslant 0}$ have the same distribution.

(c) Existence and strong uniqueness hold for (2.2). For any $T > 0$, let $\nu_T^{H_0}$ be the distribution of $(\boldsymbol{\lambda}_t)_{0 \leqslant t \leqslant T}$ with initial value the spectrum of a matrix H_0. For $0 \leqslant T \leqslant T_0$ and any given continuous trajectory $\boldsymbol{\lambda} = (\boldsymbol{\lambda}_t)_{0 \leqslant t \leqslant T_0} \subset \Sigma_N$, existence and strong uniqueness holds for (2.3) on $[0, T]$. Let $\mu_T^{H_0, \boldsymbol{\lambda}}$ be the distribution of $(\boldsymbol{u}_t)_{0 \leqslant t \leqslant T}$ with the initial matrix H_0 and the path $\boldsymbol{\lambda}$ given.

Let F be continuous bounded, from the set of continuous paths (on $[0, T]$) on $N \times N$ symmetric matrices to $\mathbb{R}$. Then for any initial matrix H_0 we have

$$\mathbb{E}^{H_0}(F((H_t)_{0 \leqslant t \leqslant T})) = \int \mathrm{d}\nu_T^{H_0}(\boldsymbol{\lambda}) \int \mathrm{d}\mu_T^{H_0, \boldsymbol{\lambda}}(\boldsymbol{u}) F((\boldsymbol{u}_t \boldsymbol{\lambda}_t \boldsymbol{u}_t^*)_{0 \leqslant t \leqslant T}). \tag{2.6}$$

The analogous statements hold in the Hermitian setting.

We will omit the subscript T when it is obvious. The previous theorem reduces the study of the eigenvector dynamics to the stochastic differential equations (2.3) and (2.5). The following lemma gives a concise form of the generators of these diffusions. It is very similar to the well-known forms of the generator for the Brownian motion on the unitary/orthogonal groups up to the following difference: weights vary depending on eigenvalue pairs.

We will need the following notations (the dependence in t will often be omitted for $c_{k\ell}$, $1 \leqslant k < \ell \leqslant N$):

$$c_{k\ell}(t) = \frac{1}{N(\lambda_k(t) - \lambda_\ell(t))^2}, \tag{2.7}$$

$$u_k\partial_{u_\ell} = \sum_{\alpha=1}^{N} u_k(\alpha)\partial_{u_\ell(\alpha)},\ u_k\partial_{\overline{u}_\ell} = \sum_{\alpha=1}^{N} u_k(\alpha)\partial_{\overline{u}_\ell(\alpha)},$$

$$X_{k\ell}^{(s)} = u_k\partial_{u_\ell} - u_\ell\partial_{u_k}, \tag{2.8}$$

$$X_{k\ell}^{(h)} = u_k\partial_{u_\ell} - \overline{u}_\ell\partial_{\overline{u}_k},\ \overline{X}_{k\ell}^{(h)} = \overline{u}_k\partial_{\overline{u}_\ell} - u_\ell\partial_{u_k}.$$

Here $\partial_{\overline{u}_\ell}$ and $\partial_{\overline{u}_\ell}$ are defined by considering u_ℓ as a complex number, i.e., if we write $u_\ell = x + iy$ then $\partial_{\overline{u}_\ell} = \frac{1}{2}\partial_x + \frac{i}{2}\partial_y$.

Lemma 2.4. *For the diffusion (2.3) (resp. (2.5)), the generators acting on smooth functions $f((u_i(\alpha))_{1\leqslant i,\alpha\leqslant N})$: $\mathbb{R}^{N^2} \to \mathbb{R}$ (resp. $\mathbb{C}^{N^2} \to \mathbb{R}$) are respectively*

$$\mathrm{L}_t^{(s)} = \sum_{1\leqslant k<\ell\leqslant N} c_{k\ell}(t)(X_{k\ell}^{(s)})^2,$$

$$\mathrm{L}_t^{(h)} = \frac{1}{2}\sum_{1\leqslant k<\ell\leqslant N} c_{k\ell}(t)\left(X_{k\ell}^{(h)}\overline{X}_{k\ell}^{(h)} + \overline{X}_{k\ell}^{(h)}X_{k\ell}^{(h)}\right). \tag{2.9}$$

The above lemma means $\mathrm{d}\mathbb{E}(g(\boldsymbol{u}_t))/\mathrm{d}t = \mathbb{E}(\mathrm{L}_t^{(s)}g(\boldsymbol{u}_t))$ (resp. $\mathrm{d}\mathbb{E}(g(\boldsymbol{u}_t))/\mathrm{d}t = \mathbb{E}(\mathrm{L}_t^{(h)}g(\boldsymbol{u}_t)))$ for the stochastic differential equations (2.3) (resp. (2.5)). It relies on a direct calculation via Itô's formula. The details are given in the appendix.

3 Eigenvector Moment Flow

3.1 The moment flow. Our observables will be moments of projections of the eigenvectors onto a given direction. More precisely, for any fixed $\mathbf{q} \in \mathbb{R}^N$ and for any $1 \leqslant k \leqslant N$, define

$$z_k(t) = \sqrt{N}\langle \mathbf{q}, u_k(t)\rangle = \sum_{\alpha=1}^{N} \mathbf{q}(\alpha)u_k(t,\alpha).$$

With this $\sqrt{N}$ normalization, the typical size of z_k is of order 1. We assume that the eigenvalue trajectory $(\lambda_k(t), 0 \leqslant t \leqslant T_0)_{k=1}^N$ in the simplex $\Sigma^{(N)}$ is given. Furthermore, $\boldsymbol{u}$ is the unique strong solution of the stochastic differential equation (2.3) (resp. (2.5)) with the given eigenvalue trajectory. Let $\mathrm{P}^{(s)}(t) = \mathrm{P}^{(s)}(z_1,\dots,z_N)(t)$ and $\mathrm{P}^{(h)} = \mathrm{P}^{(h)}(z_1,\dots,z_N)(t)$ be smooth functions. Then a simple calculation yields

$$X_{k\ell}^{(s)}\mathrm{P}^{(s)} = (z_k\partial_{z_\ell} - z_\ell\partial_{z_k})\mathrm{P}^{(s)}, \tag{3.1}$$

$$X_{k\ell}^{(h)}\mathrm{P}^{(h)} = (z_k\partial_{z_\ell} - \overline{z}_\ell\partial_{\overline{z}_k})f,\ \overline{X}_{k\ell}^{(h)}\mathrm{P}^{(h)} = (\overline{z}_k\partial_{\overline{z}_\ell} - z_\ell\partial_{z_k})\mathrm{P}^{(h)}. \tag{3.2}$$

For $m \in [\![1, N]\!]$, denote by $j_1, \dots, j_m$ positive integers and let $i_1, \dots, i_m$ in $[\![1, N]\!]$ be m distinct indices. The test functions we will consider are:

$$\mathrm{P}^{(s)}{}^{j_1,\dots,j_m}_{i_1,\dots,i_m}(z_1, \dots, z_N) = \prod_{\ell=1}^{m} z_{i_\ell}^{2j_\ell},$$
$$\mathrm{P}^{(h)}{}^{j_1,\dots,j_m}_{i_1,\dots,i_m}(z_1, \dots, z_N) = \prod_{\ell=1}^{m} z_{i_\ell}^{j_\ell} \overline{z}_{i_\ell}^{j_\ell}.$$

For any m fixed, linear combinations of such polynomial functions are stable under the action of the generator. More precisely, the following formulas hold.

(i) In the symmetric setting, one can use (3.1) to evaluate the action of the generator. If neither k nor ℓ are in $\{i_1, \dots, i_m\}$, then $(X^{(s)}_{k\ell})^2 \mathrm{P}^{(s)}{}^{j_1,\dots,j_m}_{i_1,\dots,i_m} = 0$; the other cases are covered by:

$$(X^{(s)}_{i_1\ell})^2 \mathrm{P}^{(s)}{}^{j_1,\dots,j_m}_{i_1,\dots,i_m} = 2j_1(2j_1 - 1)\mathrm{P}^{(s)}{}^{1,j_1-1,\dots,j_m}_{\ell,i_1,\dots,i_m} - 2j_1 \mathrm{P}^{(s)}{}^{j_1,\dots,j_m}_{i_1,\dots,i_m} \quad \text{when} \quad \ell \notin \{i_1, \dots, i_m\},$$
$$\begin{aligned}(X^{(s)}_{i_1 i_2})^2 \mathrm{P}^{(s)}{}^{j_1,\dots,j_m}_{i_1,\dots,i_m} &= 2j_1(2j_1 - 1)\mathrm{P}^{(s)}{}^{j_1-1,j_2+1,\dots,j_m}_{i_1,\dots,i_m} + 2j_2(2j_2 - 1)\mathrm{P}^{(s)}{}^{j_1+1,j_2-1,\dots,j_m}_{i_1,\dots,i_m} \\ &\quad - (2j_1(2j_2 + 1) + 2j_2(2j_1 + 1))\mathrm{P}^{(s)}{}^{j_1,j_2,\dots,j_m}_{i_1,i_2,\dots,i_m}.\end{aligned}$$

(ii) In the Hermitian setting, we note that the polynomials $\mathrm{P}^{(h)}$ are invariant under the permutation $z_i \to \overline{z}_i$. Thus the action of the generator $\mathrm{L}^{(h)}$ (2.9) on such functions $\mathrm{P}^{(h)}$ simplifies to

$$\mathrm{L}^{(h)}_t \mathrm{P}^{(h)} = \sum_{k<\ell} c_{k\ell} X^{(h)}_{k\ell} \overline{X}^{(h)}_{k\ell} \mathrm{P}^{(h)}.$$

Then (3.2) yields

$$X^{(h)}_{i_1\ell} \overline{X}^{(h)}_{k\ell} \mathrm{P}^{(h)}{}^{j_1,\dots,j_m}_{i_1,\dots,i_m} = j_1^2 \mathrm{P}^{(h)}{}^{1,j_1-1,\dots,j_m}_{\ell,i_1,\dots,i_m} - j_1 \mathrm{P}^{(h)}{}^{j_1,\dots,j_m}_{i_1,\dots,i_m} \quad \text{when} \quad \ell \notin \{i_1, \dots, i_m\},$$
$$\begin{aligned}X^{(h)}_{i_1 i_2 \ell} \overline{X}^{(h)}_{k\ell} \mathrm{P}^{(h)}{}^{j_1,\dots,j_m}_{i_1,\dots,i_m} &= j_1^2 \mathrm{P}^{(h)}{}^{j_1-1,j_2+1,\dots,j_m}_{i_1,\dots,i_m} + j_2^2 \mathrm{P}^{(h)}{}^{j_1+1,j_2-1,\dots,j_m}_{i_1,\dots,i_m} \\ &\quad - (j_1(j_2 + 1) + j_2(j_1 + 1))\mathrm{P}^{(h)}{}^{j_1,j_2,\dots,j_m}_{i_1,i_2,\dots,i_m}\end{aligned}$$

We now normalize the polynomials by defining

$$\mathrm{Q}^{(s)}_t{}^{j_1,\dots,j_m}_{i_1,\dots,i_m} = \mathrm{P}^{(s)}{}^{j_1,\dots,j_m}_{i_1,\dots,i_m}(t) \prod_{\ell=1}^{m} a(2j_\ell)^{-1} \text{ where } a(n) = \prod_{k \leqslant n, k \text{ odd}} k, \tag{3.3}$$
$$\mathrm{Q}^{(h)}_t{}^{j_1,\dots,j_m}_{i_1,\dots,i_m} = \mathrm{P}^{(h)}{}^{j_1,\dots,j_m}_{i_1,\dots,i_m}(t) \prod_{\ell=1}^{m} (2^{j_\ell} j_\ell!)^{-1}. \tag{3.4}$$

Note that $a(2n) = \mathbb{E}(\mathscr{N}^{2n})$ and $2^n n! = \mathbb{E}(|\mathscr{N}_1 + \mathrm{i}\mathscr{N}_2|^{2n})$, with $\mathscr{N}$, $\mathscr{N}_1$, $\mathscr{N}_2$ independent standard Gaussian random variables. The above discussion implies the following evolution of $\mathrm{Q}^{(s)}$ (resp. $\mathrm{Q}^{(h)}$) along the Dyson eigenvector flow (2.3) (resp. (2.5)).

(i) Symmetric case: $\mathrm{L}_t^{(s)}\mathrm{Q}_t^{(s)} = \sum_{k<\ell} c_{k\ell}(X_{k\ell}^{(s)})^2\mathrm{Q}_t^{(s)}$ where

$$\begin{aligned}
(X_{i_1\ell}^{(s)})^2{\mathrm{Q}_t^{(s)}}_{i_1,\dots,i_m}^{j_1,\dots,j_m} &= 2j_1{\mathrm{Q}_t^{(s)}}_{\ell,i_1,\dots,i_m}^{1,j_1-1,\dots,j_m} - 2j_1{\mathrm{Q}_t^{(s)}}_{i_1,\dots,i_m}^{j_1,\dots,j_m} \quad \text{when} \quad \ell\notin\{i_1,\dots,i_m\},\\
(X_{i_1i_2}^{(s)})^2{\mathrm{Q}_t^{(s)}}_{i_1,\dots,i_m}^{j_1,\dots,j_m} &= 2j_1(2j_2+1){\mathrm{Q}_t^{(s)}}_{i_1,\dots,i_m}^{j_1-1,j_2+1,\dots,j_m} + 2j_2(2j_1+1){\mathrm{Q}_t^{(s)}}_{i_1,\dots,i_m}^{j_1+1,j_2-1,\dots,j_m}\\
&\quad - (2j_1(2j_2+1)+2j_2(2j_1+1)){\mathrm{Q}_t^{(s)}}_{i_1,i_2,\dots,i_m}^{j_1,j_2,\dots,j_m}.
\end{aligned}$$

(ii) Hermitian case: $\mathrm{L}_t^{(h)}\mathrm{Q}_t^{(h)} = \sum_{k<\ell} c_{k\ell}X_{k\ell}^{(h)}\overline{X}_{k\ell}^{(h)}\mathrm{Q}_t^{(h)}$ where

$$\begin{aligned}
X_{i_1\ell}^{(h)}\overline{X}_{i_1\ell}^{(h)}{\mathrm{Q}_t^{(h)}}_{i_1,\dots,i_m}^{j_1,\dots,j_m} &= j_1{\mathrm{Q}_t^{(h)}}_{\ell,i_1,\dots,i_m}^{1,j_1-1,\dots,j_m} - j_1{\mathrm{Q}_t^{(h)}}_{i_1,\dots,i_m}^{j_1,\dots,j_m} \quad \text{when} \quad \ell\notin\{i_1,\dots,i_m\},\\
X_{i_1i_2}^{(h)}\overline{X}_{i_1i_2}^{(h)}{\mathrm{Q}_t^{(h)}}_{i_1,\dots,i_m}^{j_1,\dots,j_m} &= j_1(j_2+1){\mathrm{Q}_t^{(h)}}_{i_1,\dots,i_m}^{j_1-1,j_2+1,\dots,j_m} + j_2(j_1+1){\mathrm{Q}_t^{(h)}}_{i_1,\dots,i_m}^{j_1+1,j_2-1,\dots,j_m}\\
&\quad - (j_1(j_2+1)+j_2(j_1+1)){\mathrm{Q}_t^{(h)}}_{i_1,i_2,\dots,i_m}^{j_1,j_2,\dots,j_m}.
\end{aligned}$$

Thanks to the scalings (3.3) and (3.4), on the right hand sides of the above four equations, the sums of the coefficients vanish. This allows us to interpret them as multi-particle random walks (in random environments) in the next subsection.

3.2 Multi-particle random walk. Consider the following notation, $\boldsymbol{\eta} : [\![1,N]\!] \to \mathbb{N}$ where $\eta_j := \boldsymbol{\eta}(j)$ is interpreted as the number of particles at the site j. Thus $\boldsymbol{\eta}$ denotes the configuration space of particles. We denote $\mathcal{N}(\boldsymbol{\eta}) = \sum_j \eta_j$.

Define $\boldsymbol{\eta}^{i,j}$ to be the configuration by moving one particle from i to j. If there is no particle at i then $\boldsymbol{\eta}^{i,j} = \boldsymbol{\eta}$. Notice that there is a direction and the particle is moved from i to j. Given $n>0$, there is a one to one correspondence between (1) $\{(i_1,j_1),\dots,(i_m,j_m)\}$ with distinct i_k's and positive j_k's summing to n, and (2) $\boldsymbol{\eta}$ with $\mathcal{N}(\boldsymbol{\eta}) = n$: we map $\{(i_1,j_1),\dots,(i_m,j_m)\}$ to $\boldsymbol{\eta}$ with $\boldsymbol{\eta}_{i_k} = j_k$ and $\boldsymbol{\eta}_\ell = 0$ if $\ell \notin \{i_1,\dots i_m\}$. We define

$$f_{\boldsymbol{\lambda},t}^{H_0,(s)}(\boldsymbol{\eta}) = \mathbb{E}^{H_0}({\mathrm{Q}_t^{(s)}}_{i_1,\dots,i_m}^{j_1,\dots,j_m}(t) \mid \boldsymbol{\lambda}), \quad f_{\boldsymbol{\lambda},t}^{H_0,(h)}(\boldsymbol{\eta}) = \mathbb{E}^{H_0}({\mathrm{Q}_t^{(h)}}_{i_1,\dots,i_m}^{j_1,\dots,j_m} \mid \boldsymbol{\lambda}), \tag{3.5}$$

if the configuration of $\boldsymbol{\eta}$ is the same as the one given by the i,j's. Here $\boldsymbol{\lambda}$ denotes the whole path of eigenvalues for $0\leqslant t\leqslant 1$. The dependence in the initial matrix H_0 will often be omitted so that we write $f_{\boldsymbol{\lambda},t}^{(s)} = f_{\boldsymbol{\lambda},t}^{H_0,(s)}$, $f_{\boldsymbol{\lambda},t}^{(h)} = f_{\boldsymbol{\lambda},t}^{H_0,(h)}$. The following theorem summarizes the results from the previous subsection. It also defines the eigenvector moment flow, through the generators (3.7) and (3.8). They are multi-particles random walks (with $n = \mathcal{N}(\boldsymbol{\eta})$ particles) in random environments with jump rates depending on the eigenvalues.

Theorem 3.1 (Eigenvector moment flow)**.** *Let $\mathbf{q}\in\mathbb{R}^N$, $z_k = \sqrt{N}\langle \mathbf{q}, u_k(t)\rangle$ and $c_{ij}(t) = \frac{1}{N(\lambda_i-\lambda_j)^2(t)}$.*

(i) Suppose that $\boldsymbol{u}$ is the solution to the symmetric Dyson vector flow (2.3) and $f_{\boldsymbol{\lambda},t}^{(s)}(\boldsymbol{\eta})$ is given by (3.5) where $\boldsymbol{\eta}$ denote the configuration $\{(i_1,j_1),\dots,(i_m,j_m)\}$. Then $f_{\boldsymbol{\lambda},t}^{(s)}$ satisfies the equation

$$\partial_t f_{\boldsymbol{\lambda},t}^{(s)} = \mathscr{B}^{(s)}(t) f_{\boldsymbol{\lambda},t}^{(s)}, \tag{3.6}$$

$$\mathscr{B}^{(s)}(t)f(\boldsymbol{\eta}) = \sum_{i\neq j} c_{ij}(t)2\eta_i(1+2\eta_j)\left(f(\boldsymbol{\eta}^{i,j}) - f(\boldsymbol{\eta})\right). \tag{3.7}$$

(ii) Suppose that $\boldsymbol{u}$ is the solution to the Hermitian Dyson vector flow (2.5), and $f^{(h)}_{\boldsymbol{\lambda},t}$ is given by (3.5). Then it satisfies the equation

$$\partial_t f^{(h)}_{\boldsymbol{\lambda},t} = \mathscr{B}^{(h)}(t) f^{(h)}_{\boldsymbol{\lambda},t},$$
$$\mathscr{B}^{(h)}(t) f(\boldsymbol{\eta}) = \sum_{i\neq j} c_{ij}(t)\eta_i(1+\eta_j)\left(f(\boldsymbol{\eta}^{i,j}) - f(\boldsymbol{\eta})\right). \tag{3.8}$$

An important property of the eigenvector moment flow is reversibility with respect to a simple explicit equilibrium measure. In the Hermitian case, this is simply the uniform measure on the configuration space.

Recall that a measure π on the configuration space is said to be reversible with respect to a generator L if $\sum_{\boldsymbol{\eta}} \pi(\boldsymbol{\eta}) g(\boldsymbol{\eta}) \mathrm{L} f(\boldsymbol{\eta}) = \sum_{\boldsymbol{\eta}} \pi(\boldsymbol{\eta}) f(\boldsymbol{\eta}) \mathrm{L} g(\boldsymbol{\eta})$ for any functions f and g. We then define the Dirichlet form by

$$D^{\pi}(f) = -\sum_{\boldsymbol{\eta}} \pi(\boldsymbol{\eta}) f(\boldsymbol{\eta}) \mathrm{L} f(\boldsymbol{\eta}).$$

Proposition 3.2. *For the eigenvector moment flow, the following properties hold.*

(i) Define a measure on the configuration space by assigning the weight

$$\pi^{(s)}(\boldsymbol{\eta}) = \prod_{x=1}^{N} \phi(\eta_x), \ \ \phi(k) = \prod_{i=1}^{k}\left(1 - \frac{1}{2i}\right). \tag{3.9}$$

Then $\pi^{(s)}$ is a reversible measure for $\mathscr{B}^{(s)}$ and the Dirichlet form is given by

$$D^{\pi^{(s)}}(f) = \sum_{\boldsymbol{\eta}} \pi^{(s)}(\boldsymbol{\eta}) \sum_{i\neq j} c_{ij}\eta_i(1+2\eta_j)\left(f(\boldsymbol{\eta}^{i,j}) - f(\boldsymbol{\eta})\right)^2.$$

(ii) The uniform measure ($\pi^{(h)}(\boldsymbol{\eta}) = 1$ for all $\boldsymbol{\eta}$) is reversible with respect to $\mathscr{B}^{(h)}$. The associated Dirichlet form is

$$D^{\pi^{(h)}}(f) = \frac{1}{2}\sum_{\boldsymbol{\eta}}\sum_{i\neq j} c_{ij}\eta_i(1+\eta_j)\left(f(\boldsymbol{\eta}^{i,j}) - f(\boldsymbol{\eta})\right)^2.$$

Proof. We first consider (i), concerning the symmetric eigenvector moment flow. The measure $\pi^{(s)}$ is reversible for $\mathscr{B}^{(s)}$ for any choice of the coefficients satisfying $c_{ij} = c_{ji}$ if and only if, for any $i < j$,

$$\sum_{\boldsymbol{\eta}} \pi^{(s)}(\boldsymbol{\eta}) g(\boldsymbol{\eta}) \left(2\eta_i(1+2\eta_j) f(\boldsymbol{\eta}^{ij}) + 2\eta_j(1+2\eta_i) f(\boldsymbol{\eta}^{ji})\right)$$
$$= \sum_{\boldsymbol{\eta}} \pi^{(s)}(\boldsymbol{\eta}) f(\boldsymbol{\eta}) \left(2\eta_i(1+2\eta_j) g(\boldsymbol{\eta}^{ij}) + 2\eta_j(1+2\eta_i) g(\boldsymbol{\eta}^{ji})\right).$$

A sufficient condition is clearly that both of the following equations hold:

$$\sum_{\boldsymbol{\eta}} \pi^{(s)}(\boldsymbol{\eta}) g(\boldsymbol{\eta}) 2\eta_i(1+2\eta_j) f(\boldsymbol{\eta}^{ij}) = \sum_{\boldsymbol{\eta}} \pi^{(s)}(\boldsymbol{\eta}) f(\boldsymbol{\eta}) 2\eta_j(1+2\eta_i) g(\boldsymbol{\eta}^{ji}),$$
$$\sum_{\boldsymbol{\eta}} \pi^{(s)}(\boldsymbol{\eta}) g(\boldsymbol{\eta}) 2\eta_j(1+2\eta_i) f(\boldsymbol{\eta}^{ji}) = \sum_{\boldsymbol{\eta}} \pi^{(s)}(\boldsymbol{\eta}) f(\boldsymbol{\eta}) 2\eta_i(1+2\eta_j) g(\boldsymbol{\eta}^{ij}).$$

Consider the left hand side of the first one of these two equations. Let $\boldsymbol{\xi} = \boldsymbol{\eta}^{ij}$. If $\xi_j > 0$ then $\boldsymbol{\eta} = \boldsymbol{\xi}^{ji}$, $\eta_i = \xi_i + 1$ and $\eta_j = \xi_j - 1$. For the right hand side of the second equation, we make the change of variables $\boldsymbol{\xi} = \boldsymbol{\eta}^{ji}$. Finally, rename all the variables on the right hand sides by ξ. Thus the above equations are equivalent to

$$\sum_{\boldsymbol{\xi}} \pi^{(s)}(\boldsymbol{\xi}^{ji}) g(\boldsymbol{\xi}^{ji}) 2(\xi_i + 1)(2\xi_j - 1) f(\boldsymbol{\xi}) = \sum_{\boldsymbol{\xi}} \pi^{(s)}(\boldsymbol{\xi}) f(\boldsymbol{\xi}) 2\xi_j (1 + 2\xi_i) g(\boldsymbol{\xi}^{ji}),$$
$$\sum_{\boldsymbol{\xi}} \pi^{(s)}(\boldsymbol{\xi}^{ij}) g(\boldsymbol{\xi}^{ij}) 2(\xi_j + 1)(2\xi_i - 1) f(\boldsymbol{\xi}) = \sum_{\boldsymbol{\xi}} \pi^{(s)}(\boldsymbol{\xi}) f(\boldsymbol{\xi}) 2\xi_i (1 + 2\xi_j) g(\boldsymbol{\xi}^{ij}).$$

Clearly, both equations hold provided that

$$\pi^{(s)}(\boldsymbol{\xi}^{ji}) 2(\boldsymbol{\xi}_i + 1)(1 + 2(\boldsymbol{\xi}_j - 1)) = \pi^{(s)}(\boldsymbol{\xi}) 2\boldsymbol{\xi}_j (1 + 2\boldsymbol{\xi}_i). \tag{3.10}$$

If the measure is of type $\pi^{(s)}(\boldsymbol{\eta}) = \prod_x \phi(\eta_x)$ and we note $\xi_i = a, \xi_j = b$, this equation is equivalent to

$$\phi(a+1)\phi(b-1) 2(a+1)(2b-1) = \phi(a)\phi(b) 2b(2a+1),$$

and the second equation yields the same condition with the roles of a and b switched. This holds for all a and b if $\phi(k+1) = ((2k+1)/(2k+2))\phi(k)$, which gives (3.9) provided we normalize $\phi(0) = 1$. In the case (ii), the same reasoning yields that ϕ is constant.

Finally, the Dirichlet form calculation is standard: for example, for (i), $\sum_{\boldsymbol{\eta}} \pi^{(s)}(\boldsymbol{\eta}) \mathscr{B}^{(s)}(f^2)(\boldsymbol{\eta}) = 0$ by reversibility. Noting $\mathscr{B}^{(s)}(f^2)(\boldsymbol{\eta}) = 2f(\boldsymbol{\eta})\mathscr{B}^{(s)} f(\boldsymbol{\eta}) - \sum \pi^{(s)}(\boldsymbol{\eta}) 2\eta_i (1 + 2\eta_j)(f(\boldsymbol{\eta}) - f(\boldsymbol{\eta}^{ij}))^2$ allows to conclude. □

4 Maximum principle

From now on we only consider the symmetric ensemble. The Hermitian case can be treated with the same arguments and only notational changes. Given a typical path $\boldsymbol{\lambda}$, we will prove in this section that the solution to the eigenvector moment flow (3.7) converges uniformly to 1 for $t = N^{-1/4+\varepsilon}$. It is clear that the maximum (resp. minimum) of f over $\boldsymbol{\eta}$ decreases (resp. increases). We can quantify this decrease (resp. increase) in terms of the maximum and minimum themselves (see (4.17)). This yields an explicit convergence speed to 1 by a Gronwall argument.

4.1 Isotropic local semicircle law. Fix a (small) $\omega > 0$ and define

$$\mathbf{S} = \mathbf{S}(\omega, N) = \left\{ z = E + \mathrm{i}\eta \in \mathbb{C} : |E| \leqslant \omega^{-1}, \, N^{-1+\omega} \leqslant \eta \leqslant \omega^{-1} \right\}. \tag{4.1}$$

In the statement below, we will also need $m(z)$, the Stieltjes transform of the semicircular distribution, i.e.

$$m(z) = \int \frac{\varrho(s)}{s - z} \mathrm{d}s = \frac{-z + \sqrt{z^2 - 4}}{2}, \quad \varrho(s) = \frac{1}{2\pi}\sqrt{(4 - s^2)_+},$$

where the square root is chosen so that m is holomorphic in the upper half plane and $m(z) \to 0$ as $z \to \infty$. The following isotropic local semicircle law (Theorem 4.2 in [5]) gives very useful bounds on $\langle \mathbf{q}, u_k \rangle$ for any eigenvector u_k via estimates on the associated Green function.

Theorem 4.1 (Isotropic local semicircle law [5]). *Let H be an element from the generalized Wigner ensemble and $G(z) = (H-z)^{-1}$. Suppose that (1.3) holds. Then for any (small) $\xi > 0$ and (large) $D > 0$ we have, for large enough N,*

$$\sup_{|\mathbf{q}|=1, z\in\mathbf{S}} \mathbb{P}\left(|\langle \mathbf{q}, G(z)\mathbf{q}\rangle - m(z)| > N^{\xi}\left(\sqrt{\frac{\operatorname{Im} m(z)}{N\eta}} + \frac{1}{N\eta}\right)\right) \leqslant N^{-D}. \tag{4.2}$$

An important consequence of this theorem, to be used in multiple occasions, is the following isotropic delocalization of eigenvectors: under the same assumptions as Theorem 4.1, for any $\xi > 0$ and $D > 0$, we have

$$\sup_{|\mathbf{q}|=1, k\in[\![1,N]\!]} \mathbb{P}\left(|\langle \mathbf{q}, u_k\rangle| > N^{-1+\xi}\right) \leqslant N^{-D}.$$

Under the same assumptions the Stieltjes transform was shown [19] to satisfy the estimate

$$\sup_{z\in\mathbf{S}} \mathbb{P}\left(\left|\frac{1}{N}\operatorname{Tr} G(z) - m(z)\right| > \frac{N^{\xi}}{N\eta}\right) \leqslant N^{-D}. \tag{4.3}$$

4.2 Rescaling. Recall the definition (2.1) of the evolution matrix $H_t^{(s)}$. The variance $\sigma_{ij}^2(t)$ of the matrix element $h_{ij}(t)$ is given by $\sigma_{ij}^2(t) = \sigma_{ij}^2 + t/N$ if $i \neq j$, $\sigma_{ij}^2(t) = \sigma_{ij}^2 + 2t/N$ if $i = j$. Denote by $\alpha(t) = \left(1 + \frac{N+1}{N}t\right)^{-1/2}$. Then $\alpha(t)H_t^{(s)}$ is a generalized Wigner ensemble. In particular, the previously mentionned rigidity estimates hold along our dynamics if we rescale $H_t^{(s)}$ into $\alpha(t)H_t^{(s)}$. Consider the simple time change of our dynamics $u(t) = \int_0^t \alpha(s)^{-2}\mathrm{d}s$. Then $\widetilde{f}_t(\boldsymbol{\eta}) := f_{u(t)}(\boldsymbol{\eta})$ satisfies

$$\partial_t \widetilde{f}(\boldsymbol{\eta}) = \sum_{i\neq j} \frac{1}{N(\alpha(t)\lambda_i(t) - \alpha(t)\lambda_i(t))^2} 2\eta_i(1+2\eta_j)\left(\widetilde{f}(\boldsymbol{\eta}^{ij}) - \widetilde{f}(\boldsymbol{\eta})\right).$$

In the rest of the paper it will always be understood that the above time rescaling $t \to u(t)$ and matrix scaling $H_t^{(s)} \to \alpha(t)H_t^{(s)}$ are performed so that all rigidity estimates hold as presented in the previous subsection, for all time.

4.3 Maximum Principle and regularity. Let H_0 be a symmetric generalized Wigner matrix with eigenvalues $\boldsymbol{\lambda}_0$ and an eigenbasis $\boldsymbol{u}_0$. Assume that $\boldsymbol{\lambda}, \boldsymbol{u}$ satisfy (2.2) (2.3) with initial condition $\boldsymbol{\lambda}_0, \boldsymbol{u}_0$. Let $G(z) = G(z,t) = (\boldsymbol{u}^*\boldsymbol{\lambda}\boldsymbol{u} - z)^{-1}(t)$ be the Green function. For $\omega > \xi > 0$ and $\mathbf{q} \in \mathbb{R}^N$, consider the following three conditions (remember the notation (4.1) for $\mathbf{S}(\omega, N)$):

$$\begin{aligned} A_1(\mathbf{q},\omega,\xi,N) = \Big\{ &|\langle \mathbf{q}, G(z)\mathbf{q}\rangle - m(z)| < N^{\xi}\left(\sqrt{\frac{\Im m(z)}{N\eta}} + \frac{1}{N\eta}\right), \\ &\left|\frac{1}{N}\operatorname{Tr} G(z) - m(z)\right| < \frac{N^{\xi}}{N\eta} \text{ for all } t\in[0,1], z\in\mathbf{S}(\omega,N)\Big\}, \end{aligned} \tag{4.4}$$

$$A_2(\omega,N) = \left\{|\lambda_k(t) - \gamma_k| < N^{-\frac{2}{3}+\omega}(\hat{k})^{-\frac{1}{3}} \text{ for all } t\in[0,1], k\in[\![1,N]\!]\right\}, \tag{4.5}$$

$$A_3(\omega,N) = \left\{\langle \mathbf{q}, u_k(t)\rangle^2 < N^{-1+\omega} \text{ for all } t\in[0,1], k\in[\![1,N]\!]\right\}. \tag{4.6}$$

Note that the two conditions (4.5) and (4.6) follow from (4.4), i.e., $A_1 \subset A_2 \cap A_3$, by standard arguments. More precisely, (4.6) can be proved by the argument in the proof of Corollary 3.2 in [18]. The condition (4.5) is exactly the content of the rigidity of eigenvalues, i.e., Theorem 2.2 in [19]. Its proof in Section 5 of [19] used only the estimate (4.4).

The following lemma shows that these conditions hold with high probability.

Lemma 4.2. *For any $\omega > \xi > 0, D > 0$ and N large enough, we have*

$$\inf_{\mathbf{q}\in\mathbb{R}^N, |\mathbf{q}|=1} \mathbb{P}\left(A_1(\mathbf{q},\omega,\xi,N)\right) \geqslant 1 - N^{-D},$$

where the probability denotes the joint law of the random variable H_0 and the paths of $\boldsymbol{\lambda}, \boldsymbol{u}$.

Proof. For any fixed time, by (2.6) (4.2) and (4.3), the condition (4.4) holds with probability $1 - N^{-C}$ for any C. As C can be arbitrary, the same condition hold for any time and z in a discrete set of size $N^{C/2}$, say. For any two matrices H and H' with Green functions $G(z)$ and $G'(z)$, we have

$$[G(z) - G'(z)]_{ij} = -\sum_{k,\ell} G(z)_{ik}(H - H')_{k\ell} G'(z)_{\ell j}$$

Since

$$|G(z)_{ab}| \leqslant \sum_k \frac{|u_k(a)u_k(b)|}{|\lambda_k - z|} \leqslant (2\eta)^{-1} \sum_k (|u_k(a)|^2 + |u_k(b)|^2) \leqslant N\eta^{-1}, \quad \operatorname{Im} z := \eta,$$

we have

$$\Big|[G(z) - G'(z)]_{ij}\Big| = N^3 \eta^{-2} \sqrt{\sum_{k,\ell} |(H - H')_{k\ell}|^2}.$$

Applying this inequality to $H_t^{(s)}$ and $H_s^{(s)}$, we have with very high probability that

$$\sup_{|s-t|\leqslant|t-t'|} \Big|\langle \mathbf{q}, G(z)\mathbf{q}\rangle - \langle \mathbf{q}, G(z)\mathbf{q}\rangle\Big| \leqslant CN^6 |t-t'|^{1/2}$$

$N\eta \geqslant 1$. Here we have used the standard property that the sup over $[0, t-t']$ of a standard Brownian motion has size order $|t-t'|^{1/2}$ and Gaussian tails. Therefore, we can use a continuity argument to extend the estimate (4.4) to all $z \in \mathbf{S}$ and all time between 0 and 1. This proves (4.4). ☐

We define the set

$$A(\mathbf{q},\omega,\xi,\nu,N) = \Big\{(H_0,\boldsymbol{\lambda}) : \mathbb{P}\left(A_1(\mathbf{q},\omega,\xi,N)\Big|(H_0,\boldsymbol{\lambda})\right) \geqslant 1 - N^{-\nu}\Big\}. \tag{4.7}$$

From the previous lemma, one easily sees that for any $\omega > \xi$, ν and $D > 0$, we have, for large enough N,

$$\inf_{\mathbf{q}\in\mathbb{R}^N} \mathbb{P}\left(A(\mathbf{q},\omega,\xi,\nu,N)\right) \geqslant 1 - N^{-D}. \tag{4.8}$$

Theorem 4.3. *Let $n \in \mathbb{N}$ and f be a solution of the eigenvector moment flow (3.6) with initial matrix H_0 and path $\boldsymbol{\lambda}$ in $A(\mathbf{q},\omega,\xi,\nu,N)$ for some $\nu > 2$. Let $t = N^{-\frac{1}{4}+\delta}$, where $\delta \in (\frac{n\omega}{2}, 1/4]$ and we assume that $\omega > \xi$ and $n\omega < 1/2$. Then for any $\varepsilon > 0$ and large enough N we have*

$$\sup_{\boldsymbol{\eta}:\mathcal{N}(\boldsymbol{\eta})=n} |f_t(\boldsymbol{\eta}) - 1| \leqslant CN^{n\omega+\varepsilon-2\delta}. \tag{4.9}$$

The constant C depends on $\varepsilon, \omega, \delta$ and n but not on $\mathbf{q}$.

We have the following asymptotic normality for eigenvectors of a Gaussian divisible Wigner ensemble with a small Gaussian component.

Corollary 4.4. *Let δ be an arbitrarily small constant and $t = N^{-1/4+\delta}$. Let H_t be the solution to (2.1) and $(u_1(t), \dots, u_N(t))$ be an eigenbasis of H_t. The initial condition H_0 is assumed to be a symmetric generalized Wigner matrix. Then for any polynomial P in m variables and any $\varepsilon > 0$, for large enough N we have*

$$\sup_{I \subset [\![1,N]\!], |I|=m, |\mathbf{q}|=1} \left| \mathbb{E}\left(P\left((N\langle \mathbf{q}, u_k(t)\rangle^2)_{k\in I}\right)\right) - \mathbb{E}P\left((\mathscr{N}_j^2)_{j=1}^m\right)\right| \leqslant C N^{\varepsilon - 2\delta}. \tag{4.10}$$

Proof. Since H_0 is a generalized Wigner matrices, the isotropic local semicircle law, Theorem 4.1, holds for all time with ξ arbitrarily small. With $\omega = 2\xi$, and noticing that Lemma 4.2 holds for arbitrary large $\nu > 0$, (4.9) implies that (4.10) holds. □

Proof of Theorem 4.3. Because of (4.8) and $A_1 \subset A_2 \cap A_3$, we can assume in this proof that the trajectory $(H_t)_{0\leqslant t\leqslant 1}$ is in $A_1(\mathbf{q}, \omega, \xi, N) \cap A_2(\omega, N) \cap A_3(\omega, N)$; the complement of this set induces an additional error $\mathrm{O}(N^{-\nu+\xi})$ in (4.9), negligible compared to $N^{n\omega+\varepsilon-2\delta}$.

We begin with the case $n = 1$. Let $f_s(k) = f_s(\boldsymbol{\eta})$, where $\boldsymbol{\eta}$ is the configuration with one particle at the lattice point k. The equation (3.6) becomes

$$\partial_s f_s(k) = \frac{1}{N} \sum_{j\neq k} \frac{f_s(j) - f_s(k)}{(\lambda_j - \lambda_k)^2} \tag{4.11}$$

Assume that

$$\max_{k\in[\![1,N]\!]} f_s(k) = f_s(k_0)$$

for some k_0 (k_0 is not unique in general). Clearly, we have

$$\frac{f_s(j) - f_s(k_0)}{(\lambda_j - \lambda_{k_0})^2} \leqslant \frac{f_s(j) - f_s(k_0)}{(\lambda_j - \lambda_{k_0})^2 + \eta^2}$$

Together with (4.11), for any $\eta > 0$ we have

$$\partial_s f_s(k_0) = \frac{1}{N}\sum_{j\neq k_0} \frac{f_s(j) - f_s(k_0)}{(\lambda_j - \lambda_{k_0})^2} \leqslant \frac{1}{N\eta}\sum_{j\neq k_0} \frac{\eta f_s(j)}{(\lambda_j - \lambda_{k_0})^2 + \eta^2} - f_s(k_0)\frac{1}{N\eta}\sum_{j\neq k_0}\frac{\eta}{(\lambda_j - \lambda_{k_0})^2 + \eta^2}. \tag{4.12}$$

Notice that

$$\frac{1}{N}\sum_{1\leqslant j\leqslant N} \frac{\eta f_s(j)}{(\lambda_j - \lambda_{k_0})^2 + \eta^2} = \mathbb{E}\left(\sum_{j=1}^N \frac{\eta \langle \mathbf{q}, u_j\rangle^2}{(\lambda_j - \lambda_{k_0})^2 + \eta^2} \middle| (H_0, \boldsymbol{\lambda})\right).$$

From the definition of $A(\mathbf{q}, \omega, \xi, \nu, N)$, for $N^{-1+\omega} < \eta < 1$ we therefore have

$$\frac{1}{N}\sum_{1\leqslant j\leqslant N, j\neq k_0} \frac{\eta f_s(j)}{(\lambda_j - \lambda_{k_0})^2 + \eta^2} = \Im m(\lambda_{k_0} + \mathrm{i}\eta) + \mathrm{O}\left(\frac{N^{\xi}(\Im m(\lambda_{k_0} + \mathrm{i}\eta))^{1/2}}{(N\eta)^{1/2}} + \frac{N^{\omega}}{N\eta}\right),$$

where the error $N^{\omega}/(N\eta)$ comes from the missing term $j = k_0$ and we have used that for $(H_0, \lambda) \in A(\mathbf{q}, \omega, \xi, \nu, N)$, $N\langle \mathbf{q}, u_j\rangle^2$ is bounded by N^{ω} with very high probability. For the same reason, we have

$$\frac{1}{N}\sum_{1\leqslant j\leqslant N, j\neq k_0} \frac{\eta}{(\lambda_j - \lambda_{k_0})^2 + \eta^2} = \Im m(\lambda_{k_0} + \mathrm{i}\eta) + \mathrm{O}\left(\frac{N^{\omega}}{N\eta}\right).$$

Using these estimates, (4.12) yields

$$\partial_s(f_s(k_0)-1) \leqslant -c\frac{\Im m(\lambda_{k_0}+\mathrm{i}\eta)}{\eta}(f_s(k_0)-1) + \mathrm{O}\left(\frac{N^{\xi}\Im m(\lambda_{k_0}+\mathrm{i}\eta)^{1/2}}{N^{1/2}\eta^{3/2}}\right) + \mathrm{O}\left(\frac{N^{\omega}}{N\eta}\right).$$

Moreover, from the definition of $A(\mathbf{q},\omega,\xi,\nu,N)$, we know that $-2-N^{-\frac{2}{3}+\omega} \leqslant \lambda_{k_0} \leqslant 2+N^{-\frac{2}{3}+\omega}$. As our final choice of η will satisfy $N^{-\frac{2}{3}+\xi} \leqslant \eta \leqslant 1$, this implies that

$$\Im m(\lambda_{k_0}+\mathrm{i}\eta) \geqslant c\sqrt{\eta}.$$

Let $S_s = \sup_k(f_s(k)-1)$. Note that there may be some s for which S_s is not differentiable (at times when the maximum is obtained for at least two distinct indices). But if we denote

$$S'_t = \limsup_{u\to t}\frac{S_t - S_u}{t-u}, \tag{4.13}$$

the above reasoning shows

$$S'_s \leqslant -\frac{c}{\sqrt{\eta}}S_s + C\frac{N^{\xi}}{N^{1/2}\eta^{3/2}} + C\frac{N^{\omega}}{N\eta} \leqslant -\frac{c}{\sqrt{\eta}}S_s + C\frac{N^{\omega}}{N^{1/2}\eta^{3/2}}.$$

We chose $\eta = N^{-\frac{1}{2}+2\delta-\varepsilon}$ for some small $\varepsilon \in (0, 2\delta-\omega)$ and $t = N^{-\frac{1}{4}+\delta}$. The Gronwall inequality gives

$$S_t \leqslant C\left(e^{-N^{\varepsilon/2}} + N^{\omega+\varepsilon-2\delta}\right).$$

We can do the same reasoning for the minimum of f. This concludes the proof for $n=1$.

For $n \geqslant 2$ the same argument works and we will proceed by induction. Let $\boldsymbol{\xi}$ satisfy

$$\max_{\mathcal{N}(\boldsymbol{\eta})=n} f_s(\boldsymbol{\eta}) = f_s(\boldsymbol{\xi}).$$

Assume $\boldsymbol{\xi}$ is associated to j_r particles at site k_r, $1 \leqslant r \leqslant m$ for some $m \leqslant n$, where the k_r's are distinct and $j_r \geqslant 1$. Then

$$\partial_s f_s(\boldsymbol{\xi}) \leqslant C\sum_{r=1}^{m}\left(\frac{1}{N\eta}\sum_{j\neq k_r}\frac{\eta f_s(\boldsymbol{\xi}^{k_r j})}{(\lambda_{k_r}-\lambda_j)^2+\eta^2} - f_s(\boldsymbol{\xi})\frac{1}{N\eta}\sum_{j\neq k_r}\frac{\eta}{(\lambda_{k_r}-\lambda_j)^2+\eta^2}\right), \tag{4.14}$$

where $\boldsymbol{\xi}^{k_r j}$ is defined in Section 3.2. We now estimate the first term on the right hand side (the second term was estimated in the previous $n=1$ step). By (4.6), for $(H_0,\lambda) \in A(\mathbf{q},\omega,\xi,\nu,N)$, $N\langle \mathbf{q}, u_j\rangle^2$ is bounded by N^{ω} with very high probability. Thus we have

$$\frac{1}{N}\sum_{j\neq k_r}\frac{\eta f_s(\boldsymbol{\xi}^{k_r j})}{(\lambda_{k_r}-\lambda_j)^2+\eta^2} = \frac{1}{N}\sum_{j\notin\{k_1,\dots,k_m\}}\frac{\eta f_s(\boldsymbol{\xi}^{k_r j})}{(\lambda_{k_r}-\lambda_j)^2+\eta^2} + \mathrm{O}\left(\frac{N^{n\omega}}{N\eta}\right).$$

Moreover, by definition the above sum can be estimated by

$$\begin{aligned}
&\mathbb{E}\left(\left(\frac{(N\langle \mathbf{q}, u_{i_r}\rangle^2)^{j_r-1}}{a(2j_r-2)}\prod_{1\leqslant r\leqslant m, r\neq r}\frac{(N\langle \mathbf{q}, u_{i_r}\rangle^2)^{j_r}}{a(2j_r)}\right)\left(\frac{1}{N}\sum_{j\notin\{k_1,\dots,k_m\}}\frac{\eta(N\langle \mathbf{q}, u_j\rangle^2)}{(\lambda_j-\lambda_{k_r})^2+\eta^2}\right)\,\Big|\,(H_0,\boldsymbol{\lambda})\right)\\
&= \mathbb{E}\left(\left(\frac{(N\langle \mathbf{q}, u_{i_r}\rangle^2)^{j_r-1}}{a(2j_r-2)}\prod_{1\leqslant r\leqslant m, r\neq r}\frac{(N\langle \mathbf{q}, u_{i_r}\rangle^2)^{j_r}}{a(2j_r)}\right)\Im\langle \mathbf{q}, G(\lambda_{k_r}+\mathrm{i}\eta), \mathbf{q}\rangle\,\Big|\,(H_0,\boldsymbol{\lambda})\right) + \mathrm{O}\left(\frac{N^{n\omega}}{N\eta}\right),
\end{aligned} \tag{4.15}$$

where we first used that extending the indices to $1 \leqslant j \leqslant N$ induces an error $\mathrm{O}(N^{\omega}(N\eta)^{-1})$ and the bound $N\langle \mathbf{q}, u_j\rangle^2 \leqslant N^{\omega}$ holds with very high probability. We have also used that for $(H_0, \lambda) \in A(\mathbf{q}, \omega, \xi, \nu, N)$, we can replace $\Im\langle \mathbf{q}, G(\lambda_{k_r} + \mathrm{i}\eta), \mathbf{q}\rangle$ by $\Im m(\lambda_{k_r} + \mathrm{i}\eta) + \mathrm{O}(N^{\xi}(N\eta)^{-1/2})$. This yields

$$\frac{1}{N}\sum_{j\neq k_r} \frac{\eta f_s(\boldsymbol{\xi}^{k_r j})}{(\lambda_{k_r}-\lambda_j)^2+\eta^2} = f_s(\boldsymbol{\xi}\backslash k_r)\Im m(\lambda_{k_r}+\mathrm{i}\eta) + \mathrm{O}\left(\frac{N^{n\omega}}{(N\eta)^{1/2}}\right), \tag{4.16}$$

where $\boldsymbol{\xi}\backslash k_r$ stands for the configuration $\boldsymbol{\xi}$ with one particle removed from site k_r. By induction assumption, we can use (4.9) to estimate $f_s(\boldsymbol{\xi}\backslash i_r)$ for $s \in (t/2, t)$. We have thus proved that

$$\partial_s(f_s(\boldsymbol{\xi})-1) \leqslant -\frac{c}{\sqrt{\eta}}(f_s(\boldsymbol{\xi})-1) + \mathrm{O}\left(\frac{N^{n\omega}}{N^{1/2}\eta^{3/2}}\right) + \mathrm{O}\left(\frac{N^{(n-1)\omega+\varepsilon-2\delta}}{\eta}\right).$$

on $(t/2, t)$. Notice that by our assumptions on the parameters ω, δ, η and ξ, the first error term always dominates the second. One can now bound $|f_s(\boldsymbol{\xi}) - 1|$ in the same way as in the $n = 1$ case. □

If ω can be chosen arbitrarily small (this is true for generalized Wigner matrices), Theorem 4.3 gives $\sup_{\boldsymbol{\eta}:\mathcal{N}(\boldsymbol{\eta})=n} |f_t(\boldsymbol{\eta}) - 1| \to 0$ for any $t = N^{-1/4+\varepsilon}$. This could be improved to $t = N^{-1/3+\varepsilon}$ by allowing η to depend on k_0 in the previous reasoning (chose $\eta = N^{-2/3+\varepsilon}\hat{k}_0^{1/3}$).

More generally, our proof shows that the following equation (4.17) (with the convention 4.13) holds. Let

$$\begin{aligned}\Delta_1(k,\eta) &= \mathbb{E}(\langle \mathbf{q}, G(\lambda_k+\mathrm{i}\eta)\mathbf{q}\rangle - \Im m(\lambda_k+\mathrm{i}\eta) \mid (H_0,\boldsymbol{\lambda})),\\ \Delta_2(k,\eta) &= \mathbb{E}(N^{-1}\mathrm{Tr} G(\lambda_k+\mathrm{i}\eta) - \Im m(\lambda_k+\mathrm{i}\eta) \mid (H_0,\boldsymbol{\lambda})),\end{aligned}$$

where all variables depend on t (remember in particular that $G(z) = (\boldsymbol{u}_t^*\boldsymbol{\lambda}_t\boldsymbol{u}_t - z)^{-1}$). Then the following maximum inequality holds:

$$S_t' \leqslant \max_{k:S_t=f_t(k)} \inf_{\eta>0}\left\{-\frac{\Im m(\lambda_k+\mathrm{i}\eta)}{\eta}\, S_t + \frac{|\Delta_1(k,\eta)|}{\eta} + \frac{|\Delta_2(k,\eta)|(S_t+1)}{\eta} + \frac{N^{\omega}(S_t+1)}{N\eta^2}\right\}. \tag{4.17}$$

Similar inequalities for a general number of particles can be obtained.

5 Proof of the main results

5.1 A comparison theorem for eigenvectors. Corollary 4.4 asserts the asymptotic normality of eigenvector components for Gaussian divisible ensembles for t not too small. In order to prove Theorem 1.2, we need to remove the small Gaussian components of the matrix elements in this Gaussian divisible ensemble. Similar questions occurred in the proof of universality conjecture for Wigner matrices and several methods were developed for this purpose (see, e.g., [12] and [34]). Both methods can be extended to yielding similar eigenvector comparison results. In this paper, we will use the Green function comparison theorem introduced in [18, Theorem 2.3] (the parallel result following the argument of [34] was given in [35]). Roughly speaking, [22, Theorem 1.10] states that the distributions of eigenvectors for two generalized Wigner ensembles are identical provided the first four moments of the matrix elements are identical and a level repulsion estimate holds for *one* of the two ensembles. We note that the level repulsion estimates needed in [35] are substantially different. We first recall the following definition.

Definition 5.1 (Level repulsion estimate)**.** *Fix an energy E such that $\gamma_k \leqslant E \leqslant \gamma_{k+1}$ for some $k \in [\![1, N]\!]$. A generalized Wigner ensemble is said to satisfy the level repulsion at the energy E if there exist $\alpha_0 > 0$ such that for any $0 < \alpha < \alpha_0$, there exists $\delta > 0$ such that*

$$\mathbb{P}\left(|\{j : \lambda_j \in [E - N^{-2/3-\alpha}\hat{k}^{-1/3}, E + N^{-2/3-\alpha}\hat{k}^{-1/3}]\}| \geqslant 2\right) \leqslant N^{-\alpha-\delta},$$

where $\hat{k} = \min(k, N-k+1)$. A matrix ensemble is said to satisfy the level repulsion estimate uniformly if this property holds for any energy $E \in (-2, 2)$.

We note that such level repulsion estimates for generalized Wigner matrices was proved near the edge (more precisely for $0 \leqslant \hat{k} \leqslant N^{1/4}$) [7, Theorem 2.7] and in the bulk [16] via the universality of gap statistics. In the intermediate regime, the level repulsion in this sense has not been worked out although it is clear that the techniques developed in these papers can be adapted to prove such results. From now on, we will assume that this level repulsion estimate holds in the region $\mathbb{T}_N = [\![1, N^{1/4}]\!] \cup [\![N^{1-\delta}, N - N^{1-\delta}]\!] \cup [\![N - N^{1/4}, N]\!]$ needed for Theorem 1.2 and its corollaries.

The following theorem is a slight extension of [22, Theorem 1.10] with the following modifications : (1) We slightly weaken the fourth moment matching condition. (2) The original theorem was only for components of eigenvectors; we allow the eigenvector to project to a fixed direction. (3) We state it for all energies in the entire spectrum. (4) We include an error bound for the comparison. (5) We state it only for eigenvectors with no involvement of eigenvalues. Theorem 5.2 can be proved using the argument in [22]; the only modification is to replace the local semicircle law used in [22] by the isotropic local semicircle law, Theorem 4.1. Since this type of argument based on the Green function comparison theorem has been done several times, we will not repeat it here. Notice that near the edge, the four moment matching condition can be replaced by just two moments. But for applications in this paper, this improvement will not be used and so we refer the interested reader to [22].

Theorem 5.2 (Eigenvector Comparison Theorem)**.** *Let $H^{\mathbf{v}}$ and $H^{\mathbf{w}}$ be generalized Wigner ensembles where $H^{\mathbf{v}}$ satisfies the level repulsion estimate uniformly. Suppose that the first three off-diagonal moments of $H^{\mathbf{v}}$ and $H^{\mathbf{w}}$ are the same, i.e.*

$$\mathbb{E}^{\mathbf{v}}(h_{ij}^3) = \mathbb{E}^{\mathbf{w}}(h_{ij}^3) \qquad \text{for} \quad i \neq j$$

and that the first two diagonal moments of $H^{\mathbf{v}}$ and $H^{\mathbf{w}}$ are the same, i.e.

$$\mathbb{E}^{\mathbf{v}}(h_{ii}^2) = \mathbb{E}^{\mathbf{w}}(h_{ii}^2).$$

Assume also that the fourth off-diagonal moments of $H^{\mathbf{v}}$ and $H^{\mathbf{w}}$ are almost the same, i.e., there is an $a > 0$ such that

$$\left|\mathbb{E}^{\mathbf{v}}(h_{ij}^4) - \mathbb{E}^{\mathbf{w}}(h_{ij}^4)\right| \leqslant N^{-2-a} \qquad \text{for} \quad i \neq j.$$

Then there is $\varepsilon > 0$ depending on a such that for any integer k, any $\mathbf{q}_1, \dots \mathbf{q}_k$ and any choice of indices $1 \leqslant j_1, \dots, j_k \leqslant N$ we have

$$(\mathbb{E}^{\mathbf{v}} - \mathbb{E}^{\mathbf{w}})\, \Theta\Big(N\langle \mathbf{q}, u_{j_1}\rangle^2, \dots, N\langle \mathbf{q}, u_{j_k}\rangle^2\Big) = \mathrm{O}(N^{-\varepsilon}),$$

where Θ is a smooth function that satisfies

$$|\partial^m \Theta(x)| \;\leqslant\; C(1 + |x|)^C$$

for some arbitrary C and all $m \in \mathbb{N}^k$ satisfying $|m| \leqslant 5$.

5.2 Proof of Theorem 1.2 .We now summarize our situation: Given a generalized Wigner ensemble $\hat{H}$, we wish to prove that (1.5) holds for the eigenvectors of $\hat{H}$. We have proved in (4.10) that this estimate holds for any Gaussian divisible ensemble of type $H_0 + \sqrt{t}\, U$, and therefore by simple rescaling for any ensemble of type

$$H_t = e^{-t/2} H_0 + (1 - e^{-t})^{1/2}\, U,$$

where H_0 is *any* initial generalized Wigner matrix and U is an independent standard GOE matrix, as long as $t \geqslant N^{-1/4+\delta}$. We fix δ a small number, say, $\delta = 1/8$. Now we construct a generalized Wigner matrix H_0 such that the first three moments of H_t match exactly those of the target matrix $\hat{H}$ and the differences between the fourth moments of the two ensembles are less than N^{-c} for some c positive. This existence of such an initial random variable is guaranteed by, say, Lemma 3.4 of [17]. By the eigenvector comparison theorem, Theorem 5.2, we have proved (1.5) and this concludes our proof of Theorem 1.2.

5.3 Proof of Corollary 1.3. Let $\mathscr{N} = (\mathscr{N}_1, \dots, \mathscr{N}_N)$ be a Gaussian vector with covariance Id. Let $m, \ell \in \mathbb{N}$, $k \in \mathbb{T}_N$ and $\{i_1, \dots, i_\ell\} := J \subset [\![1, N]\!]$. For $\mathbf{q}$ such that $q_i = 0$ if $i \notin J$, consider the polynomial in ℓ variables:

$$Q(q_{i_1}, \dots, q_{i_\ell}) = \mathbb{E}\left((N|\langle \mathbf{q}, u_k\rangle|^2)^m\right) - \mathbb{E}\left(|\langle \mathbf{q}, \mathscr{N}\rangle|^{2m}\right).$$

From (1.5), there exists $\varepsilon > 0$ such that

$$\sup_{|q_{i_1}|^2 \leqslant \frac{1}{\ell}, \dots, |q_{i_\ell}|^2 \leqslant \frac{1}{\ell}} |Q(q_{i_1}, \dots, q_{i_\ell})| \leqslant \sup_{|q|=1} |Q(q_{i_1}, \dots, q_{i_\ell})| \leqslant N^{-\varepsilon},$$

where, for the first inequality, we note that that the maximum of Q in the unit ball is achieved on the unit sphere. Noting $R(q_{i_1}) = Q(q_{i_1}, \dots, q_{i_\ell})$ with the coefficients of the polynomial R depending on $q_{i_2}, \dots, q_{i_\ell}$, the above bound implies that all the coefficients of R are bounded by $C_1 N^{-\varepsilon}$ for some universal constant C_1 (indeed, one recovers the coefficients of R from its evaluation at $\ell + 1$ different points, by inverting a Vandermonde matrix).

By iterating the above bound on the coefficients finitely many times (ℓ iterations), we conclude that there is a universal constant C_ℓ such that all coefficients of Q are bounded by $C_\ell N^{-\varepsilon}$. This means that for any $k \in \mathbb{T}_N$ and $J \subset [\![1, N]\!]$ with $|J| = \ell$,

$$\left| \mathbb{E}\left(\prod_{\alpha \in J} \left(\sqrt{N} u_k(\alpha)\right)^{m_\alpha} \right) - \mathbb{E}\left(\prod_{\alpha \in J} (\mathscr{N}_\alpha)^{m_\alpha} \right) \right| \leqslant C\, N^{-\varepsilon}$$

whenever the integer exponents m_α satisfy $\sum m_\alpha = m$. Here C depends only on m, not on the choice of k or J. This concludes the proof of (1.6), in the case of a monomial P with even degree. If P is a monomial of odd degree, (1.6) is trivial: the left hand side vanishes thanks to the uniform phase choice $e^{\mathrm{i}\omega}$. This concludes the proof of Corollary 1.3.

5.4 Proof of Corollary 1.4. A second moment calculation yields

$$\mathbb{E}\left(\left(\frac{N}{|a_N|} \langle u_k, a_N u_k \rangle \right)^2 \right) = \frac{1}{|a_N|^2} \mathbb{E}\left(\left(\sum_\alpha a_N(\alpha)(N|u_k(\alpha)|^2 - 1) \right)^2 \right)$$

$$\leqslant \max_{\alpha \neq \beta} \mathbb{E}\left((N|u_k(\alpha)|^2 - 1)\,(N|u_k(\beta)|^2 - 1) \right) + \frac{1}{|a_N|} \max_\alpha \mathbb{E}\left(\left(N|u_k(\alpha)|^2 - 1\right)^2 \right).$$

From (1.6), the first term of the right hand side is bounded by $N^{-\varepsilon}$ and the second term is bounded by $1/|a_N|$. The Markov inequality then allows us to conclude the proof of Corollary 1.4.

6 Finite speed of propagation

In this section, we prove a finite speed of propagation estimate for the dynamics (3.6). This estimate will be a key ingredient for proving optimal relaxation time for eigenvectors in the bulk. Finite speed of propagation was first proved in [15, Section 9.6] for (3.6) when the number of particle $n=1$. But it requires a level repulsion estimate which is difficult to prove. Our estimate requires only the rigidity of eigenvalues (which holds with very high probability) and the speed of propagation obtained is nearly optimal. Our key observation is that we can construct weight functions used in the finite speed estimate depending on eigenvalues so that the singularities in the equation (3.6) are automatically cancelled by the choices of these weight functions.

We will follow the approach of [15] by decomposing the dynamics into a long range part and a short range part. The long range part can be controlled by a general argument based on decay estimate; the main new idea is in the proof of a finite speed of propagation for the short range dynamics, which is the content of Lemma 6.2.

6.1 Long and short range dynamics. We assume that for some (small) fixed parameter $\xi > 0$ there is a constant C such that for any $|i-j| \geqslant N^{\xi}$ and $0 \leqslant s \leqslant 1$ the quantity c_{ij} defined in (2.7) satisfies the following estimate

$$c_{ij}(s) \leqslant C\, \frac{N}{(i-j)^2}. \tag{6.1}$$

If M_N is distributed as a generalized Wigner matrix, then for any $\xi > 0$, (6.1) holds [19] with probability $1-e^{-c(\log N)^2}$ for some $c>0$. In this section M_N is not assumed to be distributed as a generalized Wigner matrix. Instead, we assume that (6.1) holds.

The following cutoff of the dynamics will be useful. Let $1 \ll \ell \ll N$ be a parameter to be specified later. We split the time dependent operator $\mathscr{B}$ defined in (3.7) into a short-range and a long-range part: $\mathscr{B} = \mathscr{S} + \mathscr{L}$, with

$$\begin{aligned} (\mathscr{S} f)(\boldsymbol{\eta}) &= \sum_{|j-k|\leqslant \ell} c_{jk}(s) 2\eta_j(1+2\eta_k)\left(f(\boldsymbol{\eta}^{j,k}) - f(\boldsymbol{\eta})\right), \\ (\mathscr{L} f)(\boldsymbol{\eta}) &= \sum_{|j-k|> \ell} c_{jk}(s) 2\eta_j(1+2\eta_k)\left(f(\boldsymbol{\eta}^{j,k}) - f(\boldsymbol{\eta})\right). \end{aligned} \tag{6.2}$$

Notice that $\mathscr{S}$ and $\mathscr{L}$ are time dependent. Moreover, $\mathscr{S}$ is also reversible with respect to π (the proof of Proposition 3.2 applies to any symmetric c_{ij}'s). Denote by $\mathrm{U}_{\mathscr{S}}(s,t)$ the semigroup associated with $\mathscr{S}$ from time s to time t, i.e.

$$\partial_t \mathrm{U}_{\mathscr{S}}(s,t) = \mathscr{S}(t)\mathrm{U}_{\mathscr{S}}(s,t)$$

for any $s \leqslant t$, and $\mathrm{U}_{\mathscr{S}}(s,s) = \mathrm{Id}$. The notation $\mathrm{U}_{\mathscr{B}}(s,t)$ is analogous. In the following lemma, we prove that the short-range dynamics provide a good approximation of the global dynamics. Lemmas 6.1 follows the same proof as in [15], where they were shown for $n=1$.

Lemma 6.1. *Suppose that the coefficients of $\mathscr{B}$ satisfy (6.1) for some $\xi > 0$ and let $\ell \gg N^{\xi}$. Suppose that the initial data is the delta function at an arbitrary configuration $\boldsymbol{\eta}$. Then for any $s \geqslant 0$ we have*

$$\| \left(\mathrm{U}_{\mathscr{B}}(0,s) - \mathrm{U}_{\mathscr{S}}(0,s)\right) \delta_{\boldsymbol{\eta}} \|_1 \leqslant C\, \frac{Ns}{\ell},$$

where C only depends on ξ (in particular not on $\boldsymbol{\eta}$).

Proof. By the Duhamel formula we have

$$\mathrm{U}_{\mathscr{B}}(0,s)\delta_{\boldsymbol{\eta}}=\mathrm{U}_{\mathscr{S}}(0,s)\delta_{\boldsymbol{\eta}}+\int_0^s \mathrm{U}_{\mathscr{B}}(s',s)\mathscr{L}(s')\mathrm{U}_{\mathscr{S}}(0,s')\delta_{\boldsymbol{\eta}}\mathrm{d}s'.$$

Notice that for $\ell\gg N^{\xi}$ we can use (6.1) to get

$$\|\mathscr{L}f\|_1\leqslant\sum_{\boldsymbol{\eta}}\sum_{|j-k|\geqslant\ell}c_{jk}\eta_j(1+2\eta_k)\left(|f(\boldsymbol{\eta}^{j,k})|+|f(\boldsymbol{\eta})|\right)\leqslant C\ N\ell^{-1}\|f\|_1.$$

Since $\mathrm{U}_{\mathscr{B}}$ and $\mathrm{U}_{\mathscr{S}}$ are contractions in L^1, this yields

$$\int_0^s\|\mathrm{U}_{\mathscr{B}}(s',s)\mathscr{L}(s')\mathrm{U}_{\mathscr{S}}(0,s')\delta_{\boldsymbol{\eta}}\|_1\,\mathrm{d}s'\leqslant C\ N\ell^{-1}\int_0^s\|\delta_{\boldsymbol{\eta}}\|_1\,\mathrm{d}s'\leqslant C\ \frac{Ns}{\ell},$$

which concludes the proof. □

6.2 Finite speed of propagation for the short range dynamics. Suppose that $\boldsymbol{\eta}$ is a configuration with n particles. We denote the particles in nondecreasing order by $\mathbf{x}(\boldsymbol{\eta})=(x_1(\boldsymbol{\eta}),\dots,x_n(\boldsymbol{\eta}))$ with $\alpha N\leqslant x_1\leqslant\dots\leqslant x_n\leqslant(1-\alpha)N$. We will drop the dependence on $\boldsymbol{\eta}$ and simply use $(x_1,\dots,x_n)$. In the same way, we also denote the configuration $\boldsymbol{\xi}$ by $\mathbf{y}$ with $1\leqslant y_1\leqslant\dots\leqslant y_n\leqslant N$ where we have dropped the dependence of $\boldsymbol{\xi}$ in $y_\alpha(\boldsymbol{\xi})$. This convention will be followed for the rest of this paper.

We define the following distance on the set of configurations with n particles:

$$d(\boldsymbol{\eta},\boldsymbol{\xi})=\sum_{\alpha=1}^n|x_\alpha-y_\alpha|=\min_{\sigma\in\mathscr{S}_n}\sum_{\alpha=1}^n|x_\alpha-y_{\sigma(\alpha)}|.\tag{6.3}$$

For the second equality, observe that for any $x\leqslant y$ and $a\leqslant b$, we have $|x-a|+|y-b|\leqslant|x-b|+|y-a|$.

Before stating our finite speed result, we also need the notation $r_s(\boldsymbol{\eta},\boldsymbol{\xi})=(\mathrm{U}_{\mathscr{S}}(0,s)\delta_{\boldsymbol{\eta}})(\boldsymbol{\xi})$.

Lemma 6.2. *Suppose that the eigenvalue $\boldsymbol{\lambda}$ satisfies the condition (4.5) with exponent ω such that $N^{\omega}\ll\ell$. Let $\alpha,\varepsilon>0$ and choose $\ell\geqslant Nt$ for the short range dynamics cutoff.*

(i) Uniformly in $\boldsymbol{\eta}$ supported on $[\![\alpha N,(1-\alpha)N]\!]$ and $t>0$, if $d(\boldsymbol{\eta},\boldsymbol{\xi})\geqslant N^{\varepsilon}\ell$, we have

$$\mathbb{P}\left(r_s(\boldsymbol{\eta},\boldsymbol{\xi})>e^{-N^{\varepsilon/2}}\right)=\mathrm{O}\left(N^{-D}\right)\tag{6.4}$$

for any $D>0$. Here $\mathbb{P}$ denotes integration with respect to the Dyson Brownian Motion.

(ii) Uniformly in $\boldsymbol{\eta}$ supported on $[\![1,N]\!]$ and $t>0$, if $d(\boldsymbol{\eta},\boldsymbol{\xi})\geqslant N^{\frac{1}{3}+\varepsilon}\ell^{\frac{2}{3}}$, the finite speed estimate (6.4) holds.

Proof. We first consider the case (i) corresponding to $\boldsymbol{\eta}$ supported in the bulk, but the reader may want to read first the proof of (ii), written for the simpler case $n=1$ for the sake of simplicity.

First step: definitions and dynamics. Let $\nu=N/\ell$ and $\kappa>0$ be a fixed parameter such that $-2+\kappa<\gamma_{\alpha N}$. For any $1\leqslant i\leqslant N$ and $x\in\mathbb{R}$, let $d_i(x)=|x-\gamma_i|$. Let $g_i(x)=d_i(2-\kappa)$ if $x>2-\kappa$, $g_i(x)=d(-2+\kappa)$ if $x<-2+\kappa$ and $g_i(x)=d_i(x)$ if $-2+\kappa\leqslant x\leqslant 2-\kappa$. Take χ a smooth, nonnegative, compactly supported function with $\int\chi=1$, and $\psi_i(x)=\int g_i(x-y)\nu\chi(\nu y)\mathrm{d}y$. Then ψ_i is smooth, $\|\psi_i'\|_\infty\leqslant1$ and $\|\psi_i''\|_\infty\leqslant\nu$.

Moreover, consider the stopping time

$$\tau = \inf\left\{s \geqslant 0 \mid \exists k \in [\![1, N]\!] : |\lambda_k(s) - \gamma_k| > N^{-\frac{2}{3}}(\hat{k})^{-\frac{1}{3}}\ell\right\}. \tag{6.5}$$

For any configuration $\boldsymbol{\xi}$ with n particles we define

$$\psi_s(\boldsymbol{\xi}) = \sum_{\alpha=1}^{n} \psi_{x_\alpha}(\lambda_{y_\alpha}(s\wedge\tau)) = \min_{\sigma\in\mathscr{S}_n} \sum_{\alpha=1}^{n} \psi_{x_\alpha}(\lambda_{y_{\sigma(\alpha)}}(s\wedge\tau)), \tag{6.6}$$

similarly to (6.3). For the second equality, observe that if $\alpha \leqslant \beta$ and $a \leqslant b$, then $\psi_\alpha(a)+\psi_\beta(b) \leqslant \psi_\alpha(b)+\psi_\beta(a)$ (the function $a \mapsto \psi_\alpha(a) - \psi_\beta(a)$ is nondecreasing).

We define

$$\phi_s(\boldsymbol{\xi}) = e^{\nu\psi_s(\boldsymbol{\xi})}, \quad v_s(\boldsymbol{\xi}) = \phi_s(\boldsymbol{\xi}) r_{s\wedge\tau}(\boldsymbol{\eta},\boldsymbol{\xi}).$$

Then we have (we omit the s index)

$$\mathrm{d}v(\boldsymbol{\xi}) = \sum_{|j-k|\leqslant\ell} 2\xi_k(1+2\xi_j)c_{jk}\left((v(\boldsymbol{\xi}^{kj}) - v(\boldsymbol{\xi})) + \left(\frac{\phi(\boldsymbol{\xi})}{\phi(\boldsymbol{\xi}^{kj})} - 1\right)v(\boldsymbol{\xi}^{kj})\right)\mathrm{d}(s\wedge\tau) + (\mathrm{d}\phi(\boldsymbol{\xi}))\, r(\boldsymbol{\eta},\boldsymbol{\xi})$$

$$\frac{\mathrm{d}\phi(\boldsymbol{\xi})}{\phi(\boldsymbol{\xi})} = \sum_{\alpha=1}^{n}\Bigg(\nu\psi'_{x_\alpha}(\lambda_{y_\alpha})\frac{\mathrm{d}B_{y_\alpha}(s\wedge\tau)}{\sqrt{N}} + \nu\frac{\psi'_{x_\alpha}(\lambda_{y_\alpha})}{N}\sum_{j\neq y_\alpha}\frac{\mathrm{d}(s\wedge\tau)}{\lambda_{y_\alpha}-\lambda_j}$$
$$+c_1\frac{\nu}{2N}\psi''_{x_\alpha}(\lambda_{y_\alpha})\mathrm{d}(s\wedge\tau) + c_2\frac{\nu^2}{2N}\psi'_{x_\alpha}(\lambda_{y_\alpha})^2\mathrm{d}(s\wedge\tau)\Bigg)$$

The coefficients c_1 and c_2 are non-random positive combinatorial factors depending on the locations of i, $\boldsymbol{\eta},\boldsymbol{\xi}$, but we will only need that they are uniformly bounded in N. We will adopt the convention to use indices $1 \leqslant \alpha,\beta \leqslant n, 1 \leqslant i,j,k \leqslant N$. We define

$$X_s = \sum_{\boldsymbol{\xi}} \pi(\boldsymbol{\xi}) v_s(\boldsymbol{\xi})^2,$$

where $\pi = \pi^{(s)}$ is the reversible measure defined in (3.9) for the symmetric eigenvector moment flow (which is also reversible w.r.t its short-range cutoff version). Then

$$\mathrm{d}X_s = 2\sum_{\boldsymbol{\xi}}\pi(\boldsymbol{\xi})v(\boldsymbol{\xi})\sum_{|j-k|\leqslant\ell}2\xi_k(1+2\xi_j)c_{jk}\left((v(\boldsymbol{\xi}^{kj}) - v(\boldsymbol{\xi})) + \left(\frac{\phi(\boldsymbol{\xi})}{\phi(\boldsymbol{\xi}^{kj})} - 1\right)v(\boldsymbol{\xi}^{kj})\right)\mathrm{d}(s\wedge\tau) \tag{6.7}$$

$$+ 2\sum_{\boldsymbol{\xi}}\pi(\boldsymbol{\xi})v(\boldsymbol{\xi})\,(\mathrm{d}\phi(\boldsymbol{\xi}))\, r(\boldsymbol{\eta},\boldsymbol{\xi}) \tag{6.8}$$

$$+ \sum_{\boldsymbol{\xi}}\pi(\boldsymbol{\xi})\mathrm{d}\langle v(\boldsymbol{\xi})\rangle_{s\wedge\tau}. \tag{6.9}$$

Second step: bound on (6.7) and (6.9). Using reversibility with respect to π, the first term can be written

$$(6.7) = -\sum_{\boldsymbol{\xi}}\pi(\boldsymbol{\xi})\sum_{|j-k|\leqslant\ell}2\xi_k(1+2\xi_j)c_{jk}(v(\boldsymbol{\xi}^{kj}) - v(\boldsymbol{\xi})))^2\mathrm{d}(s\wedge\tau) \tag{6.10}$$

$$+ \sum_{\boldsymbol{\xi}}\pi(\boldsymbol{\xi})\sum_{|j-k|\leqslant\ell}2\xi_k(1+2\xi_j)c_{jk}\left(\frac{\phi(\boldsymbol{\xi}^{kj})}{\phi(\boldsymbol{\xi})} + \frac{\phi(\boldsymbol{\xi})}{\phi(\boldsymbol{\xi}^{kj})} - 2\right)v(\boldsymbol{\xi})v(\boldsymbol{\xi}^{kj})\mathrm{d}(s\wedge\tau). \tag{6.11}$$

Here the equality (6.10) is a direct application of the reversibility property, while (6.11) also follows from the reversibility as follows. Notice that

$$\sum_{\boldsymbol{\xi}} \pi(\boldsymbol{\xi}) v(\boldsymbol{\xi}) \sum_{|j-k|\leqslant \ell} 2\xi_k(1+2\xi_j) c_{jk} \frac{\phi(\boldsymbol{\xi})}{\phi(\boldsymbol{\xi}^{kj})} v(\boldsymbol{\xi}^{kj}) = \langle g, \mathscr{S} r\rangle_\pi + \langle g, r\rangle_\pi, \quad g = \phi^2 r \tag{6.12}$$

One can check that (6.11) follows from $\langle g, \mathscr{S} r\rangle_\pi = \langle \mathscr{S} g, r\rangle_\pi$.

We now estimate the term $\frac{\phi(\boldsymbol{\xi}^{kj})}{\phi(\boldsymbol{\xi})} + \frac{\phi(\boldsymbol{\xi})}{\phi(\boldsymbol{\xi}^{kj})} - 2$ in (6.11). If it is nonzero (and we assume first that $j < k$) then there exists $1 \leqslant p < q \leqslant n$ such that $y_p \leqslant j < y_{p+1}$, $y_{q-1} < k = y_q$ (recall $y_q = y_q(\boldsymbol{\xi})$) and

$$\begin{aligned}
|\psi_s(\boldsymbol{\xi}^{kj}) - \psi_s(\boldsymbol{\xi})| &= |(\psi_{x_{p+1}}(\lambda_j) + \psi_{x_{p+2}}(\lambda_{y_{p+1}})) + \cdots + \psi_{x_q}(\lambda_{y_{q-1}}) - (\psi_{x_{p+1}}(\lambda_{y_{p+1}}) + \cdots + \psi_{x_q}(\lambda_{y_q}))| \\
&\leqslant \sum_{\alpha=p+1}^{q} \left|\psi_{x_\alpha}(\lambda_{y_{\alpha-1}\vee j}) - \psi_{x_\alpha}(\lambda_{y_\alpha})\right| \\
&\leqslant C \min(|\lambda_j(s\wedge\tau) - \lambda_k(s\wedge\tau)|, \nu^{-1}).
\end{aligned} \tag{6.13}$$

Here we have used the definition (6.6) in the first equality and for the second inequality we used: (i) $|\psi'_{x_\alpha}|_\infty \leqslant 1$, (ii) ψ_{x_α} is flat close to the edges and (iii) if $|k-j| \leqslant \ell$ are bulk indices, then $|\lambda_k(s\wedge\tau) - \lambda_j(s\wedge\tau)| \leqslant C\nu^{-1}$ by definition of the stopping time τ. Note that (6.13) also holds if $j > k$, with a proof being identical to the case $j < k$ up to notations.

Thanks to (6.13), we obtain

$$\left|\frac{\phi(\boldsymbol{\xi}^{kj})}{\phi(\boldsymbol{\xi})} + \frac{\phi(\boldsymbol{\xi})}{\phi(\boldsymbol{\xi}^{kj})} - 2\right| \leqslant C\ \nu^2 |\lambda_k - \lambda_j|^2.$$

This allows us to bound

$$(6.11) \leqslant C\ \frac{\nu^2}{N} \sum_{\boldsymbol{\xi}} \pi(\boldsymbol{\xi}) \sum_{k:\xi_k>0} \sum_{|j-k|\leqslant\ell} \nu(\boldsymbol{\xi})\nu(\boldsymbol{\xi}^{kj}) \mathrm{d}(s\wedge\tau) \leqslant \frac{\nu^2 \ell}{N} e^{\nu \frac{\ell}{N}} X_s \mathrm{d}(s\wedge\tau).$$

Moreover, the bracket term (6.9) is easily bounded by

$$(6.9) \leqslant C\ \sum_{\boldsymbol{\xi}} \pi(\boldsymbol{\xi}) v(\boldsymbol{\xi})^2 \sum_{\alpha=1}^{n} \nu^2 \frac{\psi'_{x_\alpha}(\lambda_{y_\alpha})^2}{N} \mathrm{d}(s\wedge\tau) \leqslant C\ \frac{\nu^2}{N} X_s \mathrm{d}(s\wedge\tau).$$

Third step: bound on (6.8). We can bound (6.8) by

$$\begin{aligned}
&2\sum_{\boldsymbol{\xi}} \pi(\boldsymbol{\xi}) v(\boldsymbol{\xi})^2 \sum_{\alpha=1}^{n} \left(\nu \frac{\psi'_{x_\alpha}(\lambda_{y_\alpha})}{N} \sum_{j\neq y_\alpha} \frac{1}{\lambda_{x_\alpha} - \lambda_j} + c_1 \frac{\nu}{2N} \psi''_{x_\alpha}(\lambda_{\boldsymbol{\xi}(i)}) + c_2 \frac{\nu^2}{2N} \psi'_{x_\alpha}(\lambda_{\boldsymbol{\xi}(i)})^2 \right) \mathrm{d}(s\wedge\tau) \\
&\leqslant C\ \frac{\nu^2}{N} X_s \mathrm{d}(s\wedge\tau) + 2\sum_{\boldsymbol{\xi}} \pi(\boldsymbol{\xi}) v(\boldsymbol{\xi})^2 \sum_{1\leqslant\alpha\leqslant n, |j-y_\alpha|>\ell} \nu \frac{|\psi'_{x_\alpha}(\lambda_{y_\alpha})|}{N} \frac{\mathrm{d}(s\wedge\tau)}{|\lambda_{y_\alpha} - \lambda_j|} \\
&\quad + 2\sum_{\boldsymbol{\xi}} \pi(\boldsymbol{\xi}) v(\boldsymbol{\xi})^2 \sum_{1\leqslant\alpha\leqslant n, |j-y_\alpha|\leqslant\ell} \frac{\nu}{N} \frac{\psi'_{x_\alpha}(\lambda_{y_\alpha})}{\lambda_{y_\alpha} - \lambda_j} \mathrm{d}(s\wedge\tau).
\end{aligned} \tag{6.14}$$

As rigidity holds when $\tau > s$, the above sum over $|j - y_\alpha| > \ell$ is at most $C\ \nu(\log N)\mathrm{d}(s\wedge\tau)$.

To bound the contribution of $|j-y_\alpha|\leqslant \ell$, we symmetrize the summands of (6.14) into

$$\frac{\nu}{N}\sum_{i<j:|i-j|\leqslant\ell}\frac{1}{\lambda_i-\lambda_j}\sum_{\boldsymbol{\xi}}\pi(\boldsymbol{\xi})v(\boldsymbol{\xi})^2\sum_{\alpha:y_\alpha=i}\psi'_{x_\alpha}(\lambda_i)+\frac{\nu}{N}\sum_{i>j:|i-j|\leqslant\ell}\frac{1}{\lambda_i-\lambda_j}\sum_{\boldsymbol{\xi}}\pi(\boldsymbol{\xi})v(\boldsymbol{\xi})^2\sum_{\alpha:y_\alpha=i}\psi'_{x_\alpha}(\lambda_i)$$
$$=\frac{\nu}{N}\sum_{i<j:|i-j|\leqslant\ell}\frac{1}{\lambda_i-\lambda_j}\sum_{\boldsymbol{\xi}}\pi(\boldsymbol{\xi})v(\boldsymbol{\xi})^2\left(\sum_{\alpha:y_\alpha=i}\psi'_{x_\alpha}(\lambda_i)-\sum_{i:y_\alpha=j}\psi'_{x_\alpha}(\lambda_j)\right)$$
$$\leqslant\frac{\nu}{N}\sum_{i<j:|i-j|\leqslant\ell}\frac{1}{\lambda_i-\lambda_j}\sum_{\boldsymbol{\xi}}\pi(\boldsymbol{\xi})v(\boldsymbol{\xi})^2\left(\sum_{\alpha:y_\alpha=i}\psi'_{x_\alpha}(\lambda_i)-\sum_{i:y_\alpha=j}\psi'_{x_\alpha}(\lambda_i)\right)+C\,\nu^2\frac{\ell}{N}X_s,\quad (6.15)$$

where we just replaced $\psi'_{x_\alpha}(\lambda_j)$ with $\psi'_{x_\alpha}(\lambda_i)$, up to an error at most $C\,\nu^2\frac{\ell}{N}X_s$, obtained by using $|\psi'_{x_\alpha}(\lambda_j)-\psi'_{x_\alpha}(\lambda_i)|/|\lambda_j-\lambda_i|\leqslant\|\psi'_{x_\alpha}\|_\infty\leqslant\nu$. In all the following bounds, we consider i and j as fixed indices. We also introduce the following subsets of configurations with n particles, for any $0\leqslant q\leqslant p\leqslant n$:

$$\mathcal{A}_p=\{\boldsymbol{\xi}:\xi_i+\xi_j=p\},\ \mathcal{A}_{p,q}=\{\boldsymbol{\xi}\in\mathcal{A}_p:\xi_i=q\}.$$

Denote $\bar{\boldsymbol{\xi}}$ the configuration exchanging all particles from sites i and j, i.e. $\bar\xi_i=\xi_j$, $\bar\xi_j=\xi_i$ and $\bar\xi_k=\xi_k$ if $k\neq i,j$. Using $\pi(\boldsymbol{\xi})=\pi(\bar{\boldsymbol{\xi}})$, we can bound the sum over $\boldsymbol{\xi}$ in (6.15) by

$$\frac{1}{\lambda_i-\lambda_j}\sum_{p=0}^{n}\sum_{q=0}^{p}\sum_{\boldsymbol{\xi}\in\mathcal{A}_{p,q}}\pi(\boldsymbol{\xi})v(\boldsymbol{\xi})^2\left(\sum_{\alpha:y_\alpha=i}\psi'_{x_\alpha}(\lambda_i)-\sum_{\alpha:y_\alpha=j}\psi'_{x_\alpha}(\lambda_i)\right)$$
$$=\frac{1}{\lambda_i-\lambda_j}\sum_{p=0}^{n}\sum_{q=0}^{\lfloor p/2\rfloor}c_q\sum_{\boldsymbol{\xi}\in\mathcal{A}_{p,q}}\pi(\boldsymbol{\xi})\Bigg[v(\boldsymbol{\xi})^2\left(\sum_{\alpha:y_\alpha=i}\psi'_{x_\alpha}(\lambda_i)-\sum_{\alpha:y_\alpha=j}\psi'_{x_\alpha}(\lambda_i)\right)$$
$$-v(\bar{\boldsymbol{\xi}})^2\left(\sum_{\alpha:\bar y_\alpha=j}\psi'_{x_\alpha}(\lambda_i)-\sum_{\alpha:\bar y_\alpha=i}\psi'_{x_\alpha}(\lambda_i)\right)\Bigg],\quad (6.16)$$

where the constant $c_q=0$ if p is even and $q=p/2$, and $c_q=1$ otherwise. Remember that for any $a\leqslant b$, we have $\psi'_a\geqslant\psi'_b$. This implies that $\sum_{\alpha:y_\alpha=i}\psi'_{x_\alpha}(\lambda_i)\geqslant\sum_{\alpha:\bar y_\alpha=j}\psi'_{x_\alpha}(\lambda_i)$ and $\sum_{\alpha:\bar y_\alpha=i}\psi'_{x_\alpha}(\lambda_i)\geqslant\sum_{\alpha:y_\alpha=j}\psi'_{x_\alpha}(\lambda_i)$ so that

$$\sum_{\alpha:y_\alpha=i}\psi'_{x_\alpha}(\lambda_i)-\sum_{\alpha:y_\alpha=j}\psi'_{x_\alpha}(\lambda_i)\geqslant\sum_{\alpha:\bar y_\alpha=j}\psi'_{x_\alpha}(\lambda_i)-\sum_{\alpha:\bar y_\alpha=i}\psi'_{x_\alpha}(\lambda_i).\quad (6.17)$$

Equations (6.16) and (6.17) together with $\lambda_i<\lambda_j$ give

$$\frac{1}{\lambda_i-\lambda_j}\sum_{p=0}^{n}\sum_{q=0}^{p}\sum_{\boldsymbol{\xi}\in\mathcal{A}_{p,q}}\pi(\boldsymbol{\xi})v(\boldsymbol{\xi})^2\left(\sum_{\alpha:y_\alpha=i}\psi'_{x_\alpha}(\lambda_i)-\sum_{\alpha:y_\alpha=j}\psi'_{x_\alpha}(\lambda_i)\right)$$
$$\leqslant\frac{C}{\lambda_i-\lambda_j}\sum_{p=0}^{n}\sum_{q=0}^{\lfloor p/2\rfloor}\sum_{\boldsymbol{\xi}\in\mathcal{A}_{p,q}}\pi(\boldsymbol{\xi})\left(v(\boldsymbol{\xi})^2-v(\bar{\boldsymbol{\xi}})^2\right)\left(\sum_{\alpha:y_\alpha=i}\psi'_{x_\alpha}(\lambda_i)-\sum_{\alpha:y_\alpha=j}\psi'_{x_\alpha}(\lambda_i)\right)$$
$$\leqslant\frac{C}{|\lambda_i-\lambda_j|}\sum_{\boldsymbol{\xi}}\pi(\boldsymbol{\xi})\left|v(\boldsymbol{\xi})^2-v(\bar{\boldsymbol{\xi}})^2\right|.$$

where we used, in the second inequality, $\|\psi'_{x_\alpha}\|_\infty \leqslant 1$. Note that transforming $\boldsymbol{\xi}$ into $\bar{\boldsymbol{\xi}}$ can be achieved by transferring a particle for i to j (or j to i) one by one at most n times, so that

$$\frac{1}{|\lambda_i-\lambda_j|}\sum_{\boldsymbol{\xi}}\pi(\boldsymbol{\xi})\left|v(\boldsymbol{\xi})^2-v(\bar{\boldsymbol{\xi}})^2\right| \leqslant \frac{C}{|\lambda_i-\lambda_j|}\sum_{\boldsymbol{\xi}}\pi(\boldsymbol{\xi})\left(|v(\boldsymbol{\xi})^2-v(\boldsymbol{\xi}^{ij})^2|+|v(\boldsymbol{\xi})^2-v(\boldsymbol{\xi}^{ji})^2|\right)$$
$$\leqslant CM\sum_{\boldsymbol{\xi}}\pi(\boldsymbol{\xi})\frac{(v(\boldsymbol{\xi})-v(\boldsymbol{\xi}^{ij}))^2+(v(\boldsymbol{\xi})-v(\boldsymbol{\xi}^{ji}))^2}{(\lambda_i-\lambda_j)^2}+CM^{-1}\sum_{\boldsymbol{\xi}}\pi(\boldsymbol{\xi})\left((v(\boldsymbol{\xi})+v(\boldsymbol{\xi})^{ij})^2+(v(\boldsymbol{\xi})+v(\boldsymbol{\xi})^{ji})^2\right)$$

for any $M>0$. We finally proved that the drift term from (6.8) is bounded above by

$$C\ M\frac{\nu}{N}\sum_{\boldsymbol{\xi}}\pi(\boldsymbol{\xi})\sum_{|i-j|\leqslant\ell}\frac{(v(\boldsymbol{\xi})-v(\boldsymbol{\xi}^{ij}))^2}{(\lambda_i-\lambda_j)^2}+CM^{-1}\frac{\nu}{N}\sum_{\boldsymbol{\xi}}\pi(\boldsymbol{\xi})\sum_{|i-j|\leqslant\ell}(v(\boldsymbol{\xi})+v(\boldsymbol{\xi}^{ij}))^2+C(\nu\log N+\nu^2\frac{\ell}{N})X_s.$$

We chose $M=c\nu^{-1}$ with c small enough so that the first sum above can be absorbed into the dissipative term (6.10). The second sum above is then bounded by $\frac{\nu^2\ell}{N}e^{\nu\frac{\ell}{N}}X_s$.

Fourth step: conclusion. All together, the above estimates give

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathbb{E}(X_s)\leqslant C(\nu\log N+\frac{\nu^2\ell}{N})e^{\nu\frac{\ell}{N}}\mathbb{E}(X_s),$$

so for our choice $\nu=N/\ell$ we have $\mathbb{E}(X_s)\leqslant Ce^{C\frac{N}{\ell}(\log N)s}$. In particular,

$$\mathbb{E}(e^{2\frac{N}{\ell}\sum_{\alpha=1}^n\psi_{x_\alpha}(\lambda_{y_\alpha}(s\wedge\tau))}r_{t\wedge\tau}(\boldsymbol{\eta},\boldsymbol{\xi})^2)\leqslant Ce^{\frac{N}{\ell}(\log N)t}.$$

If $d(\boldsymbol{\xi},\boldsymbol{\eta})\geqslant N^{\varepsilon}\ell$, then $\sum_{\alpha=1}^n\psi_{x_\alpha}(\lambda_{y_\alpha}(s\wedge\tau))>\ell\frac{N^{\varepsilon}}{N}$, so that (remember $\ell\geqslant Nt$)

$$\mathbb{E}\left(r_{t\wedge\tau}(\boldsymbol{\eta},\boldsymbol{\xi})^2\right)\leqslant Ce^{-cN^{\varepsilon}}.$$

One concludes using Markov's inequality and $\mathbb{P}(\tau<t)\leqslant N^{-D}$.

The proof of (ii) proceeds in exactly the same way with only two differences: 1. $g_i(x)=d_i(x)$ for any $x\in\mathbb{R}$ (in particular ψ_i is not made flat near the edges); 2. ν is chosen to be $\nu=(N/\ell)^{2/3}$. Since the full proof for edge case is parallel to the bulk case, we give all details hereafter only for $n=1$.

Let $\nu=(N/\ell)^{2/3}$. Assume that the initial configuration $\boldsymbol{\eta}$ consists in one particle at $k_0\in[\![1,N]\!]$. Let $d(x)=|x-\gamma_{k_0}|$ and χ as in the proof of (i). Define $\psi(x)=\int d(x-y)\nu\chi(\nu y)\mathrm{d}y$ and

$$\psi_s(k)=\psi(\lambda_k(s\wedge\tau)),\ \phi_s(k)=e^{\nu\psi_s(k)},\ v_s(k)=\phi_s(k)r_{s\wedge\tau}(k_0,k),$$

where τ is defined by (6.5). Then by definition of the dynamics and the Itô formula, we have (here we drop the time parameter s whenever it is obvious)

$$\mathrm{d}v(k)=2\sum_{|j-k|\leqslant\ell}c_{jk}\left((v(j)-v(k))+\left(\frac{\phi(k)}{\phi(j)}-1\right)v(j)\right)\mathrm{d}(s\wedge\tau)+(\mathrm{d}\phi(k))\,r(k_0,k)$$
$$\frac{\mathrm{d}\phi(k)}{\phi(k)}=\nu\psi'(\lambda_k)\frac{\mathrm{d}B_k(s\wedge\tau)}{\sqrt{N}}+\left(\nu\frac{\psi'(\lambda_k)}{N}\sum_{j\neq k}\frac{1}{\lambda_k-\lambda_j}+\frac{\nu}{2N}\psi''(\lambda_k)+\frac{\nu^2}{2N}\psi'(\lambda_k)^2\right)\mathrm{d}(s\wedge\tau)$$

Thus if we define $X_s = \sum_{k=1}^N v_s(k)^2$, we obtain

$$\mathrm{d}X_s = -2 \sum_{|j-k|\leqslant \ell} c_{jk}(v(j)-v(k))^2 \mathrm{d}(s\wedge\tau) \tag{6.18}$$

$$+2\sum_{|j-k|\leqslant\ell} c_{jk}\left(\frac{\phi(k)}{\phi(j)}+\frac{\phi(j)}{\phi(k)}-2\right)v(j)v(k)\mathrm{d}(s\wedge\tau) \tag{6.19}$$

$$+\frac{\nu}{N}\sum_k \psi''(\lambda_k)v(k)^2\mathrm{d}(s\wedge\tau) \tag{6.20}$$

$$+\frac{\nu^2}{N}\sum_k \psi'(\lambda_k)^2 v(k)^2\mathrm{d}(s\wedge\tau) \tag{6.21}$$

$$+2\frac{\nu}{N}\sum_{j<k}\frac{\psi'(\lambda_j)v(j)^2-\psi'(\lambda_k)v(k)^2}{\lambda_j-\lambda_k}\mathrm{d}(s\wedge\tau) \tag{6.22}$$

$$+2\nu\sum_k \frac{\mathrm{d}B_k(s\wedge\tau)}{\sqrt{N}}\psi'(\lambda_k)v(k)^2.$$

From $\|\phi'\|_\infty \leqslant 1$, the definition of τ and ν, we have $\nu|\phi(\lambda_k)-\phi(\lambda_j| \leqslant \nu|\lambda_k-\lambda_j| \leqslant C\nu|\gamma_\ell+2| = \mathrm{O}(1)$ (this is where we critically used that $\nu \leqslant (N/\ell)^{2/3}$), so that $\left|\frac{\phi(k)}{\phi(j)}+\frac{\phi(j)}{\phi(k)}-2\right| \leqslant C\ \nu^2|\lambda_k-\lambda_j|^2$. One concludes easily that (6.19) is bounded above by $C\nu^2\frac{\ell}{N}\mathrm{d}(s\wedge\tau)X_s$. The terms (6.20) and (6.21) are of smaller order by $\|\psi'\|_\infty \leqslant 1$ and $\|\psi''\|_\infty \leqslant \nu$.

Finally, (6.22) is of order at most

$$\frac{\nu}{N}\sum_{j<k:|j-k|>\ell}\frac{v(k)^2}{|\lambda_j-\lambda_k|}+\frac{\nu}{N}\sum_{j<k:|j-k|\leqslant\ell}|\psi'(\lambda_j)|\frac{|v(j)^2-v(k)^2|}{|\lambda_j-\lambda_k|}+\frac{\nu^2}{N}\sum_{|j-k|\leqslant\ell}\|\psi''\|_\infty v(k)^2$$

By rigidity, the first sum above has order $\nu(\log N)X_s$. The third sum is at most $\nu^2\ell/NX_s$. Finally, the second sum is bounded using

$$2\frac{|v(j)^2-v(k)^2|}{|\lambda_j-\lambda_k|} \leqslant M^{-1}(v(j)+v(k))^2+M\frac{(v(j)-v(k))^2}{(\lambda_j-\lambda_k)^2}$$

Choosing $M = c\nu^{-1}$ for c small enough, this proves that (6.22) can be absorbed into the dissipative term (6.18) plus an error of order $(\nu^2\ell/N)X_s$.

Using $\nu \leqslant (N/\ell)^{2/3}$, we have thus proved that $\frac{\mathrm{d}}{\mathrm{d}s}\mathbb{E}(X_s) \leqslant C\left(\nu\log N+\frac{\nu^2\ell}{N}\right)\mathbb{E}(X_s) \leqslant C\nu(\log N)\mathbb{E}(X_s)$. In particular,

$$\mathbb{E}(e^{2\nu\psi(\lambda_k(s\wedge\tau))}r_{t\wedge\tau}(k_0,k)^2) \leqslant e^{C\nu(\log N)t}.$$

If $|k-k_0| \geqslant N^{1/3+\varepsilon}\ell^{2/3}$, then $\psi(\lambda_k(s\wedge\tau)) \geqslant N^\varepsilon(\ell/N)^{2/3} = N^\varepsilon\nu^{-1}$, so we obtained

$$\mathbb{E}(r_{t\wedge\tau}(k_0,k)^2) \leqslant e^{C\nu(\log N)t-N^\varepsilon},$$

which is exponentially small: $(N/\ell)^{2/3}(\log N)t = \mathrm{O}(\log N)$ as $\ell \geqslant Nt$ and $t \leqslant 1$. By the Markov inequality, we have thus proved the part (ii) of the lemma. □

7 Relaxation to equilibrium for $t \gtrsim N^{-1}$

The maximum inequality (4.17) allowed to prove convergence of the eigenvector moment flow along the whole spectrum, in Section 4, for $t \gtrsim N^{-1/4}$. Assume that, for some reason, the maximum of this flow is always obtained for configurations supported in the bulk. Then we can make the approximation $\Im m(\lambda_k + \mathrm{i}\eta) \sim 1$ in (4.17), and we obtain

$$S_t' \leqslant -\frac{1}{\eta} S_t + \frac{N^{\xi}}{N^{1/2}\eta^{3/2}}$$

assuming the optimal isotropic local semicircle law with a tiny error $N^{\xi}/\sqrt{N\eta}$. Choosing $\eta = N^{-1+\varepsilon}$ for some small $\varepsilon > 0$ then gives, by Gronwall, a relaxation time of order $\gtrsim N^{-1}$. The purpose of this section is to make this argument rigorous by using the finite speed of propagation for the eigenvector moment flow, i.e., Lemma 6.2.

7.1 Statement of the result. The initial matrix is denoted $M_N = M_N(0)$, it satisfies the local semicircle law, and its eigenvalues follow the usual Dyson Brownian motion dynamics.

Let $G_N^{(s)}$ (resp. $G_N^{(h)}$) be a sequence of $N \times N$ random matrices from the Gaussian orthogonal (resp. unitary) ensemble (normalized with limiting spectral measure supported on $(-2, 2)$, for example). Note that in this section G stands for a Gaussian matrix, not its Green function.

Theorem 7.1. *Let ε be any arbitrarily small positive constant and $t = N^{-1+\varepsilon}$. Assume that, for a deterministic sequence of matrices $(M_N)_{N\geqslant 1}$ and a sequence of unit vectors $\mathbf{q} = \mathbf{q}_N$, we have, for any $\omega > \xi > 0, D > 0$ and N large enough (depending on these parameters),*

$$\mathbb{P}\left(A_1(\mathbf{q}, \omega, \xi, N) \mid M_N\right) \geqslant 1 - N^{-D}. \tag{7.1}$$

Here we used the notation (4.4) and $\mathbb{P}(\cdot \mid M_N)$ denotes probability with respect to the matrix Dyson Brownian motion path with the initial condition fixed by M_N. Then the asymptotic normality of bulk eigenvectors of $M_N + \sqrt{t}\, G_N^{(s)}$ holds. More precisely, if $\boldsymbol{u}$ is the eigenbasis of $M_N + \sqrt{t}\, G_N^{(s)}$, for any polynomial P there exists $c > 0$ such that

$$\sup_{I \subset [\![\alpha N, (1-\alpha)N]\!], |I| = m, |\mathbf{q}|=1} \left| \mathbb{E}\left(P\left(\left(N|\langle \mathbf{q}, u_k\rangle|^2\right)_{k\in I}\right)\right) - \mathbb{E}\left(P\left((|\mathscr{N}_j|^2)_{j=1}^m\right)\right)\right| \leqslant N^{-c}. \tag{7.2}$$

If moreover (7.1) holds for any given sequence $(\mathbf{q}_N)_{N\geqslant 1}$, then any bulk eigenvector of $M_N + \sqrt{t}\, G_N^{(s)}$ have asymptotically independent normal entries (the analogue of Corollary 1.3) and each eigenvector satisfy local quantum unique ergodicity (the analogue of Corollary 1.4).

Similar results hold for the Hermitian matrices $M_N + \sqrt{t}\, G_N^{(h)}$.

The Green function in A_1 appearing in (7.1) is with respect to the matrix $(M_N(s))_{s\geqslant 0}$ with $M_N = M_N(0)$ being the initial matrix and $M_N(s)$ the value at time t of a (matrix) Dyson Brownian Motion.

Theorem 7.1 means that, the initial structure of bulk eigenvectors completely disappears with the addition of a small noise, provided that the initial matrix satisfies a strong form of semicircle law. If the initial condition is a generalized Wigner matrix, the matrix Dyson Brownian motion is again a generalized Wigner ensemble after rescaling. In this case, the asymptotic normality of the eigenvectors was already proved in Theorem 1.2 and therefore the conclusion of Theorem 7.4 was proved as well. The key point of Theorem 7.1 lies in that it holds for deterministic initial matrices, provided that the local isotropic semicircle law holds.

Note that by standard perturbation theory Theorem 7.1 in general does not hold for $t \ll N^{-1}$. Recall that Dyson's conjecture states that the relaxation time to local equilibrium for bulk eigenvalues under the DBM is $t \sim N^{-1}$. Thus Theorem 7.1 is the analogue of this conjecture in the context of bulk eigenvectors.

Remark 7.2. *Theorem 7.1 gives optimal relaxation speed for dynamics of bulk eigenvectors provided that the local law holds along the whole spectrum, i.e. condition (7.1) holds. One may be interested in the dynamics relaxation only locally, i.e. proving QUE only for certain eigenvectors with corresponding energy λ_i around $E_0 = \gamma_{k_0} \in (-2.2)$. Then as an input, the local law is only needed in a small window around E.*

More precisely, let $c > \varepsilon > 0$ be fixed (remember $t = N^{-1+\varepsilon}$). Assume that (7.1) holds in the sense that:

(i) for any $\omega > \xi > 0$ in the smaller domain (replacing the original domain defined in (4.1))

$$\widetilde{\mathbf{S}} = \widetilde{\mathbf{S}}(\omega, N) = \big\{z = E + \mathrm{i}\eta \in \mathbb{C} : |E - E_0| \leqslant N^{-1+c}\,,\ N^{-1+\omega} \leqslant \eta \leqslant \omega^{-1}\big\}. \tag{7.3}$$

(ii) replacing the Stieltjes transform $m(z)$ in (4.4) by any smooth function uniformly bounded away from zero, uniformly in N and $\widetilde{\mathbf{S}}(\omega, N)$.

Then the conclusion (7.2) holds after restricting the sup *to $I \subset [\![k_0 - N^c/10, k_0 + N^c/10]\!]$.*

To summarize, the optimal time relaxation result, Theorem 7.1, can be made local in the spectrum, because the key input in this result, the finite speed of propagation Lemma 6.2, holds locally. The modifications needed to prove these local versions are obvious and we leave them to interested readers.

We will prove Theorem 7.1 by using the maximum principle locally. For this purpose, we will use the finite speed of propagation estimate, Lemma 6.2. This will be explained in the next subsections.

7.2 Flattening of initial condition at the edge. Let $\alpha > 0$ be a fixed small number. We define the following flattening and averaging operators on the space of functions of configurations with n points: any $a \in [\![1, N/2]\!]$,

$$\begin{aligned}
&(\mathrm{Flat}_a(f))(\boldsymbol{\eta}) = f(\boldsymbol{\eta}) \text{ if } \boldsymbol{\eta} \subset [\![a, N+1-a]\!],\ 1 \text{ otherwise},\\
&\mathrm{Av}(f) = \frac{1}{|[\![\alpha N, 2\alpha N]\!]|} \sum_{a \in [\![\alpha N, 2\alpha N]\!]} \mathrm{Flat}_a(f).
\end{aligned}$$

We can write

$$\mathrm{Av}(f)(\boldsymbol{\eta}) = a_{\boldsymbol{\eta}} f(\boldsymbol{\eta}) + (1 - a_{\boldsymbol{\eta}}) \tag{7.4}$$

for some coefficient $a_{\boldsymbol{\eta}} \in [0,1]$ ($a_{\boldsymbol{\eta}} = 0$ if $\boldsymbol{\eta} \not\subset [\![\alpha N, (1-\alpha)N]\!]$, 1 if $\boldsymbol{\eta} \subset [\![2\alpha N, (1-2\alpha)N]\!]$). We will only use the elementary property

$$|a_{\boldsymbol{\eta}} - a_{\boldsymbol{\xi}}| \leqslant C\, \frac{d(\boldsymbol{\eta}, \boldsymbol{\xi})}{N}. \tag{7.5}$$

For a general number of particles n, consider now the following modification of the eigenvector moment flow (3.6). We only keep the short-range dynamics (depending on a parameter ℓ) and modify the initial condition to be flat when there is a particle close to the edge:

$$\begin{aligned}
&\partial_t g_{\boldsymbol{\lambda},t} = \mathscr{S}(t) g_{\boldsymbol{\lambda},t},\\
&g_{\boldsymbol{\lambda},0}(\boldsymbol{\eta}) = (\mathrm{Av} f_{\boldsymbol{\lambda},0})(\boldsymbol{\eta}),
\end{aligned} \tag{7.6}$$

We will abbreviate $g_{\boldsymbol{\lambda},t}(\boldsymbol{\eta})$ by $g_t(\boldsymbol{\eta})$, and $f_{\boldsymbol{\lambda},t}(\boldsymbol{\eta})$ by $f_t(\boldsymbol{\eta})$ (for $n = 1$, we write these functions as $f_t(k)$ and $g_t(k)$ where $\boldsymbol{\eta}$ is the configuration with 1 particle at k). We remind the reader that $f_t(\boldsymbol{\eta})$ can be define either by (3.5) or by the solution of the equation (3.6). In particular, $f_t(k)$ is the conditional expectation of $|\langle \mathbf{q}, u_k(t)\rangle|^2$ given $\boldsymbol{\lambda}$, i.e.,

$$f_t(k) = N\mathbb{E}\left(|\langle \mathbf{q}, u_k(t)\rangle|^2 \mid \boldsymbol{\lambda}\right) \tag{7.7}$$

where $\mathbf{q}$ is a fixed unit vector. In all our application, the initial data $f_{\boldsymbol{\lambda},0})(\boldsymbol{\eta})$ is independent of $\boldsymbol{\lambda}$ and given by (3.5) with $t = 0$. For $g_{\boldsymbol{\lambda},t}$, we can only understand it as the solution to (7.6).

For small time t, by finite speed of propagation we will prove that $g = 1$ (up to exponentially small corrections) close to the edge, so that the maximum principle for the dynamics (7.6) can be localized in the bulk.

We first prove that for these modified dynamics, the isotropic law holds in the following sense. The following result is deterministic.

Lemma 7.3. *Let $\varepsilon > 0$ be a fixed small number, $t = N^{-1+\varepsilon}$ and $\ell = N^{\delta}Nt$ for some $\delta > 0$ (here ℓ is the short-range dynamics cutoff parameter). Then there exist (small) positive constants ω_0, ξ_0 such that the following holds. Assume that for some $0 < \omega < \omega_0$, $0 < \xi < \xi_0$, $(M_N(s))_{0\leqslant s\leqslant 1}$ is in $A_1(\mathbf{q}, \omega, \xi, N)$. Assume moreover that* (7.17) *holds. Let z satisfy $-3 < \Re(z) < 3$ and $N^{-1+2\omega} < \Im(z) < \min(N^{-1+\delta/2}, N^{-3/4})$. Then we have*

$$\left|\Im\sum_{k=1}^{N}\frac{1}{N}\frac{g_t(k)}{z-\lambda_k} - \Im m(z)\right| \leqslant CN^{\xi+\omega}\left(\sqrt{\frac{\Im m(z)}{N\eta}} + \frac{1}{N\eta}\right) + C\frac{\ell N^{2\omega}}{N} \tag{7.8}$$

where C depends only on $\xi, \omega, \nu, \varepsilon$. Moreover, consider the case of n particles. Let $k_0 \in [\![1, N]\!]$ and $z = \lambda_{k_0} + \mathrm{i}\eta$. Then for any configuration $\boldsymbol{\eta}$ containing at least one particle at k_0 we have

$$\Im\sum_{k=1}^{N}\frac{1}{N}\frac{g_t(\boldsymbol{\eta}^{k_0k})}{z-\lambda_k} - \Im m(z)\left(a_{\boldsymbol{\eta}} f_t(\boldsymbol{\eta}\backslash k_0) + (1-a_{\boldsymbol{\eta}})\right) \leqslant C\left(N^{\xi+n\omega}\left(\sqrt{\frac{\Im m(z)}{N\eta}} + \frac{1}{N\eta}\right) + \frac{\ell N^{2\omega}}{N}\right) \tag{7.9}$$

where $\boldsymbol{\eta}\backslash k_0$ stands for the configuration $\boldsymbol{\eta}$ with one particle removed from site k_0.

Proof. We first show that the difference between $g_t(k) = (\mathrm{U}_{\mathscr{S}}(0,t)\mathrm{Av} f_0)(k)$ and $(\mathrm{Av}\mathrm{U}_{\mathscr{B}}(0,t)f_0)(k)$ is small. More precisely, we can bound the left hand side of (7.8) by $|(\mathrm{i})| + |(\mathrm{ii})| + |(\mathrm{iii})|$ where

$$(\mathrm{i}) = \Im\sum_{k=1}^{N}\frac{1}{N}\frac{(\mathrm{U}_{\mathscr{S}}(0,t)\mathrm{Av} f_0)(k) - (\mathrm{Av}\mathrm{U}_{\mathscr{S}}(0,t)f_0)(k)}{z-\lambda_k},$$

$$(\mathrm{ii}) = \Im\sum_{k=1}^{N}\frac{1}{N}\frac{(\mathrm{Av}\mathrm{U}_{\mathscr{S}}(0,t)f_0)(k) - (\mathrm{Av}\mathrm{U}_{\mathscr{B}}(0,t)f_0)(k)}{z-\lambda_k},$$

$$(\mathrm{iii}) = \Im\sum_{k=1}^{N}\frac{1}{N}\frac{(\mathrm{Av}\mathrm{U}_{\mathscr{B}}(0,t)f_0)(k)}{z-\lambda_k} - \Im m(z).$$

The term (i) will be controlled by finite speed of propagation; (ii) will be controlled by Lemma 6.1, and (iii) by the isotropic local semicircle law.

To bound (i), we write

$$(\mathrm{U}_{\mathscr{S}}(0,t)\mathrm{Av} f_0)(k) - (\mathrm{Av}\mathrm{U}_{\mathscr{S}}(0,t)f_0)(k) = \frac{1}{\alpha N}\sum_{a\in[\![\alpha N, 2\alpha N]\!]}\left(\mathrm{U}_{\mathscr{S}}(0,t)\mathrm{Flat}_a f_0 - \mathrm{Flat}_a \mathrm{U}_{\mathscr{S}}(0,t)f_0\right)(k). \tag{7.10}$$

To control the above terms, first assume that $a+\ell N^{\omega}<k$ Denote $(f\mathbb{1}_{\geqslant a})(x)=f(x)\mathbb{1}_{x\geqslant a}$, and similarly for $f\mathbb{1}_{<a}$. Then

$$\begin{aligned}(\mathrm{U}_{\mathscr{S}}(0,t)\mathrm{Flat}_a f_0)(k) &= \left(\mathrm{U}_{\mathscr{S}}(0,t)(f_0\mathbb{1}_{\geqslant a})\right)(k)+\left(\mathrm{U}_{\mathscr{S}}(0,t)\mathbb{1}_{<a}\right)(k)\\ &= \left(\mathrm{U}_{\mathscr{S}}(0,t)(f_0\mathbb{1}_{\geqslant a})\right)(k)+\mathrm{O}(e^{-N^c})\\ &= \left(\mathrm{U}_{\mathscr{S}}(0,t)(f_0\mathbb{1}_{\geqslant a})\right)(k)+\left(\mathrm{U}_{\mathscr{S}}(0,t)(f_0\mathbb{1}_{<a})\right)(k)+\mathrm{O}(e^{-N^c})\\ &= \left(\mathrm{U}_{\mathscr{S}}(0,t)f_0\right)(k)+\mathrm{O}(e^{-N^c})\\ &= (\mathrm{Flat}_a\mathrm{U}_{\mathscr{S}}(0,t)f_0)(k)+\mathrm{O}(e^{-N^c})\end{aligned} \tag{7.11}$$

In the above lines, we used the finite speed of propagation Lemma 6.2 in the second and third equalities, namely $(\mathrm{U}_{\mathscr{S}}(0,t)\delta_x)(k)=\mathrm{O}(e^{-N^c})$ for any $x\leqslant a$ and $k\geqslant a+\ell N^{\omega}$ (the case $x>a/2$ follows from part (i) of Lemma 6.2, the case $x\leqslant a/2$ from part (ii) and $a\in[\![\alpha N,2\alpha N]\!]$).
For $k<a-\ell N^{\omega}$, in the same way we obtain

$$(\mathrm{U}_{\mathscr{S}}(0,t)\mathrm{Flat}_a f_0)(k)=1+\mathrm{O}(e^{-N^c})=(\mathrm{Flat}_a\mathrm{U}_{\mathscr{S}}(0,t)f_0)(k)+\mathrm{O}(e^{-N^c}). \tag{7.12}$$

For $a-\ell N^{\omega}\leqslant k\leqslant a+\ell N^{\omega}$, as $\mathrm{U}_{\mathscr{S}}$ is a bounded in L_∞ we have

$$|(\mathrm{U}_{\mathscr{S}}(0,t)\mathrm{Flat}_a f_0)(k)-(\mathrm{Flat}_a\mathrm{U}_{\mathscr{S}}(0,t)f_0)(k)|\leqslant 2\sup_k f_0(k)\leqslant CN^{\omega}. \tag{7.13}$$

Equations (7.11), (7.12), (7.13) together imply that (7.10) and therefore (i) are bounded by $C\ \ell N^{2\omega}/N$.

To bound the term (ii), define the reversed dynamics $\mathrm{U}^*_{\mathscr{S}}$ by

$$\partial_\sigma \mathrm{U}^*_{\mathscr{S}}(s,\sigma)=\mathscr{S}(t-\sigma)\mathrm{U}^*_{\mathscr{S}}(s,\sigma), \tag{7.14}$$

and s is always set to be $=0$] and similarly for $\mathrm{U}^*_{\mathscr{B}}$. Notice that Lemma 6.1 holds for these time-reversed dynamics, the proof is unchanged. Thus we have

$$\begin{aligned}|(\mathrm{Av}\mathrm{U}_{\mathscr{S}}(0,t)f_0)(k)-(\mathrm{Av}\mathrm{U}_{\mathscr{B}}(0,t)f_0)(k)| &\leqslant |(\mathrm{U}_{\mathscr{S}}(0,t)f_0)(k)-(\mathrm{U}_{\mathscr{B}}(0,t)f_0)(k)|\\ &= \frac{1}{\pi(k)}|\langle f_0,(\mathrm{U}^*_{\mathscr{S}}(0,t)-\mathrm{U}^*_{\mathscr{B}}(0,t))\delta_k\rangle_\pi|\leqslant C\ N^{\omega}\frac{Nt}{\ell},\end{aligned}$$

where the first inequality follows from (7.4), and the second follows from Lemma 6.1. This proves that $|(\mathrm{ii})|\leqslant N^{\omega}\frac{Nt}{\ell}(\frac{N^{\xi}}{N\eta}+\sqrt{\frac{\Im m(z)}{N\eta}}+\operatorname{Im} m(z))$, where we used the local semicircle law, i.e. our matrix is in $A_1(\mathbf{q},\omega,\xi,N)$ from (4.4). We therefore have $|(\mathrm{ii})|\leqslant N^{\xi+\omega}\sqrt{\frac{\Im m(z)}{N\eta}}$ provided that $Nt/\ell\leqslant 1/(N\eta)^{1/2}$, which follows from our assumptions on t,ℓ and $\Im(z)\geqslant N^{-1+2\delta}$.

Concerning the error term (iii), we proceed as follows. Let m_0 be the index such that $|\Re(z)-\gamma_{m_0}|=\inf_{1\leqslant i\leqslant N}\{|\Re(z)-\gamma_i|\}$. Then

$$\Im\sum_{k=1}^{N}\frac{1}{N}\frac{(\mathrm{Av}\mathrm{U}_{\mathscr{B}}(0,t)f_0)(k)}{z-\lambda_k}=\Im\sum_{|k-m_0|\leqslant N\sqrt{\eta}}\frac{1}{N}\frac{(\mathrm{Av}\mathrm{U}_{\mathscr{B}}(0,t)f_0)(k)}{z-\lambda_k}+\mathrm{O}\left(\frac{N^{\omega}}{N}\sum_{i>N\sqrt{\eta}}\frac{\eta}{\eta^2+(i/N)^2}\right)$$

where we use that $\|f_0\|_\infty \leqslant N^\omega$ (which follows from the condition (4.6)). For any function f we write $(\mathrm{Av})f(k) = a_k f(k) + (1 - a_k)$ with the notation from (7.4). We obtain

$$\Im \sum_{k=1}^N \frac{1}{N} \frac{(\mathrm{Av}\mathrm{U}_{\mathscr{B}}(0,t)f_0)(k)}{z - \lambda_k} = \Im \sum_{|k-m_0|\leqslant N\sqrt{\eta}} \frac{1}{N} \frac{a_k f_t(k) + (1-a_k)}{z-\lambda_k} + \mathrm{O}(N^\omega \sqrt{\eta})$$
$$= \Im \sum_{|k-m_0|\leqslant N\sqrt{\eta}} \frac{1}{N} \frac{a_{m_0} f_t(k) + (1-a_{m_0})}{z-\lambda_k} + \Im \sum_{|k-m_0|\leqslant N\sqrt{\eta}} \frac{1}{N} \frac{(a_k - a_{m_0}) f_t(k) + (a_{m_0} - a_k)}{z-\lambda_k} + \mathrm{O}(N^\omega \sqrt{\eta}). \tag{7.15}$$

Moreover, the first sum above is equal to

$$a_{m_0} \Im \sum_{k=1}^N \frac{1}{N} \frac{f_t(k)}{z-\lambda_k} + (1-a_{m_0}) \Im \sum_{k=1}^N \frac{1}{N} \frac{1}{z-\lambda_k} + \mathrm{O}(N^\omega \sqrt{\eta}) = \Im m(z) + \mathrm{O}\left(N^\xi \sqrt{\frac{\Im m(z)}{N\eta}}\right) + \mathrm{O}\left(N^\omega \sqrt{\eta}\right)$$

where we used $(M_N, \boldsymbol{\lambda}) \in A(\mathbf{q}, \omega, \xi, \nu, N)$. From (7.5), we have $|a_k - a_{m_0}| \leqslant \sqrt{\eta} N^\omega$ and the second sum in (7.15) can be bounded by $\mathrm{O}(N^\omega \sqrt{\eta})$, which is smaller than $N^\omega/(N\eta)$ for $\eta \leqslant N^{-3/4}$. Gathering all estimates, we obtain that (7.8) holds.

In the case of general n, to prove (7.9), we proceed in the same way. As the term of type (i) is also bounded by finite speed of propagation, we just need to prove that

$$\Im \sum_{k=1}^N \frac{1}{N} \frac{\mathrm{Av}\mathrm{U}_{\mathscr{S}}(0,t) f_0(\boldsymbol{\eta}^{k_0 k})}{z-\lambda_k} = m(z)\left(a_{\boldsymbol{\eta}} f_t(\boldsymbol{\eta}/k_0) + (1-a_{\boldsymbol{\eta}})\right) + \mathrm{O}\left(N^{\xi+n\omega}\left(\sqrt{\frac{\Im m(z)}{N\eta}} + \frac{1}{N\eta}\right)\right).$$

Thanks to Lemma 6.1 it is sufficient to prove the above estimate replacing $\mathrm{U}_{\mathscr{S}}$ by $\mathrm{U}_{\mathscr{B}}$. We also can restrict the summation to $|k - k_0| \leqslant N\sqrt{\eta}$. Then, similarly to the $n = 1$ case, we write

$$\mathrm{Av}\mathrm{U}_{\mathscr{B}}(0,t) f_0(\boldsymbol{\eta}^{k_0 k}) = \mathrm{Av} f_t(\boldsymbol{\eta}^{k_0 k}) = a_{\boldsymbol{\eta}^{k_0 k}} f_t(\boldsymbol{\eta}^{k_0 k}) + (1 - a_{\boldsymbol{\eta}^{k_0 k}})$$
$$= \left(a_{\boldsymbol{\eta}} f_t(\boldsymbol{\eta}^{k_0 k}) + (1-a_{\boldsymbol{\eta}})\right) + \left((a_{\boldsymbol{\eta}^{k_0 k}} - a_{\boldsymbol{\eta}}) f_t(\boldsymbol{\eta}^{k_0 k}) + (a_{\boldsymbol{\eta}} - a_{\boldsymbol{\eta}^{k_0 k}})\right).$$

Using (7.5) and $|k - k_0| \leqslant N\sqrt{\eta}$ to bound the above second term, we are left with proving that

$$a_{\boldsymbol{\eta}} \Im \sum_{k=1}^N \frac{1}{N} \frac{f_t(\boldsymbol{\eta}^{k_0 k})}{z-\lambda_k} + (1-a_{\boldsymbol{\eta}}) \Im \sum_{k=1}^N \frac{1}{N} \frac{1}{z-\lambda_k} = m(z)\left(a_{\boldsymbol{\eta}} f_t(\boldsymbol{\eta}/k_0) + (1-a_{\boldsymbol{\eta}})\right) + \mathrm{O}\left(N^{\xi+n\omega}\left(\sqrt{\frac{\Im m(z)}{N\eta}} + \frac{1}{N\eta}\right)\right).$$

The second sum above is properly estimated by $m(z)$ because we are in a good set. Concerning the first sum, its contribution is not trivial if $a_{\boldsymbol{\eta}} \neq 0$, in particular $k_0 \in [\![\alpha N, (1-\alpha)N]\!]$. Then $\Im m(z) \sim 1$ and this first sum can be estimated exactly as in (4.15), (4.16). This concludes the proof. □

7.3 Localized maximum principle. The following result states that, for a typical initial conditions and a generic eigenvalue path, the relaxation time of the bulk eigenvectors is of order at most $N^{-1+\varepsilon}$ for any small $\varepsilon > 0$.

Theorem 7.4. *Let $n \in \mathbb{N}$, $\alpha, \varepsilon > 0$ be arbitrarily small constants and $t = N^{-1+\varepsilon}$. Then there exists a constant ω_0 such that the following holds.*

Assume that for some $0 < \xi < \omega < \omega_0$, $(M_N(s))_{0\leqslant s\leqslant 1}$ is in $A_1(\mathbf{q}, \omega, \xi, N)$. Assume moreover that (7.17) holds. Let f be a solution of the eigenvector moment flow (3.6) with initial matrix M_N and path $\boldsymbol{\lambda}$. Then there exists $c > 0$ such that for large enough N we have

$$\sup_{\boldsymbol{\eta}:\mathcal{N}(\boldsymbol{\eta})=n, \boldsymbol{\eta}\subset[\![\alpha N,(1-\alpha)N]\!]} |f_t(\boldsymbol{\eta}) - 1| \leqslant CN^{-c}. \tag{7.16}$$

Proof. As α is arbitrary we just need to prove the result for α replaced by 3α. Moreover, we only need to prove (7.16) with $f_t(\boldsymbol{\eta})$ replaced by $g_t(\boldsymbol{\eta})$ solving the cutoff dynamics (7.6). Indeed, we have

$$f_t(\boldsymbol{\eta}) - g_t(\boldsymbol{\eta}) = \frac{1}{\pi(\boldsymbol{\eta})}\langle f_0, (\mathrm{U}^*_{\mathscr{B}}(0,t) - \mathrm{U}^*_{\mathscr{S}}(0,t))\delta_{\boldsymbol{\eta}}\rangle_\pi + \frac{1}{\pi(\boldsymbol{\eta})}\langle \mathrm{U}_{\mathscr{S}}(0,t)(f_0 - g_0), \delta_{\boldsymbol{\eta}}\rangle_\pi.$$

where we used the notation (7.14) for the time-reversed dynamics. From Lemma 6.1 (which holds also for the time-reversed dynamics) and the bound $\|f_0\|_\infty \leqslant N^\omega$, the first term on the right hand side of the equation is bounded by $N^{1+\omega}t/\ell$. By the finite speed of propagation Lemma 6.2, the second term is exponentially small (remember that $f_0(\boldsymbol{\xi}) - g_0(\boldsymbol{\xi}) = 0$ if $\boldsymbol{\xi} \subset [\![2\alpha N, (1-2\alpha)N]\!]$ and $\boldsymbol{\eta}$ is supported in $[\![3\alpha N, (1-3\alpha)N]\!]$). We therefore just need to show that

$$\sup_{\boldsymbol{\eta}:\mathcal{N}(\boldsymbol{\eta})=n, \boldsymbol{\eta}\subset[\![3\alpha N,(1-3\alpha)N]\!]} |g_t(\boldsymbol{\eta}) - 1| \leqslant CN^{-\varepsilon}.$$

We will prove that such an estimate holds for any $\alpha > 0$ by induction on n. Assume there is just one particle. Following the idea from the proof of Theorem 4.3, for a given $0 \leqslant s \leqslant t$ let k_0 be an index such that $g_s(k_0) = \sup_k\{g_s(k)\}$. We consider two possible cases: if $g_s(k_0) - 1 \leqslant N^{-10}$ then there is nothing to prove. If $g_s(k_0) - 1 \geqslant N^{-10}$, then from the finite speed of propagation assumption (i.e., we are in the set $\mathcal{A}$), k_0 is in the bulk, i.e., $k_0 \in [\![\frac{\alpha}{2}N, (1-\frac{\alpha}{2})N]\!]$ (the reason is that if k_0 were near the edges, then $g_s(k_0) - 1$ is exponentially small). We then have

$$\begin{aligned}\partial_s g_s(k_0) = (\mathscr{S}(s)g_s)(k_0) &= \frac{1}{N}\sum_{j\neq k_0, |j-k_0|\leqslant\ell} \frac{g_s(j) - g_s(k_0)}{(\lambda_j - \lambda_{k_0})^2} \\ &\leqslant \frac{1}{\eta}\sum_{j\neq k_0, |j-k_0|\leqslant\ell} \frac{1}{N}\frac{\eta g_s(j)}{(\lambda_j - \lambda_{k_0})^2 + \eta^2} - \frac{g_s(k_0)}{\eta}\sum_{j\neq k_0, |j-k_0|\leqslant\ell}\frac{1}{N}\frac{\eta}{(\lambda_j - \lambda_{k_0})^2+\eta^2}.\end{aligned}$$

If $\ell \gg N\eta$ (which we obviously can assume, as we will chose $\eta = N^{-1+c}$ for some small $c > 0$, extending the above sums to all indices j induces an error $\eta N^{1+\omega}/\ell$ where we have used that $\|g_s\|_\infty \leqslant \|g_0\|_\infty \leqslant N^\omega$. Combining this fact with Lemma 7.3 and the rigidity of eigenvalues which follows from that the path $\boldsymbol{\lambda}$ is assumed to be in the set $A_2(\omega, N)$ defined in (4.5), we have proved (here $z = \lambda_{k_0} + \mathrm{i}\eta$) that

$$\partial_s(g_s(k_0) - 1) \leqslant -\frac{\Im m(z)}{\eta}(g_s(k_0) - 1) + \mathrm{O}\left(N^{\omega+\xi}\left(\frac{(\Im m(z))^{1/2}}{\eta^{3/2}N^{1/2}} + \frac{1}{N\eta^2}\right) + \frac{\ell N^{2\omega}}{N\eta}\right) + \mathrm{O}\left(\frac{N^{1+\omega}}{\ell}\right).$$

As $k_0 \in [\![\frac{\alpha}{2}N, (1-\frac{\alpha}{2})N]\!]$, we have $\Im m(z) \sim 1$. Moreover, the second error term $\mathrm{O}\left(\frac{N^{1+\omega}}{\ell}\right)$ is dominated by the first one (recall that the cutoff parameter ℓ in Lemma 7.3 satisfies $\ell = N^\delta Nt$ and $\eta \leqslant N^{-1+\delta/2}$). Denote by $S_s = \sup_k(g_s(k) - 1)$ and we choose the parameters so that $\eta = N^{-1+\frac{\varepsilon}{2}}$ and $\omega_0 \leqslant \varepsilon/10$. We proved that if $S_s \geqslant N^{-10}$ then

$$\partial_s S_s \leqslant -\frac{c}{\eta}S_s + C\left(\frac{N^{\omega+\xi}}{\eta^{3/2}N^{1/2}} + \frac{\ell N^{2\omega}}{N\eta}\right) \leqslant -cN^{1-\frac{\varepsilon}{2}}S_s + CN^{1-3\varepsilon/4}.$$

By Gronwall's lemma, we obtain $S_t = \mathrm{O}(N^{-\varepsilon/4})$. This concludes the proof for $n = 1$.

For general n, as in the 1-particle case we can assume that $\sup_{\boldsymbol{\eta}} g_t(\boldsymbol{\eta})$ is achieved for some $\boldsymbol{\xi} \subset [\![\frac{\alpha}{2}N, (1-\frac{\alpha}{2})N]\!]$. Then the analogue of (4.14) holds with f replaced by g. The first sum in (4.14) then can be evaluated using (7.9):

$$\frac{1}{N\eta}\sum_{j\neq k_r}\frac{\eta g_s(\boldsymbol{\xi}^{k_r j})}{(\lambda_{k_r}-\lambda_j)^2+\eta^2} = \Im m(\lambda_{k_r}+\mathrm{i}\eta)(a_{\boldsymbol{\xi}} f_s(\boldsymbol{\xi}\backslash k_r)+(1-a_{\boldsymbol{\xi}})).$$

From the result at rank $n-1$ with α replaced by $\alpha/10$, we know that for $s \in [t/2, t]$ we have

$$f_s(\boldsymbol{\xi}\backslash k_r) = g_s(\boldsymbol{\xi}\backslash k_r) + \mathrm{O}(N^{-c}) = 1 + \mathrm{O}(N^{-c}).$$

This proves that

$$\partial_s(g_s(\boldsymbol{\xi})-1) \leqslant -\frac{\Im m(z)}{\eta}(g_s(\boldsymbol{\xi})-1) + \mathrm{O}\left(\frac{(\Im m(z))^{1/2}}{\eta^{3/2}N^{1/2}}\ell N^{n\omega+\xi} + \frac{\Im m(z)}{\eta}N^{-c} + \frac{\ell N^{2\omega}}{N\eta}\right).$$

One now can conclude the proof as in the $n = 1$ case. □

Proof of Theorem 7.1 We can assume that the trajectory $(M_t)_{0\leqslant t\leqslant 1}$ is in $A_1(\mathbf{q},\omega,\xi,N)\cap A_2(\omega,N)\cap A_3(\omega,N)$. Indeed, as noted in Section 4, $A_1 \subset A_2 \cap A_3$, and the complement of A_1 has measure at most N^{-D}, which induces negligible error terms in the universality statements. For the same reason, thanks to Lemma 6.2, we can assume that the following finite speed of propagation holds: for any small $c, \alpha > 0$, uniformly $\boldsymbol{\eta}$ supported in the $[\![\alpha N, (1-\alpha)N]\!]$ and $d(\boldsymbol{\eta},\boldsymbol{\xi}) > N^c\ell$, for large enough N we have

$$r_s(\boldsymbol{\eta},\boldsymbol{\xi}) < e^{-N^{c/2}}. \tag{7.17}$$

Under the above two assumptions, we apply Theorem 7.4, which proves the first statement of Theorem 7.1. The last two statements of Theorem 7.1 easily follow by the arguments used in Sections 5.3 and 5.4.

Appendix A Continuity estimate for $t \lesssim N^{-1/2}$

The main result in Section 7, Theorem 7.4, asserts the asymptotic normality of eigenvector components for Gaussian divisible ensembles for $t \gtrsim N^{-1}$. To prove Theorem 1.2 for bulk eigenvectors, in this appendix we remove the small Gaussian components of the matrix elements. As we saw in Section 5, one way to proceed consists in a Green function comparison theorem. Here, we proceed in a different way: the Dyson Brownian motion preserves the local structure of generalized Wigner matrices up to time $N^{-1/2}$ (see the lemma hereafter). This approach is much more direct and there is no need to construct moment matching matrices. It provides a completely dynamical proof of Theorem 1.2 for bulk eigenvectors.

We remark that although this proof is very simple, the fact that the Dyson Brownian motion preserves the detailed behaviour of eigenvalues and eigenvectors is surprising and even contradictory. Consider for example the eigenvalue flow. It was proved that this spectral dynamics take very general initial data to local equilibrium for any time $t \gtrsim N^{-1}$. So how can we prove that the changes of the eigenvalues up to time $N^{-1/2}$ is less than the accuracy N^{-1}? The answer is that we only prove the preservation of the Dyson Brownian motion for matrix models. In other words, the matrix structure gives this preservation of the local structure.

We start with the following matrix stochastic differential equation which is an Ornstein-Uhlenbeck version of the Dyson Brownian motion. Let $H_t = (h_{ij}(t))$ be a symmetric $N \times N$ matrix. The dynamics of the matrix entries are given by the stochastic differential equations

$$\mathrm{d}h_{ij}(t) = \frac{\mathrm{d}B_{ij}(t)}{\sqrt{N}} - \frac{1}{2Ns_{ij}} h_{ij}(t)\mathrm{d}t, \tag{A.1}$$

where B is symmetric with $(B_{ij})_{i\leqslant j}$ a family of independent Brownian motions. The parameter $s_{ij} > 0$ can take any positive values, but in this paper, we choose s_{ij} to be the variance of $h_{ij}(0)$. Clearly, for any $t \geqslant 0$ we have $\mathbb{E}(h_{ij}(t)^2) = s_{ij}$ and thus the variance of the matrix element is preserved in this flow. We will call this system of stochastic differential equations (A.1) a generalized Dyson Brownian motion. For this flow, the following continuity estimate holds.

Lemma A.1. *Suppose that we have $c/N \leqslant s_{ij} \leqslant C/N$ for some fixed constants c and C, uniformly in i and j. Denote $\partial_{ij} = \partial_{h_{ij}}$. Suppose that F is a smooth function of the matrix elements $(h_{ij})_{i\leqslant j}$ satisfying*

$$\sup_{0\leqslant s\leqslant t, i\leqslant j, \boldsymbol{\theta}} \mathbb{E}\left((N^{3/2}|h_{ij}(s)^3| + \sqrt{N}|h_{ij}(s)|) \left|\partial_{ij}^3 F(\boldsymbol{\theta} H_s)\right| \right) \leqslant M, \tag{A.2}$$

where $(\boldsymbol{\theta} H)_{ij} = \theta_{ij} h_{ij}$, $\theta_{k\ell} = 1$ unless $\{k,\ell\} = \{i,j\}$ and $0 \leqslant \theta_{ij} \leqslant 1$. Then

$$\mathbb{E}F(H_t) - \mathbb{E}F(H_0) = \mathrm{O}(tN^{1/2})M.$$

Proof. By Itô's formula, we have

$$\partial_t \mathbb{E}F(H_t) = -\frac{1}{2N}\sum_{i\leqslant j}\left(\frac{1}{s_{ij}} \mathbb{E}\left(h_{ij}(t)\partial_{ij}F(H_t)\right) - \mathbb{E}\left(\partial_{ij}^2 F(H_t)\right)\right).$$

A Taylor expansion yields

$$\begin{aligned}
\mathbb{E}\left(h_{ij}(t)\partial_{ij}F(H_t)\right) &= \mathbb{E}h_{ij}(t)\partial_{ij}F_{h_{ij}(t)=0} + \mathbb{E}\left(h_{ij}(t)^2 \partial_{ij}^2 F_{h_{ij}(t)=0}\right) + \mathrm{O}\left(\sup_{\boldsymbol{\theta}} \mathbb{E}\left(|h_{ij}(t)^3 \partial_{ij}^3 F(\boldsymbol{\theta} H_t)|\right)\right) \\
&= s_{ij}\mathbb{E}\left(\partial_{ij}^2 F_{h_{ij}(t)=0}\right) + \mathrm{O}\left(\sup_{\boldsymbol{\theta}} \mathbb{E}\left(|h_{ij}(t)^3 \partial_{ij}^3 F(\boldsymbol{\theta} H_t)|\right)\right), \\
\mathbb{E}\left(\partial_{ij}^2 F(H_t)\right) &= \mathbb{E}\left(\partial_{ij}^2 F_{h_{ij}(t)=0}\right) + \mathrm{O}\left(\sup_{\boldsymbol{\theta}} \mathbb{E}\left(|h_{ij}(t)(\partial_{ij}^3 F(\boldsymbol{\theta} H)|)\right)\right).
\end{aligned}$$

Together with the condition $c/N \leqslant s_{ij} \leqslant C/N$, we have

$$\partial_t \mathbb{E}F(H_t) = N^{1/2}\,\mathrm{O}\left(\sup_{i\leqslant j,\boldsymbol{\theta}} \mathbb{E}(N^{3/2}|h_{ij}(t)^3| + N^{1/2}|h_{ij}(t)|)|\partial_{ij}^3 F(\boldsymbol{\theta} H_t)|\right).$$

Integration over time finishes the proof. □

The previous lemma implies the following eigenvalues and eigenvectors continuity estimate for the dynamics (A.1).

Corollary A.2. *Let $\alpha > 0$ be arbitrarily small, $\delta \in (0, 1/2)$ and $t = N^{-1+\delta}$. Denote by H_t the solution of (A.1) with a symmetric generalized Wigner matrix H_0 as the initial condition. Let μ_t be the law of H_t. Let m be any positive integer and $\Theta : \mathbb{R}^{2m} \to \mathbb{R}$ be a smooth function satisfying*

$$\sup_{k \in [\![0,5]\!], x \in \mathbb{R}} |\Theta^{(k)}(x)|(1 + |x|)^{-C} < \infty \tag{A.3}$$

for some $C > 0$. Denote by $(u_1(t), \ldots, u_N(t))$ the eigenvectors of H_t associated with the eigenvalues $\lambda_1(t) \leqslant \ldots \leqslant \lambda_N(t)$. Then there exists $\varepsilon > 0$ (depending only on Θ, δ and α) such that, for large enough N,

$$\sup_{I \subset [\![\alpha N, (1-\alpha)N]\!], |I| = m, |\mathbf{q}| = 1} \left| (\mathbb{E}^{\mu_t} - \mathbb{E}^{\mu_0}) \Theta \left((N(\lambda_k - \gamma_k), N \langle \mathbf{q}, u_k \rangle^2)_{k \in I} \right) \right| \leqslant N^{-\varepsilon}.$$

Proof. One may try to apply Lemma A.1 directly for $F(H) = (\boldsymbol{\lambda}, \mathbf{u})$, but the third derivative of this function seems hard to bound. Instead, we can prove the continuity estimate when F is a product of Green functions of H, which in turn implies the continuity estimate for eigenvalues and eigenvectors. In the following, the fact that (i) and (ii) imply (A.4) relies on classical techniques [22]. The crucial condition is (i), i.e., comparison of Green functions up to some scale smaller than microscopic, $\eta = N^{-1-\varepsilon}$. In Section 5 such a comparison was shown by moment matching. Hereafter, Lemma A.1 allows to prove this Green function comparison by a dynamic approach.

Let $\mathbf{v}$ and $\mathbf{w}$ refer to two generalized Wigner ensembles. Consider the following statements.

(i) Green functions comparison up to a very small scale. For any $\kappa > 0$ there exists $\xi, \varepsilon > 0$ such that for any $N^{-1-\xi} < \eta < 1$ and any smooth function F with polynomial growth, we have

$$\sup_{|\mathbf{q}|=1, E_1, \ldots, E_m \in (-2+\kappa, 2-\kappa)^m} |(\mathbb{E}^{\mathbf{v}} - \mathbb{E}^{\mathbf{w}}) F\left((\langle \mathbf{q}, G(z_k) \mathbf{q} \rangle)_{k=1}^m \right)| \leqslant C N^{-\varepsilon} \left(\frac{1}{N\eta} + \frac{1}{\sqrt{N\eta}} \right),$$

for some $C = C(\kappa, F) > 0$. Here $z_k = E_k + \mathrm{i}\eta$.

(ii) Level repulsion estimate. For both ensembles $\mathbf{v}$ and $\mathbf{w}$ and for any $\kappa > 0$ the following holds. There exists $\xi_0 > 0$ such that for any $0 < \xi < \xi_0$ there exists $\delta > 0$ satisfying

$$\mathbb{P}\left(|\{\lambda_i \in [E - N^{-1-\xi}, E + N^{-1-\xi}]\}| \geqslant 2 \right) \leqslant N^{-\xi-\delta},$$

for any $E \in (-2 + \kappa, 2 - \kappa)$. Here the probability measure can be either the ensemble $\mathbf{v}$ or $\mathbf{w}$.

From Section 5 in [22], if (i) and (ii) hold then for any $\alpha > 0$ and Θ satisffying (A.3) there exists $\varepsilon > 0$ such that for large enough N we have

$$\sup_{I \subset [\![\alpha N, (1-\alpha)N]\!], |I| = m, |\mathbf{q}| = 1} \left| (\mathbb{E}^{\mathbf{v}} - \mathbb{E}^{\mathbf{w}}) \Theta \left((N(\lambda_k - \gamma_k), N \langle \mathbf{q}, u_k \rangle^2)_{k \in I} \right) \right| \leqslant N^{-\varepsilon}. \tag{A.4}$$

The level repulsion condition condition (ii) was proved in the generalized Wigner context [15, equation (5.32)]. We therefore only need to check the main assumption (i), which is a consequence of Lemma A.1 and the isotropic local semicircle law, Theorem 4.1. Indeed, we need to find a good bound M in (A.2) for a function F of type given in (i). For simplicity we only consider the case

$$F(H) = \langle \mathbf{q}, G(z) \mathbf{q} \rangle,$$

where $z = E + \mathrm{i}\eta$ with $N^{-1-\xi} < \eta < 1$ and $-2 + \kappa < E < 2 - \kappa$. The general case

$$F(H) = \langle \mathbf{q}_1, G(z_1) \mathbf{q}_1 \rangle \ldots \langle \mathbf{q}_k, G(z_k) \mathbf{q}_k \rangle$$

is analogous. We have

$$\partial_{ij}^3\langle \mathbf{q}, G(z)\mathbf{q}\rangle = -\sum_{a,b}\sum_{\boldsymbol{\alpha},\boldsymbol{\beta}} q_a G(z)_{a\alpha_1} G(z)_{\beta_1,\alpha_2} G(z)_{\beta_2,\alpha_3} G(z)_{\beta_3,b}\, q_b$$

where $\{\alpha_k,\beta_k\}=\{i,j\}$ or $\{j,i\}$. From the isotropic local semicircle law (4.2) the following four expressions

$$\sum_a q_a\, G(z)_{a\alpha_1},\ G(z)_{\beta_1,\alpha_2},\ G(z)_{\beta_2,\alpha_3},\ \sum_b G(z)_{\beta_3,b}\, q_b$$

are bounded by $N^{2\xi}((N\eta)^{-1}+(N\eta)^{-1/2})$ with very high probability provided that $N^{-1+\xi}\leqslant\eta\leqslant 1$. Moreover, by a dyadic argument explained in [18] Section 8, we have for any $y\leqslant\eta$

$$|\langle \mathbf{q}, G(E+\mathrm{i}y)\mathbf{q}\rangle| \leqslant C\log N\frac{\eta}{y}\Im\langle \mathbf{q}, G(E+\mathrm{i}\eta)\mathbf{q}\rangle.$$

Consequently, we proved that uniformly in $E\in(-2+\kappa,2-\kappa)$, $N^{-1-\xi}\leqslant\eta\leqslant 1$, we have

$$\partial_{ij}^3\langle \mathbf{q}, G(E+\mathrm{i}\eta)\mathbf{q}\rangle = \mathrm{O}(N^{5\xi}(N\eta)^{-1}+(N\eta)^{-1/2}))$$

with very high probability. The hypothesis (A.2) therefore holds with $M=C(\varepsilon)N^{5\xi}((N\eta)^{-1}+(N\eta)^{-1/2})$. As ξ is arbitrarily small, Lemma A.1 proves that for any $\delta\in(0,1/2)$ and $t=N^{-1+\delta}$ there exists some $\varepsilon>0$ with

$$|\mathbb{E}F(H_t)-\mathbb{E}F(H_0)|\leqslant N^{-\varepsilon}((N\eta)^{-1}+(N\eta)^{-1/2}).$$

Thus assumption (i) holds and the Corollary is proved. □

To complete the proof of Theorem 1.2 for bulk eigenvectors by a dynamical approach, we proceed as follows. Let H_0 be a generalized Wigner matrix. For $\delta\in(0,1/2)$ and $t=N^{-1+\delta}$, let H_t be the solution of (A.1) at time t. On the one hand, from Corollary A.2 we have

$$\sup_{I\subset[\![\alpha N,(1-\alpha)N]\!],|I|=m,|\mathbf{q}|=1}\left|\mathbb{E}\left(P\left((N\langle \mathbf{q},u_k(t)\rangle^2)_{k\in I}\right)\right)-\mathbb{E}\left(P\left((N\langle \mathbf{q},u_k\rangle^2)_{k\in I}\right)\right)\right|\leqslant N^{-\varepsilon}.$$

On the other hand, the entry $h_{ij}(t)$ of H_t is distributed as

$$e^{-\frac{t}{2Ns_{ij}}}h_{ij}(0)+\left(s_{ij}\left(1-e^{-\frac{t}{Ns_{ij}}}\right)\right)^{1/2}\mathscr{N}^{(ij)} \tag{A.5}$$

where $(\mathscr{N}^{(ij)})_{i\leqslant j}$ are independent standard Gaussian random variables. For any $\nu<\frac{1}{2}\inf_{i,j}s_{ij}\left(1-e^{-\frac{t}{Ns_{ij}}}\right)$, let W_0 be a random matrix with entry $(W_0)_{ij}$ distributed as

$$\begin{aligned}
&e^{-\frac{t}{2Ns_{ij}}}h_{ij}(0)+\left(s_{ij}\left(1-e^{-\frac{t}{Ns_{ij}}}\right)-\nu\right)^{1/2}\mathscr{N}_1^{(ij)}\quad \text{if } i\neq j,\\
&e^{-\frac{t}{2Ns_{ij}}}h_{ij}(0)+\left(s_{ij}\left(1-e^{-\frac{t}{Ns_{ij}}}\right)-2\nu\right)^{1/2}\mathscr{N}_1^{(ij)}\quad \text{if } i=j,
\end{aligned}$$

where $(\mathscr{N}_1^{(ij)})_{i\leqslant j}$ are independent standard Gaussian random variables, independent from H_0. Then W_0 is a generalized Wigner matrix modulo scaling: for any i we have $\sum_j \mathrm{Var}(W_0)_{ij}=1-(N+1)\nu$. Moreover from (A.5) $h_{ij}(t)$ is distributed as

$$\begin{aligned}
&(W_0)_{ij}+\nu^{1/2}\mathscr{N}_2^{(ij)}\quad \text{if } i\neq j,\\
&(W_0)_{ij}+(2\nu)^{1/2}\mathscr{N}_2^{(ij)}\quad \text{if } i=j,
\end{aligned}$$

where $(\mathscr{N}_2^{(ij)})_{i\leqslant j}$ are independent standard Gaussian random variables, independent of W_0. This proves that H_t is distributed as $W_{t'}$, where $(W_s)_{s\geqslant 0}$ satisfies (2.1) and $t' = N\nu$. We choose $\nu = N^{-2+\xi}$ for some $\xi \in (0,1)$ and apply Theorem 7.4 to $W_{t'}$: this yields

$$\sup_{I\subset[\![\alpha N,(1-\alpha)N]\!],|I|=m,|\mathbf{q}|=1}\left|\mathbb{E}\left(P\left((N\langle\mathbf{q},u_k(t)\rangle^2)_{k\in I}\right)\right)-\mathbb{E}P\left((\mathscr{N}_j^2)_{j=1}^m\right)\right|\leqslant N^{-\varepsilon}.$$

We have thus proved Theorem 1.2 by a dynamic approach, in the bulk case.

Appendix B Generator of the Dyson vector flow

B.1 Proof of Theorem 2.3. We first consider the symmetric case.

(*a*) For any $\varepsilon>0$, let $\tau_\varepsilon=\inf\{t\geqslant 0\mid |\lambda_i-\lambda_j|=\varepsilon \text{ for some } i\neq j\ or|\lambda_i|=\varepsilon^{-1} \text{ for some } i\}$ and ϕ_ε be a sooth function on $\mathbb{R}$ such that $\phi_\varepsilon(x)=x^{-1}$ if $x\geqslant\varepsilon$. Then, as all of the following coefficients are Lipschitz, pathwise existence and uniqueness holds for the system of stochastic differential equations

$$\begin{aligned}
\mathrm{d}\lambda_k &= \frac{\mathrm{d}B^{(s)}_{kk}}{\sqrt{N}}+\frac{1}{N}\sum_{\ell\neq k}\phi_\varepsilon(\lambda_k-\lambda_\ell)\mathrm{d}t,\\
\mathrm{d}u_k &= \frac{1}{\sqrt{N}}\sum_{\ell\neq k}(\mathrm{d}B^{(s)}_{k\ell})\phi_\varepsilon(\lambda_k-\lambda_\ell)u_\ell-\frac{1}{2N}\sum_{\ell\neq k}\phi_\varepsilon(\lambda_k-\lambda_\ell)^2u_k\mathrm{d}t.
\end{aligned}$$

Consequently, if one can prove that $\tau_\varepsilon\to\infty$ almost surely as $\varepsilon\to 0$, then existence and strong uniqueness for the system (2.2), (2.3) easily follow. This non-explosion nor collision result follows from Proposition 1 in [29]. It immediately yields $\boldsymbol{\lambda}_t\in\Sigma_N$ for ant $t\geqslant 0$.

To prove that $\boldsymbol{u}_t\in\mathrm{O}(N)$ for any $t\geqslant 0$, we consider the stochastc differential equations satisfied by $u_i\cdot u_j$, $1\leqslant i\leqslant j\leqslant N$. Itô's formula yields

$$\begin{aligned}
\mathrm{d}(u_i\cdot u_j) &= \frac{1}{\sqrt{N}}\sum_{k\notin\{i,j\}}\left(\frac{\mathrm{d}B^{(s)}_{jk}}{\lambda_j-\lambda_k}u_i\cdot u_k+\frac{\mathrm{d}B^{(s)}_{ik}}{\lambda_i-\lambda_k}u_j\cdot u_k\right)+\frac{1}{\sqrt{N}}\frac{\mathrm{d}B^{(s)}_{ji}}{\lambda_j-\lambda_i}(|u_i|^2-|u_j|^2)\\
&\quad -\frac{1}{2N}\left(\sum_{k\neq j}\frac{1}{(\lambda_j-\lambda_k)^2}+\sum_{k\neq i}\frac{1}{(\lambda_i-\lambda_k)^2}+\frac{1}{(\lambda_i-\lambda_j)^2}\right)u_i\cdot u_j\mathrm{d}t,\ i\neq j,\\
\mathrm{d}(|u_i|^2) &= \frac{2}{\sqrt{N}}\sum_{k\neq i}\frac{\mathrm{d}B^{(s)}_{ik}}{\lambda_i-\lambda_k}u_i\cdot u_k+\frac{1}{N}\sum_{k\neq i}\frac{|u_k|^2-|u_i|^2}{(\lambda_i-\lambda_k)^2}.
\end{aligned}$$

For the same reason as previously, existence and strong uniqueness hold for the above system, and $u_i\cdot u_j=0$ $(i\neq j)$, $|u_i|^2=1$ is an obvious solution (remember that $\boldsymbol{u}_0\in\mathrm{O}(N)$), which completes the proof.

(b) Let $\tilde{H}^{(s)}_t=\boldsymbol{u}_t\boldsymbol{\lambda}_t\boldsymbol{u}_t^*$. On the one hand, Itô's formula gives

$$\mathrm{d}\tilde{H}^{(s)}_{km}=(\boldsymbol{u}\boldsymbol{\lambda}(\mathrm{d}\boldsymbol{u})^*+\boldsymbol{u}(\mathrm{d}\boldsymbol{\lambda})\boldsymbol{u}^*+(\mathrm{d}\boldsymbol{u})\boldsymbol{\lambda}\boldsymbol{u}^*)_{km}+\sum_{\ell\neq s}\frac{1}{N}\frac{\lambda_\ell}{(\lambda_s-\lambda_\ell)^2}u_s(k)u_s(m)\mathrm{d}t. \tag{B.1}$$

On the other hand, the evolution equations for $\boldsymbol{\lambda}$ and $\boldsymbol{u}$ is

$$\mathrm{d}\boldsymbol{\lambda} = \mathrm{d}M_{\boldsymbol{\lambda}} + \mathrm{d}D_{\boldsymbol{\lambda}} \qquad (\mathrm{d}M_{\boldsymbol{\lambda}})_{ij} = \frac{\mathrm{d}B^{(s)}{}_{ii}}{\sqrt{N}}\mathbb{1}_{i=j},\ (\mathrm{d}D_{\boldsymbol{\lambda}})_{ij} = \left(\frac{1}{2}\sum_{\ell\neq i}\frac{1}{\lambda_i-\lambda_\ell}\right)\mathrm{d}t\mathbb{1}_{i=j},$$

$$\mathrm{d}\boldsymbol{u} = \boldsymbol{u}(\mathrm{d}M_{\boldsymbol{u}} + \mathrm{d}D_{\boldsymbol{u}}) \qquad (\mathrm{d}M_{\boldsymbol{u}})_{ij} = \frac{1}{\sqrt{N}}\frac{\mathrm{d}B^{(s)}_{ij}}{\lambda_i-\lambda_j}\mathbb{1}_{i\neq j},\ (\mathrm{d}D_{\boldsymbol{u}})_{ij} = -\frac{1}{2N}\sum_{\ell\neq i}\frac{\mathrm{d}t}{(\lambda_i-\lambda_j)^2}\mathbb{1}_{i\neq j}.$$

Consequently, after defining the diagonal matrix process D by

$$(\mathrm{d}D)_{ij} = \frac{1}{N}\sum_{\ell\neq i}\frac{\lambda_\ell}{(\lambda_\ell-\lambda_i)^2}\mathrm{d}t\mathbb{1}_{i=j},$$

the equation (B.1) can be written

$$\mathrm{d}\tilde{S} = \boldsymbol{u}(\boldsymbol{\lambda}(\mathrm{d}M_{\boldsymbol{u}})^* + (\mathrm{d}M_{\boldsymbol{u}})\boldsymbol{\lambda} + \mathrm{d}M_{\boldsymbol{\lambda}})\boldsymbol{u}^* + \boldsymbol{u}(\boldsymbol{\lambda}(\mathrm{d}D_{\boldsymbol{u}})^* + (\mathrm{d}D_{\boldsymbol{u}})\boldsymbol{\lambda} + \mathrm{d}D_{\boldsymbol{\lambda}} + \mathrm{d}D)\boldsymbol{u}^*.$$

We have $\boldsymbol{\lambda}(\mathrm{d}M_{\boldsymbol{u}})^* + (\mathrm{d}M_{\boldsymbol{u}})\boldsymbol{\lambda} + \mathrm{d}M_{\boldsymbol{\lambda}} = \frac{1}{\sqrt{N}}\mathrm{d}B^{(s)}$ and $\boldsymbol{\lambda}(\mathrm{d}D_{\boldsymbol{u}})^* + (\mathrm{d}D_{\boldsymbol{u}})\boldsymbol{\lambda} + \mathrm{d}D_{\boldsymbol{\lambda}} + \mathrm{d}D = 0$, so

$$\mathrm{d}\tilde{S} = \frac{1}{\sqrt{N}}\boldsymbol{u}(\mathrm{d}B^{(s)})\boldsymbol{u}^*.$$

As $\boldsymbol{u}_t \in \mathrm{O}(N)$ almost surely for any $t \geqslant 0$, by Lévy's criterion, the process M defined by $M_0 = 0$ and $\mathrm{d}M_t = \boldsymbol{u}(\mathrm{d}B^{(s)})\boldsymbol{u}^*$ is a symmetric Dyson Brownian motion. This concludes the proof: $(\tilde{H}^{(s)}_t)_{t\geqslant 0}$ and $(H^{(s)}_t)_{t\geqslant 0}$ have the same law, as they are both solution of the same stochastic differential equation, for which weak uniqueness holds.

(c) Existence and strong uniqueness for (2.2) has a proof strictly identical to (a). For a given continuous trajectory $(\boldsymbol{\lambda}_t)_{t\geqslant 0} \subset \Sigma_N$, existence and strong uniqueness for (2.3) is elementary, because $\sup_{t\in[0,T],i\neq j}|\lambda_i - \lambda_j|^{-1} < \infty$ and the coefficients are Lipschitz for any given $t \in [0,T]$.

Let $\boldsymbol{\lambda}'$ be the solution of (2.2), and $(\boldsymbol{u}^{(\boldsymbol{\lambda}')}_t)_{t\geqslant 0}$ be the solution of (2.3) for given $\boldsymbol{\lambda}'$. If the initial conditions match, we have

$$\mathbb{P}((\boldsymbol{\lambda}'_t, \boldsymbol{u}^{(\boldsymbol{\lambda}')}_t) = (\boldsymbol{\lambda}_t, \boldsymbol{u}_t) \text{ for all } t \geqslant 0) = 1, \tag{B.2}$$

because $(\boldsymbol{\lambda}'_t, \boldsymbol{u}^{(\boldsymbol{\lambda}')}_t)$ is a solution of the system of stochastic differential equations (2.2,2.3) for which strong uniqueness holds. Equations (B.2) together with (b) yields

$$\mathbb{E}(F((H^{(s)}_t)_{0\leqslant t\leqslant T})) = \mathbb{E}(F((\boldsymbol{u}^{(\boldsymbol{\lambda}')}_t\boldsymbol{\lambda}'_t(\boldsymbol{u}^{(\boldsymbol{\lambda}')}_t)^*)_{0\leqslant t\leqslant T})). \tag{B.3}$$

As strong uniqueness holds, $(\boldsymbol{\lambda}'_t)_{0\leqslant t\leqslant T}$ is a measurable function (called f) of $((B^{(s)}_{ii})_{0\leqslant t\leqslant T})_{i=1}^N$, and $(\boldsymbol{u}^{\boldsymbol{\lambda}'}_t)_{0\leqslant t\leqslant T}$ is a measurable function of $((B^{(s)}_{ij})_{0\leqslant t\leqslant T})_{i<j}$ and $(\boldsymbol{\lambda}'_t)_{0\leqslant t\leqslant T}$ (called g). We therefore have (for some Wiener measures ω_1, ω_2) for any bounded continuous function G

$$\begin{aligned}\mathbb{E}(G((\boldsymbol{\lambda}'_t)_{0\leqslant t\leqslant T}, (\boldsymbol{u}^{(\boldsymbol{\lambda}')}_t)_{0\leqslant t\leqslant T})) &= \iint \mathrm{d}\omega_1(B_1)\mathrm{d}\omega_2(B_2)G(f(B_1), g(f(B_1), B_2))\\ &= \iint \mathrm{d}\nu_T(\boldsymbol{\lambda})\mathrm{d}\omega_2(B_2)G(\boldsymbol{\lambda}, g(\boldsymbol{\lambda}, B_2)) = \iint \mathrm{d}\nu_T(\boldsymbol{\lambda})\mathrm{d}\mu_T(\boldsymbol{u}^{(\boldsymbol{\lambda})})G(\boldsymbol{\lambda}, \boldsymbol{u}^{(\boldsymbol{\lambda})}).\end{aligned}$$

Together with (B.3), this concludes the proof. We used the independence of the diagonal of $B^{(s)}$ with the other entries in the first equality above.

B.2 Proof of Lemma 2.4. We consider the Hermitian setting, the symmetric one being slightly easier. Let f be a smooth function of the matrix entries, $u_k(\alpha) = x_{k\alpha} + \mathrm{i} y_{k\alpha}$, $1 \leqslant k, \alpha \leqslant N$. We denote $\langle \cdot, \cdot \rangle' = (\mathrm{d}/\mathrm{d}t)\langle \cdot, \cdot \rangle$. Itô's formula yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}(f) = \mathbb{E}((\mathrm{I}) + (\mathrm{II}) + (\mathrm{III})),$$

$$(\mathrm{I}) = \sum_{k,\alpha}(-\frac{1}{2}\sum_{\ell \neq k} c_{k\ell})(x_{k\alpha}\partial_{x_{k\alpha}} + y_{k\alpha}\partial_{y_{k\alpha}})f,$$

$$(\mathrm{II}) = \frac{1}{2}\sum_{k,\alpha,\beta}\left(\langle x_{k\alpha}, x_{k\beta}\rangle'\partial_{x_{k\alpha}x_{k\beta}} + \langle y_{k\alpha}, y_{k\beta}\rangle'\partial_{y_{k\alpha}y_{k\beta}} + \langle x_{k\alpha}, y_{k\beta}\rangle'\partial_{x_{k\alpha}y_{k\beta}} + \langle y_{k\alpha}, x_{k\beta}\rangle'\partial_{y_{k\alpha}x_{k\beta}}\right) f,$$

$$(\mathrm{III}) = \sum_{k<\ell,\alpha,\beta}\left(\langle x_{k\alpha}, x_{\ell\beta}\rangle'\partial_{x_{k\alpha}x_{\ell\beta}} + \langle y_{k\alpha}, y_{\ell\beta}\rangle'\partial_{y_{k\alpha}y_{\ell\beta}} + \langle x_{k\alpha}, y_{\ell\beta}\rangle'\partial_{x_{k\alpha}y_{\ell\beta}} + \langle y_{k\alpha}, x_{\ell\beta}\rangle'\partial_{y_{k\alpha}x_{\ell\beta}}\right) f.$$

Substituting $\partial_x = \partial_u + \partial_{\overline{u}}$ and $\partial_y = \mathrm{i}(\partial_u - \partial_{\overline{u}})$ gives

$$(\mathrm{I}) = -\frac{1}{2}\sum_{k<\ell,\alpha} c_{k\ell}\left(u_k(\alpha)\partial_{u_k(\alpha)} + \overline{u}_k(\alpha)\partial_{\overline{u}_k(\alpha)} + u_\ell(\alpha)\partial_{u_\ell(\alpha)} + \overline{u}_\ell(\alpha)\partial_{\overline{u}_\ell(\alpha)}\right) f$$

$$= -\frac{1}{2}\sum_{k<\ell} c_{k\ell}\left(u_k\partial_{u_k} + \overline{u}_k\partial_{\overline{u}_k} + u_\ell\partial_{u_\ell} + \overline{u}_\ell\partial_{\overline{u}_\ell}\right) f.$$

Moreover, from the stochastic differential equation (2.5), we obtain

$$\langle x_{k\alpha}, x_{k\beta}\rangle' = \langle y_{k\alpha}, y_{k\beta}\rangle' = \frac{1}{2}\sum_{\ell \neq k} c_{k\ell}\Re(u_\ell(\alpha)\overline{u}_\ell(\beta)), \langle x_{k\alpha}, y_{k\beta}\rangle' = -\langle y_{k\alpha}, x_{k\beta}\rangle' = \frac{1}{2}\sum_{\ell \neq k} c_{k\ell}\Im(\overline{u}_\ell(\alpha)u_\ell(\beta)).$$

It implies that

$$(\mathrm{II}) = \frac{1}{2}\sum_{k<\ell,\alpha,\beta} c_{k\ell}\left(u_\ell(\alpha)\overline{u}_\ell(\beta)\partial_{u_k(\alpha)\overline{u}_k(\beta)} + \overline{u}_\ell(\alpha)u_\ell(\beta)\partial_{\overline{u}_k(\alpha)u_k(\beta)} + u_k(\alpha)\overline{u}_k(\beta)\partial_{u_\ell(\alpha)\overline{u}_\ell(\beta)} + \overline{u}_k(\alpha)u_k(\beta)\partial_{\overline{u}_\ell(\alpha)u_\ell(\beta)}\right) f$$

$$= \frac{1}{2}\sum_{k<\ell} c_{k\ell}\left(u_\ell\partial_{u_k}\overline{u}_\ell\partial_{\overline{u}_k} + \overline{u}_\ell\partial_{\overline{u}_k}u_\ell\partial_{u_k} + \overline{u}_k\partial_{\overline{u}_\ell}u_k\partial_{u_\ell} + u_k\partial_{u_\ell}\overline{u}_k\partial_{\overline{u}_\ell}\right) f.$$

Finally, concerning the term (III), a calculation yields, for $k \neq \ell$,

$$\langle x_{k\alpha}, x_{\ell\beta}\rangle' = -\langle y_{k\alpha}, y_{\ell\beta}\rangle' = -\frac{1}{2}c_{k\ell}\Re(u_\ell(\alpha)u_k(\beta)), \langle x_{k\alpha}, y_{\ell\beta}\rangle' = \langle x_{\ell\beta}, y_{k\alpha}\rangle' = -\frac{1}{2}c_{k\ell}\Im(u_\ell(\alpha)u_k(\beta)).$$

We therefore get

$$(\mathrm{III}) = -\frac{1}{2}\sum_{k<\ell,\alpha,\beta} c_{k\ell}\left(u_\ell(\alpha)u_k(\beta)\partial_{u_k(\alpha)u_\ell(\beta)} + \overline{u}_\ell(\alpha)\overline{u}_k(\beta)\partial_{\overline{u}_k(\alpha)\overline{u}_\ell(\beta)} + u_k(\alpha)u_\ell(\beta)\partial_{u_\ell(\alpha)u_k(\beta)} + \overline{u}_k(\alpha)\overline{u}_\ell(\beta)\partial_{\overline{u}_\ell(\alpha)\overline{u}_k(\beta)}\right) f$$

$$= -\frac{1}{2}\sum_{k<\ell} c_{k\ell}\left(u_\ell\partial_{u_k}u_k\partial_{u_\ell} - u_\ell\partial_{u_\ell} + \overline{u}_\ell\partial_{\overline{u}_k}\overline{u}_k\partial_{\overline{u}_\ell} - \overline{u}_\ell\partial_{\overline{u}_\ell} + u_k\partial_{u_\ell}u_\ell\partial_{u_k} - u_k\partial_{u_k} + \overline{u}_k\partial_{\overline{u}_\ell}\overline{u}_\ell\partial_{\overline{u}_k} - \overline{u}_k\partial_{\overline{u}_k}\right) f.$$

Gathering our estimates for (I), (II) and (III) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}(f) = \frac{1}{2}\sum_{k<\ell} c_{k\ell}\mathbb{E}\left(\left((u_k\partial_{u_\ell} - \overline{u}_\ell\partial_{\overline{u}_k})(\overline{u}_k\partial_{\overline{u}_\ell} - u_\ell\partial_{u_k}) + (\overline{u}_k\partial_{\overline{u}_\ell} - u_\ell\partial_{u_k})(u_k\partial_{u_\ell} - \overline{u}_\ell\partial_{\overline{u}_k})\right) f\right).$$

This completes the proof.

Appendix C Covariance matrices

Because of motivations in statistics, we will only define the eigenvector moment flow for real-valued covariance matrices. The eigenvector dynamics were already considered in [9]. The normalization constants follow our convention and are different from [9].

Let B be a $M \times N$ real matrix Brownian motion: $B_{ij} (1 \leqslant i \leqslant N, 1 \leqslant j \leqslant M)$ are independent standard Brownian motions. We define the $M \times N$ matrix M by

$$M_t = M_0 + \frac{1}{\sqrt{N}} B_t,$$

Then the real Wishart process is X is defined by $X_t = M_t^* M_t$. In the following, we will assume for simplicity that $M \geqslant N$, to avoid trivial eigenvalues of X (the case $M \leqslant N$ admits similar results, up to trivial adjustements). The eigenvalues and eigenvectors dynamics were given in [9], i.e. the direct analogue of definitions (2.2), (2.3) and Theorem 2.3 hold for the following stochastic differential equations:

$$\begin{aligned} \mathrm{d}\lambda_k &= 2\sqrt{\lambda_k}\frac{\mathrm{d}B^{(s)}_{kk}}{\sqrt{N}} + \left(\frac{1}{N}\sum_{\ell \neq k}\frac{\lambda_k + \lambda_\ell}{\lambda_k - \lambda_\ell} + \frac{M}{N}\right)\mathrm{d}t, \\ \mathrm{d}u_k &= \frac{1}{\sqrt{N}}\sum_{\ell\neq k}\frac{\sqrt{\lambda_k+\lambda_\ell}}{\lambda_k-\lambda_\ell}(\mathrm{d}B^{(s)}_{k\ell})u_\ell - \frac{1}{2N}\sum_{\ell\neq k}\frac{\lambda_k+\lambda_\ell}{(\lambda_k-\lambda_\ell)^2}(\mathrm{d}t)u_k. \end{aligned} \tag{C.1}$$

where $B^{(s)}$ is a (symmetric) $N \times N$ Dyson Brownian motion.

After conditioning on the eigenvalues trajectory, in the same way as Lemma 2.4, the generator for the above eigenvector dynamics can be shown to be

$$\mathrm{L}_t = \sum_{1\leqslant k<\ell\leqslant N} d_{k\ell}(t)(X^{(s)}_{k\ell})^2$$

where we used the notations (2.8) and

$$d_{k\ell}(t) = \frac{\lambda_k + \lambda_\ell}{N(\lambda_k(t) - \lambda_\ell(t))^2}$$

The definition and utility of the eigenvector moment flow for covariance matrices are then summarized as follows.

Theorem C.1 (Eigenvector moment flow for covariance matrices). *Let $\mathbf{q} \in \mathbb{R}^N$, $z_k = \sqrt{N}\langle \mathbf{q}, u_k(t)\rangle$. Suppose that $\boldsymbol{u}$ is the solution to the stochastic differential equation (C.1) and $f_{\boldsymbol{\lambda},t}(\boldsymbol{\eta})$ is defined analogously to* (3.5), *where $\boldsymbol{\eta}$ denote the configuration $\{(i_1, j_1), \dots, (i_m, j_m)\}$. Then $f_{\boldsymbol{\lambda},t}$ satisfies the equation*

$$\begin{aligned} &\partial_t f_{\boldsymbol{\lambda},t} = \mathscr{B}^{(s)}(t) f_{\boldsymbol{\lambda},t}, \\ &\mathscr{B}^{(s)}(t) f(\boldsymbol{\eta}) = \sum_{i\neq j} d_{ij}(t) 2\eta_i(1+2\eta_j)\left(f(\boldsymbol{\eta}^{i,j}) - f(\boldsymbol{\eta})\right). \end{aligned}$$

As in the case of symmetric matrices, the above eigenvector moment flow is reversible with respect to the measure $\pi^{(s)}$ defined in (3.9). Thus analogues of Theorems 1.2, Corollary 1.3, 1.4 and Theorem 7.1 for covariance matrices can be proved with arguments parallel to those used in Sections 4, 5 and 6.

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