MAXIMUM OF DIRICHLET L FUNCTIONS ON A SHORT INTERVAL OF THE CRITICAL LINE

CAMIL L SUCIU

1. Introduction

Following the groundbreaking paper [2] in which strong probabilistic bounds on short intervals of the critical line were established for the Riemann zeta function, we extend the main theorem there to L functions corresponding to Dirichlet characters, obtaining both upper and lower bounds for individual L functions on random intervals of constant length on the critical line. Our results do not take into account dependence on the modulus q of the character which is treated as a constant, but a more careful analysis of the method of proof should allow for results in which this dependence is made explicit.

In [2], Arguin, Belius, Bourgade, Radziwill and Soundararajan showed that for any $\varepsilon > 0$, as $T \to \infty$ the following holds:

$$\frac{1}{T}\text{meas}\Big\{T \leq t \leq 2T : (1-\varepsilon)\log\log T < \max_{|t-u|\leq 1} \log\left|\zeta\left(\tfrac{1}{2}+\mathrm{i}u\right)\right| < (1+\varepsilon)\log\log T\Big\} \to 1.$$

Let now $q \geq 3$ and χ a fixed non-principal character on $(\mathbb{Z}/q\mathbb{Z})^*$. Hence χ is completely multiplicative, periodic with period q, $\chi(n) = 0, (n,q) > 1$ and $|\chi(n)| = 1, (n,q) = 1$, while χ non-principal implies that $\sum_{n=1}^{q} \chi(n) = 0$.

As usual we define

$$L(s) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \Pi_{p\ \text{prime}}(1 - \frac{\chi(p)}{p^s})^{-1},\ \operatorname{Re} s > 1$$

the L function associated with χ. The fact that $\sum_{n=1}^{q} \chi(n) = 0$ implies that L originally defined and analytic for Re $s > 1$ where the series above is absolutely convergent, extends analytically to $\operatorname{Re} s > 0$ with conditional convergence on $0 < \operatorname{Re} s \leq 1$. It is well known that L extends to an entire function which satisfies a functional equation relating $L(s)$ with $L(1-s)$ but since our results concern only the critical strip $0 < \operatorname{Re} s < 1$ and actually the critical line $\operatorname{Re} s = \frac{1}{2}$ and its immediate neighborhood, we will not be needing that.

Our main result is:

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Theorem 1.1. *For any $\varepsilon > 0$, as $T \to \infty$ we have*

$$\frac{1}{T}\text{meas}\Big\{T \le t \le 2T : (1-\varepsilon)\log\log T < \max_{|t-u|\le 1} \log\left|L\left(\tfrac{1}{2}+\mathrm{i}u\right)\right| < (1+\varepsilon)\log\log T\Big\} \to 1.$$

The proof will follow closely the proof of the result above in [2], using corresponding results for the second and fourth momentum of L, L' as well as for various approximations of L, L' by Dirichlet polynomials and related functions that are well known and proven in literature for the Riemann zeta function. We will sketch proofs of these results as we proceed.

First let's note that we can reduce the proof of Theorem 1.1 to the case of χ primitive character, which means that χ is not induced by a character χ_1 with modulus $q_1|q, q_1 < q$, where χ would be induced by χ_1 if $\chi(n) = \chi_1(n)$ when $(n, q) = 1$.

In the induced case one clearly has that

$$L_\chi(s) = \Pi_{p|q}(1 - \frac{\chi_1(p)}{p^s})L_{\chi_1}(s)$$

for all s by analytic continuation from the Euler product for $\mathrm{Re}\, s > 1$. Since there are finitely many primes $p|q$ and for each of them we have $A \le |(1 - \frac{\chi_1(p)}{p^s})| \le B$ for $\mathrm{Re}\, s \ge \delta > 0$ and some constants $A, B > 0$, it follows that

$$A_1 < \log\left|L_\chi\left(\tfrac{1}{2}+\mathrm{i}t\right)\right| - \log\left|L_{\chi_1}\left(\tfrac{1}{2}+\mathrm{i}t\right)\right| < A_2$$

for some constants A_1, A_2 so the result for χ_1 implies the result for χ. So from now on, we assume χ primitive.

1.1. **About the proof.** Theorem 1.1 asserts two statements: first an *upper bound* that for typical $t \in [T, 2T]$ one has $\max_{|t-u|\le 1} \log|L(\frac{1}{2}+\mathrm{i}u)| \le (1+\varepsilon)\log\log T$, and second a *lower bound* that this maximum is also typically $\ge (1-\varepsilon)\log\log T$. The upper bound in Theorem 1.1 admits a short proof based on a Sobolev type inequality and second moment estimates for $L(s)$ and $L'(s)$. This argument is given in section 2, and indeed in Proposition 2.1 we establish the stronger assertion that for any function $V = V(T)$ tending to infinity with T we have

$$\frac{1}{T}\text{meas}\Big\{\max_{|t-u|\le 1} \log\left|L\left(\tfrac{1}{2}+\mathrm{i}u\right)\right| < \log(V\log T)\Big\} \to 1.$$

The lower bound in Theorem 1.1 requires substantially more work, and forms the bulk of the paper. In Section 3, we reduce the proof of Theorem 1.1 to two propositions. The first step, Proposition 3.1, transforms the problem to the study of Dirichlet polynomials supported on the primes below $X = \exp((\log T)^{1-\kappa})$ for a suitable $\kappa = \kappa(\varepsilon) > 0$. The second step, Proposition 3.2, establishes lower bounds for the Dirichlet polynomials over primes.

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2. Proof of the upper bound

The upper bound implicit in our theorem will be a simple consequence of estimates for the second moment of L functions and their derivatives, together with a Sobolev-type inequality. Let f (possibly complex valued) be continuously differentiable on $[-1,1]$. For any $u \in [-1,1]$, note that

$$f(u)^2 = \frac{f(1)^2 + f(-1)^2}{2} + \int_{-1}^{u} f'(v)f(v)\mathrm{d}v - \int_{u}^{1} f'(v)f(v)\mathrm{d}v,$$

so that using the triangle inequality we get the Sobolev inequality:

$$\max_{u\in[-1,1]} |f(u)|^2 \le \frac{|f(1)|^2 + |f(-1)|^2}{2} + \int_{-1}^{1} |f'(v)f(v)|\mathrm{d}v. \tag{1}$$

Proposition 2.1. *Let $V = V(T)$ be any function that tends to infinity as $T \to \infty$. Then*

$$\mathbb{P}\Big(\max_{|t-u|\le 1} |L(1/2 + \mathrm{i}u)| > V \log T\Big) = \mathrm{O}(1/V^2) = \mathrm{o}(1),$$

where we recall that t is sampled uniformly in the range $[T, 2T]$.

Proof. Chebyshev's inequality implies that

$$\mathbb{P}\Big(\max_{|t-u|\le 1} |L(1/2 + \mathrm{i}u)| > V \log T\Big) \le \frac{1}{V^2(\log T)^2}\mathbb{E}\left[\max_{|t-u|\le 1} |L(1/2 + \mathrm{i}u)|^2\right]. \tag{2}$$

Applying (1) with $f(v) = L(1/2 + \mathrm{i}t + \mathrm{i}v)$, we obtain

$$\max_{|t-u|\le 1} |L(1/2+\mathrm{i}u)|^2 \ll |L(\tfrac{1}{2}+\mathrm{i}(t+1))|^2+|L(\tfrac{1}{2}+\mathrm{i}(t-1))|^2+\int_{-1}^{1} |L'(\tfrac{1}{2}+\mathrm{i}(t+v))L(\tfrac{1}{2}+\mathrm{i}(t+v))|\mathrm{d}v,$$

Integrating the above on $[T, 2T]$ and switching the double integral gives:

$$\mathbb{E}\left[\max_{|t-u|\le 1} |L(1/2 + \mathrm{i}u)|^2\right] \ll \frac{1}{T}\int_{T-1}^{2T+1} \Big(|L(\tfrac{1}{2} + \mathrm{i}t)|^2 + |L'(\tfrac{1}{2} + \mathrm{i}t)L(\tfrac{1}{2} + \mathrm{i}t)|\Big)\mathrm{d}t.$$

Now we use bounds for the second momentum of L, L' which follow similarly to the ones for the Riemann zeta function and for which we will sketch a proof below.

$$\int_{T-1}^{2T+1} |L(\tfrac{1}{2} + \mathrm{i}t)|^2\mathrm{d}t \ll T\log T, \qquad \text{and} \qquad \int_{T-1}^{2T+1} |L'(\tfrac{1}{2} + \mathrm{i}t)|^2\mathrm{d}t \ll T(\log T)^3. \tag{3}$$

Using these estimates and Cauchy-Schwarz inequality, we conclude that

$$\mathbb{E}\left[\max_{|t-u|\le 1} |L(1/2 + \mathrm{i}u)|^2\right] \ll (\log T)^2,$$

which, in view of (2), yields the proposition. □

Proof of Second Momentum Bounds for L,L'

First we prove (3) using approximations (proven below) of L, L' by a Dirichlet polynomial similar to Theorem 4.11.1 in [11]. So we assume:

$$L(s) = \sum_{n\le qx} \frac{\chi(n)}{n^s} + O(x^{-\sigma}) \tag{4}$$

uniformly for $s=\sigma+it$, $\sigma\geq\delta>0$, $|t|<\frac{2\pi x}{C}$ for a given constant $C>1$.

Now in (4), we let $\sigma=\frac{1}{2}$, $\delta=\frac{1}{4}$, $C=\frac{2\pi}{3}$, $x=T$.

Letting $Q(s)=\sum_{n\leq qT}\frac{\chi(n)}{n^s}$, we see that $L(\frac{1}{2}+it)=Q(\frac{1}{2}+it)+R$ where $R>0$ and $R=O(T^{-\frac{1}{2}})$ for all $t\in[T-1,2T+1]$ since our choice of C gives us $|t|<3T$. Hence:

$$(5)\quad \int_{T-1}^{2T+1}|L(\frac{1}{2}+it)|^2dt\leq\int_{T-1}^{2T+1}|Q(\frac{1}{2}+it)|^2dt+2R\int_{T-1}^{2T+1}|Q(\frac{1}{2}+it)|dt+(T+2)R^2$$

By Cauchy-Schwarz:

$$2R\int_{T-1}^{2T+1}|Q(\frac{1}{2}+it)|dt\leq 2R\sqrt{T+2}\left(\int_{T-1}^{2T+1}|Q(\frac{1}{2}+it)|^2dt\right)^{\frac{1}{2}}\ll\left(\int_{T-1}^{2T+1}|Q(\frac{1}{2}+it)|^2dt\right)^{\frac{1}{2}}$$

while $(T+2)R^2\ll 1$
So to prove our estimate for L, it is enough to prove $\int_{T-1}^{2T+1}|Q(\frac{1}{2}+it)|^2dt=O(T\log T)$

Since $|Q|^2=Q\bar{Q}$ and $|\chi(n)|=0$ or 1, multiplying and integrating term by term, we get:

$$\int_{T-1}^{2T+1}|Q(\frac{1}{2}+it)|^2dt\ll(T+2)\sum_{n\leq qT}\frac{1}{n}+\sum_{1\leq n<k\leq qT}\frac{1}{\log(k/n)\sqrt{nk}}$$

But $(T+2)\sum_{n\leq qT}\frac{1}{n}\ll(T+2)\log(qT)\ll T\log T$. We split the second sum into $\frac{k}{n}\geq\frac{3}{2}$ and $\frac{k}{n}<\frac{3}{2}$
Now the subsum when $\frac{k}{n}\geq\frac{3}{2}$ is at most:

$$\frac{1}{\log\frac{3}{2}}\sum_{1\leq n<k\leq qT}\frac{1}{\sqrt{nk}}\ll(\sum_{1\leq n\leq qT}\frac{1}{\sqrt{n}})^2\ll(\sqrt{qT})^2\ll T$$

.
For the second subsum, when $\frac{k}{n}<\frac{3}{2}$, we split it further into sums where $k-n=c$ for some constant $1\leq c<qT$. Since $\frac{k}{n}<\frac{3}{2}$, $\frac{c}{n}<\frac{1}{2}$ so $\log(\frac{k}{n})=\log(1+\frac{c}{n})\gg\frac{c}{n}$.
So for a fixed c, the corresponding subsum is at most $\frac{qT}{c}$ since $\sqrt{\frac{n}{k}}\leq 1$. Thus the full subsum over all $c's$ for $\frac{k}{n}\geq\frac{3}{2}$ is at most: $qT\sum_{1\leq c<qT}\frac{1}{c}\ll qT\log T\ll T\log T$.
So the estimate is proved.

For L', instead of (4), we have:

$$(6)\qquad L'(s)=\sum_{n\leq qx}\frac{-\log(n)\chi(n)}{n^s}+O(x^{-\sigma}\log x)$$

uniformly for $s=\sigma+it$, $\sigma\geq\delta>0$, $|t|<\frac{2\pi x}{C}$ for a given constant $C>1$ with the proof following by applying Cauchy to (4).

With the same choices as above and a similar procedure with $Q_1(s)=\sum_{n\leq qT}\frac{-\chi(n)\log(n)}{n^s}$, it follows that:

$$\int_{T-1}^{2T+1} |Q_1(\frac{1}{2}+it)|^2 dt \ll (T+2)\sum_{n\le qT}\frac{(\log n)^2}{n} + \sum_{1\le n<k\le qT}\frac{\log(q)\log(k)}{\log(k/n)\sqrt{nk}} \ll$$

$$\ll (\log qT)^2((T+2)\sum_{n\le qT}\frac{1}{n} + \sum_{1\le n<k\le qT}\frac{1}{\log(k/n)\sqrt{nk}}) \ll T(\log T)^3$$

Hence,

$$\int_{T-1}^{2T+1} |L'(\frac{1}{2}+it)|^2 dt \ll T(\log T)^3$$

as the error is negligible as above. This completes the proof of bounds for the second moments of L and L'.

Note that by using the more precise mean value theorem for Dirichlet polynomials, Theorem 5.2 in [5]

$$\int_T^{2T} |\sum_{n\le N} a_n n^{it}|^2 = T\sum_{n\le N}|a_n|^2 + O(\sum_{n\le N} n|a_n|^2)$$

we can actually obtain asymptotics for the second momentum of both L, L'. We also note that here we do not need that χ is primitive, but only non-principal, though we will need primitivity in the lower bound part.

Sketch of Proof of (4), (6)

Define the Hurwitz zeta function $\zeta(s,a) = \sum_{n=0}^{\infty}\frac{1}{(n+a)^s}$ for $0 < a \le 1$ and $\operatorname{Re} s > 1$ where $\zeta(s,1)$ is the Riemann zeta function. It follows that

$$L(s) = q^{-s}\sum_{r=1}^{q}\chi(r)\zeta(s,\frac{r}{q}) \tag{7}$$

Applying Euler's summation formula with $f(t) = (t+a)^{-s}$ we get for $\operatorname{Re} s > 1$:

$$\zeta(s,a) = \sum_{n=0}^{N}\frac{1}{(n+a)^s} + \frac{(N+a)^{1-s}}{s-1} - s\int_N^{\infty}\frac{x-\lfloor x\rfloor}{(x+a)^{s+1}}dx \tag{8}$$

Since $0 \le x - \lfloor x\rfloor < 1$, it follows that the integral is absolutely and uniformly convergent for $\operatorname{Re} s \ge \delta > 0$. So the above equation is valid for $\operatorname{Re} s > 0$ and can be used to define the analytic continuation of $\zeta(s,a)$ up to $\operatorname{Re} s > 0$.
Similarly to the proof of Theorem 4.11.1 in [11], we have (with $|t| < \frac{2\pi x}{C}, C > 1$):

$$\sum_{x<n\le N}\frac{1}{(n+a)^s} = \int_x^N \frac{du}{(u+a)^s} + O(x^{-\sigma}) = \frac{(N+a)^{1-s} - x^{1-s}}{1-s} + O(x^{-\sigma})$$

Hence $\zeta(s,a) = \sum_{n\le x}\frac{1}{(n+a)^s} - \frac{(x+a)^{1-s}}{1-s} + O(x^{-\sigma}) + O(\frac{|s|+1|}{N^{\sigma}})$.

Taking N to infinity, we get:

$$\zeta(s,a)=\sum_{n\le x}\frac{1}{(n+a)^s}-\frac{x^{1-s}}{1-s}+O(x^{-\sigma}) \tag{9}$$

because $\frac{(x+a)^{1-s}-x^{1-s}}{1-s}=O(x^{-\sigma})$.

Applying (9) for each $a=\frac{1}{q},...,\frac{q-1}{q},1$ and plugging into (7), we get (4) noting that $\sum_{k=1}^{q}\chi(k)=0$.

For proving (6), we note that (4) gives us (under the corresponding assumptions on x,σ,t) that

$$L(s)=\sum_{n\le qx}\frac{\chi(n)}{n^s}+f(s)$$

where f is analytic and $f(s)=O(x^{-\sigma})$. Applying Cauchy to a small circle around s of radius $\frac{1}{\log x}$ we get that

$$f'(s)=O(x^{-\sigma}\log x\int_0^{2\pi}x^{-\frac{\cos\theta}{\log x}}d\theta)=O(x^{-\sigma}\log x)$$

and we are done by differentiating (4) term by term.

3. Plan of the proof of the lower bound

Here we assume χ is a primitive character with modulus q. Given $\varepsilon>0$, we will fix a large integer $K=K(\varepsilon)$, and divide the primes below

$$X=\exp((\log T)^{1-\frac{1}{K}}) \tag{10}$$

into $K-1$ ranges depending on their size, as follows. Take $J_0=[2,\exp((\log T)^{\frac{1}{K}})]$, and for $1\le j\le K-2$ set

$$J_j=(\exp((\log T)^{\frac{j}{K}}),\exp((\log T)^{\frac{j+1}{K}})]. \tag{11}$$

For each $0\le j\le K-2$, we define the Dirichlet polynomial

$$P_j(u)=\mathrm{Re}\sum_{p\in J_j}\frac{\chi(p)}{p^{\sigma_0+\mathrm{i}u}}, \tag{12}$$

where

$$\sigma_0=\frac{1}{2}+\frac{(\log T)^{\frac{3}{2K}}}{\log T}. \tag{13}$$

By taking T large enough we can assume that all primes $p|q$ are in J_0.

Using the prime number theorem (see for example Theorem 6.9 of [7]) and partial summation it follows that for some constant $c>0$, and any $\sigma=\frac{1}{2}+\delta$ with $\delta>0$

$$\sum_{x\le p\le y}\frac{1}{p^{2\sigma}}=\int_x^y\frac{1}{u^{2\sigma}\log u}\mathrm{d}u+\mathrm{O}(e^{-c\sqrt{\log x}})=\log\frac{\log y}{\log x}+\mathrm{O}(\delta\log y+e^{-c\sqrt{\log x}}). \tag{14}$$

Since $(\sigma_0 - 1/2) \times \log(\sup J_{K-3}) = (\log T)^{-\frac{1}{2K}}$ it follows that, for all $1 \leq j \leq K-3$,

$$\sum_{p \in J_j} \frac{1}{p^{2\sigma_0}} = \frac{1}{K} \log\log T + \mathrm{O}((\log T)^{-\frac{1}{2K}}), \tag{15}$$

and the sums above for $j = 1 \ldots K-3$ are all on primes that do not divide q by our assumption above.

Proposition 3.1. *Let $\varepsilon > 0$ be given, and let $K = K(\varepsilon)$ be a suitably large integer. Then*

$$\mathbb{P}\Big(\max_{|t-u| \leq 1} \log |L(\tfrac{1}{2} + \mathrm{i}u)| > (1-2\varepsilon) \log\log T \Big)$$
$$\geq \mathbb{P}\Big(\max_{|t-u| \leq \frac{1}{4}} \sum_{j=1}^{K-3} P_j(u) > (1-\varepsilon) \log\log T \Big) + \mathrm{o}(1).$$

Proposition 3.2. *Let $K > 3$ be a natural number, and $0 < \lambda < 1$ be a real number. Then*

$$\mathbb{P}\Big(\max_{|t-u| \leq \frac{1}{4}} \Big(\min_{1 \leq j \leq K-3} P_j(u) \Big) > \frac{\lambda}{K} \log\log T \Big) = 1 + \mathrm{o}(1). \tag{16}$$

Once Propositions 3.1, 3.2 are proven, Theorem 1.1 follows immediately:

Proof of Theorem 1.1. If Proposition 3.2 holds, then

$$\max_{|u-t| \leq \frac{1}{4}} \sum_{j=1}^{K-3} P_j(u) > \lambda \Big(1 - \frac{3}{K} \Big) \log\log T.$$

Taking λ sufficiently close to 1, and K large enough, the lower bound of the theorem now follows from Proposition 3.1. □

Before proceeding to the proofs of the proposition, we record some results on mean values of Dirichlet polynomials which will be repeatedly used below and are proven in the main paper [2].

Lemma 3.3. *For any complex numbers $a(n)$ and $b(n)$, and $N \leq T$ we have*

$$\int_T^{2T} \Big(\sum_{m \leq N} a(m) m^{-\mathrm{i}t} \Big) \Big(\sum_{n \leq N} b(n) n^{\mathrm{i}t} \Big) \mathrm{d}t = T \sum_{n \leq N} a(n) b(n) + \mathrm{O}\Big(N \log N \sum_{n \leq N} (|a(n)|^2 + |b(n)|^2) \Big).$$

Lemma 3.4. *Let $x \geq 2$ be a real number, and suppose that for primes $p \leq x$, $a(p)$ and $b(p)$ are complex numbers with $|a(p)|$ and $|b(p)|$ both at most 1. Then for any natural number k we have*

$$\mathbb{E}\Big[\Big(\frac{1}{2} \sum_{p \leq x} (a(p) p^{-\mathrm{i}t} + b(p) p^{\mathrm{i}t}) \Big)^k \Big] = \partial_z^k \Big(\prod_{p \leq x} I_0(\sqrt{a(p) b(p)}\, z) \Big) \Big|_{z=0} + \mathrm{O}\Big(\frac{x^{2k}}{T} \Big)$$

where $I_0(z) = \sum_{n \geq 0} z^{2n}/(2^{2n}(n!)^2)$ denotes the Bessel function. In particular, the expression is $\mathrm{O}\left(x^{2k}/T\right)$ for odd k.

Lemma 3.5. *Let $x \geq 2$ be a real number, and suppose $\sigma \geq \frac{1}{2}$. Let k be a natural number such that $x^k \leq T(\log T)^{-1}$. Then, for any sequence of complex numbers $a(p)$ defined on the primes p below x,*

$$\frac{1}{T}\int_T^{2T} \Big|\sum_{p\leq x} \frac{a(p)}{p^{\sigma+\mathrm{i}t}}\Big|^{2k} \mathrm{d}t \ll k!\Big(\sum_{p\leq x}\frac{|a(p)|^2}{p^{2\sigma}}\Big)^k.$$

4. Proof of Proposition 3.1

4.1. **Step 1.** We divide the proof of the proposition into three parts, the first of which bounds the maximum of the L function over intervals of the critical line in terms of the maximum over intervals lying slightly to the right of the critical line.

Lemma 4.1. *Let $\varepsilon > 0$ be given, and suppose $\frac{1}{2} \leq \sigma \leq \frac{1}{2} + (\log T)^{-1/2-\varepsilon}$. Then, for any real number $V \geq 2$,*

$$\mathbb{P}\Big(\max_{|t-u|\leq 1} |L\left(1/2+\mathrm{i}u\right)| > V\Big) \geq \mathbb{P}\Big(\max_{|t-u|\leq \frac{1}{4}} |L(\sigma+\mathrm{i}u)| > 2V\Big) + \mathrm{o}(1).$$

Proof. From (4) we recall that for $\sigma \geq \frac{1}{2}$

$$L(\sigma+\mathrm{i}t) = \sum_{n\leq qT}\frac{\chi(n)}{n^{\sigma+\mathrm{i}t}} + \mathrm{O}(T^{-\frac{1}{2}}). \tag{17}$$

Using knowledge of the Fourier transform of the function $e^{-|x|}$, we may write

$$\frac{1}{n^{\sigma-\frac{1}{2}}} = \frac{1}{\pi}\int_{-\infty}^{\infty} n^{-iv}\frac{(\sigma-1/2)}{(\sigma-1/2)^2+v^2}\mathrm{d}v = \frac{1}{\pi}\int_{-T/2}^{T/2} n^{-iv}\frac{(\sigma-1/2)}{(\sigma-1/2)^2+v^2}\mathrm{d}v + O(T^{-1}).$$

Thus by multiplying both sides by $\frac{\chi(n)}{n^{\frac{1}{2}+it}}$ and adding $n = 1, ..., qT$ we see that

$$L(\sigma+\mathrm{i}t) = \frac{1}{\pi}\int_{-T/2}^{T/2} L\left(1/2+\mathrm{i}(t+v)\right)\frac{\sigma-1/2}{(\sigma-1/2)^2+v^2}\mathrm{d}v + O(T^{-\frac{1}{2}}). \tag{18}$$

Consider $t \in [T, 2T]$ such that $\max_{|v|\leq\frac{1}{4}} |L\left(\sigma+\mathrm{i}(t+v)\right)| > 2V$ but $\max_{|v|<1} |L(1/2+\mathrm{i}(t+v))| \leq V$; we must show that the measure of the set of such points t is $\mathrm{o}(T)$. If t is such a point, then denote by $v^\star = v^\star(t)$ the $v \in [-\frac{1}{4}, \frac{1}{4}]$ where the maximum of $|L(\sigma+\mathrm{i}(t+v))|$ is attained. Applying (18) to the point $\sigma+\mathrm{i}(t+v^\star)$ we obtain

$$2V < |L(\sigma+\mathrm{i}(t+v^\star))| \leq \frac{1}{\pi}\int_{-T/2}^{T/2} |L\left(1/2+\mathrm{i}(t+v^\star+v)\right)|\frac{(\sigma-1/2)}{(\sigma-1/2)^2+v^2}\mathrm{d}v + \mathrm{O}(T^{-\frac{1}{2}}).$$

Since $|L(1/2+iu)| \leq V$ for $|t-u| \leq 1$ (by assumption), the portion of the integral above with $|v| \leq \frac{3}{4}$ is less than V. Therefore it follows that

$$V + \mathrm{O}(T^{-\frac{1}{2}}) \leq \frac{1}{\pi}\int_{\frac{3}{4}\leq|v|\leq\frac{T}{2}} |L\left(1/2+\mathrm{i}(t+v^\star+v)\right)|\frac{(\sigma-1/2)}{(\sigma-1/2)^2+v^2}\mathrm{d}v.$$

Using the Cauchy-Schwarz inequality, we deduce that for such t,

$$\Big(\frac{V}{(\sigma-1/2)}\Big)^2 \ll \Big(\int_{\frac{3}{4}\leq|v|\leq\frac{T}{2}} |L(1/2+\mathrm{i}(t+v^\star+v))|\frac{\mathrm{d}v}{v^2}\Big)^2 \ll \int_{\frac{1}{2}\leq|v|\leq\frac{T}{2}} |L(1/2+\mathrm{i}(t+v))|^2\frac{\mathrm{d}v}{v^2}.$$

Therefore, by Chebyshev's inequality, the measure of the set of such points $t \in [T, 2T]$ is

$$\ll \Big(\frac{(\sigma-1/2)}{V}\Big)^2 \int_T^{2T} \int_{\frac12 \le |v| \le \frac{T}{2}} |L(1/2+\mathrm{i}(t+v))|^2 \frac{\mathrm{d}v}{v^2}\mathrm{d}t \ll \Big(\frac{(\sigma-1/2)}{V}\Big)^2 \int_{T/2}^{5T/2} |L(1/2+\mathrm{i}t)|^2 \mathrm{d}t,$$

which, by (3) and the assumption on σ, is

$$\ll (\sigma-1/2)^2 T \log T = \mathrm{o}(T).$$

□

4.2. **Step 2.** The second part of the attack will consist of showing that on the σ_0 line, one can typically invert $L(\sigma_0+it)$ and replace it by a suitable Dirichlet polynomial. We define

$$M(s) = \sum_n \frac{\mu(n)a(n)\chi(n)}{n^s}, \tag{19}$$

where the factor $a(n)$ equals 1 if all primes factors of n are smaller than X and $\Omega(n) \le 100K \log\log T =: \nu$, while $a(n) = 0$ otherwise. Recall that μ denotes the Möbius function, $\Omega(n)$ counts the number of prime factors of n (with multiplicity), and X was defined in (10).

Lemma 4.2. *With the above notation*

$$\int_T^{2T} |L(\sigma_0+\mathrm{i}t)M(\sigma_0+\mathrm{i}t)-1|^2 \mathrm{d}t = \mathrm{O}\Big(\frac{T}{(\log T)^{100}}\Big).$$

Proof. From its definition, $a(n) = 0$ unless $n \le X^\nu < T^\varepsilon$ ($\varepsilon > 0$ is a fixed arbitrarily small constant), and therefore estimating trivially one has $M(\sigma_0+\mathrm{i}t) \ll T^\varepsilon$. Combining this with (17), we see that

$$\int_T^{2T} L(\sigma_0+\mathrm{i}t)M(\sigma_0+\mathrm{i}t)\mathrm{d}t = \int_T^{2T} \sum_{n\le qT} \frac{\chi(n)}{n^{\sigma_0+\mathrm{i}t}} \sum_m \frac{\mu(m)a(m)\chi(m)}{m^{\sigma_0+\mathrm{i}t}}\mathrm{d}t + \mathrm{O}(T^{\frac12+\varepsilon}).$$

Carrying out the integral over t and using $\chi(1)=1, |\chi(n)| \le 1$, this is

$$T + \mathrm{O}\Big(\sum_{\substack{n\le qT, m\le X^\nu\\ mn>1}} \frac{1}{(mn)^{\sigma_0}}\Big) + \mathrm{O}(T^{\frac12+\varepsilon}) = T + \mathrm{O}(T^{\frac12+\varepsilon}).$$

Thus, expanding out the square in the desired integral, we see that it equals

$$\int_T^{2T} |L(\sigma_0+\mathrm{i}t)M(\sigma_0+\mathrm{i}t)|^2 \mathrm{d}t - T + \mathrm{O}(T^{\frac12+\varepsilon}). \tag{20}$$

To estimate the second moment in (20), we invoke a restricted L version of the classical Selberg lemma for which we will sketch a proof at the end of the section using Selberg's method from [9] (one can find [9] as 15 in Selberg's Collected Papers, Vol 1, [10]).

Proposition 4.3. *For any fixed $\varepsilon > 0$ and $1 \le h, k \le T^\varepsilon, (h,q) = (k,q) = 1$ and $1/2 < \sigma \le 1$, we have*

$$\int_T^{2T} \chi(h)\bar\chi(k)\Big(\frac{h}{k}\Big)^{\mathrm{i}t} |L(\sigma+\mathrm{i}t)|^2 \mathrm{d}t =$$

$$\int_T^{2T} (L_{0,q}(2\sigma)\Big(\frac{(h,k)^2}{hk}\Big)^{\sigma} + \Big(\frac{qt}{2\pi}\Big)^{1-2\sigma} L_{0,q}(2-2\sigma)\Big(\frac{(h,k)^2}{hk}\Big)^{1-\sigma} \mathrm{d}t + \mathrm{O}(T^{1-\sigma/2+4\varepsilon})$$

Where $L_{0,q}(\sigma+it) = \zeta(\sigma+it)\prod_{p|q}(1-\frac{1}{p^s})$ is the L function corresponding to the trivial/principal character of modulus q.

Using this result, we may write

$$\int_T^{2T} |L(\sigma_0+\mathrm{it})M(\sigma_0+\mathrm{it})|^2 \mathrm{d}t = \sum_{h,k} \frac{\mu(h)a(h)\mu(k)a(k)}{h^{\sigma_0}k^{\sigma_0}} \int_T^{2T} \chi(h)\bar{\chi}(k)\Big(\frac{h}{k}\Big)^{\mathrm{it}} |L(\sigma_0+\mathrm{it})|^2 \mathrm{d}t$$

$$= S_1 + S_2 + E, \tag{21}$$

say, with

$$S_1 = TL_{0,q}(2\sigma_0) \sum_{(h,q)=1,(k,q)=1} \frac{\mu(h)a(h)\mu(k)a(k)}{(hk)^{\sigma_0}} \Big(\frac{(h,k)^2}{hk}\Big)^{\sigma_0}, \tag{22}$$

$$S_2 = L_{0,q}(2-2\sigma_0)\Big(\int_T^{2T} \Big(\frac{qt}{2\pi}\Big)^{1-2\sigma_0} \mathrm{d}t\Big) \sum_{(h,q)=1,(k,q)=1} \frac{\mu(h)a(h)\mu(k)a(k)}{(hk)^{\sigma_0}} \Big(\frac{(h,k)^2}{hk}\Big)^{1-\sigma_0}, \tag{23}$$

and

$$E = \mathrm{O}\Big(T^{\frac{3}{4}+4\varepsilon} \sum_{h,k\le T^{\varepsilon}} \frac{1}{(hk)^{\sigma_0}}\Big) = \mathrm{O}(T^{\frac{3}{4}+5\varepsilon}). \tag{24}$$

Now consider the quantity S_1. Here the sum is over all h and k whose prime factors do not divide q, are below X, and with $\Omega(h)$ and $\Omega(k)$ below ν. If we drop the Ω condition, then the contribution to S_1 would be (upon considering whether a prime p divides neither h nor k, or divides exactly one of h or k, or divides both h and k)

$$TL_{0,q}(2\sigma_0) \sum_{\substack{(h,q)=1,(k,q)=1 \\ p|hk \implies p\le X}} \frac{\mu(h)\mu(k)}{(hk)^{\sigma_0}} \Big(\frac{(h,k)^2}{hk}\Big)^{\sigma_0} = TL_{0,q}(2\sigma_0) \prod_{p\le X,(p,q)=1} \Big(1 - \frac{1}{p^{2\sigma_0}} - \frac{1}{p^{2\sigma_0}} + \frac{1}{p^{2\sigma_0}}\Big)$$

$$= T\zeta(2\sigma_0) \prod_{p\le X} \Big(1 - \frac{1}{p^{2\sigma_0}}\Big). \tag{25}$$

since X is large, $q \le X$ hence all the primes $p|q$ satisfy $p \le X$ too.

The difference between S_1 and (25) comes from the terms with either $\Omega(h)$ or $\Omega(k)$ being larger than ν, and these terms give a contribution bounded by (assuming that $\Omega(h)$ is larger than ν)

$$\ll TL_{0,q}(2\sigma_0) \sum_{\substack{h,k \\ \Omega(h)>\nu \\ p|hk \implies p\le X}} \frac{|\mu(h)\mu(k)|}{(hk)^{\sigma_0}}\Big(\frac{(h,k)^2}{hk}\Big)^{\sigma_0}$$

$$\ll TL_{0,q}(2\sigma_0)e^{-\nu} \sum_{\substack{h,k \\ p|hk \implies p\le X}} \frac{|\mu(h)\mu(k)|}{(hk)^{\sigma_0}}\Big(\frac{(h,k)^2}{hk}\Big)^{\sigma_0} e^{\Omega(h)},$$

since $e^{\Omega(h)-\nu} \ge 1$ when $\Omega(h) \ge \nu$, and is non-negative for other terms. As $L_{0,q}(2\sigma_0) \approx \zeta(2\sigma_0)$, the sum over h and k may now be expressed as a product over the primes below X, yielding

$$T\zeta(2\sigma_0)e^{-\nu}\prod_{p\le X}\Big(1+\frac{e}{p^{2\sigma_0}}+\frac{1}{p^{2\sigma_0}}+\frac{e}{p^{2\sigma_0}}\Big) \ll T(\log T)e^{-\nu}\prod_{p\le X}\Big(1+\frac{7}{p}\Big) \ll \frac{T}{(\log T)^{100}}.$$

Thus

$$S_1 = T\zeta(2\sigma_0)\prod_{p\le X}\Big(1-\frac{1}{p^{2\sigma_0}}\Big)+O\Big(\frac{T}{(\log T)^{100}}\Big) = T\prod_{p>X}\Big(1-\frac{1}{p^{2\sigma_0}}\Big)^{-1}+\mathrm{O}\Big(\frac{T}{(\log T)^{100}}\Big).$$

Recalling the definitions of σ_0 and X, we find $(\sigma_0 - 1/2)\log X = (\log T)^{\frac{1}{2K}}$, and so

$$\sum_{p>X}\log\Big(1-\frac{1}{p^{2\sigma_0}}\Big)^{-1} \ll \sum_{p>X}\frac{1}{p^{2\sigma_0}} \ll X^{-(\sigma_0-1/2)}\sum_{p>X}\frac{1}{p^{\sigma_0+1/2}} \ll (\log T)^{-100},$$

which enables us to conclude that $S_1 = T + O(T/(\log T)^{100})$.

Arguing similarly, we see that

$$S_2 \sim L_{0,q}(2-2\sigma_0)\Big(\int_T^{2T}\Big(\frac{qt}{2\pi}\Big)^{1-2\sigma_0}\mathrm{d}t\Big)\prod_{p\le X,(p,q)=1}\Big(1-\frac{2}{p}+\frac{1}{p^{2\sigma_0}}\Big) \ll T^{2-2\sigma_0}\log T \ll \frac{T}{(\log T)^{100}}.$$

Inserting the evaluation of S_1 with the estimates for S_2 and E into (21), and then into (20), we obtain the lemma. □

Lemma 4.2 ensures that for most t one has $L(\sigma_0+\mathrm{i}t)M(\sigma_0+\mathrm{i}t)\approx 1$, and we next refine this to ensure that for most t one has $L(\sigma_0+\mathrm{i}u)M(\sigma_0+\mathrm{i}u)\approx 1$ for all u with $|u-t|\le 1$.

Lemma 4.4. *For any $\varepsilon > 0$, we have*

$$\mathbb{P}\Big(\max_{|t-u|\le 1}|M(\sigma_0+\mathrm{i}u)L(\sigma_0+\mathrm{i}u)-1| > \varepsilon\Big) = \mathrm{o}(1).$$

Proof. We deduce this from Lemma 4.2 and a Sobolev inequality argument. Note that by (1), we have

$$\max_{|t-u|\le 1}|LM(\sigma_0+\mathrm{i}u)-1|^2 \ll |LM(\sigma_0+\mathrm{i}(t+1))-1|^2+|LM(\sigma_0+\mathrm{i}(t-1))-1|^2$$
$$+\int_{t-1}^{t+1}|LM(\sigma_0+\mathrm{i}v)-1||(L'M+LM')(\sigma_0+\mathrm{i}v)|\mathrm{d}v.$$

Ignoring the end cases $t \in [T, T+1]$ or $t \in [2T-1, 2T]$, by Chebyshev's inequality the probability we want to bound is (using the above estimate)

$$\ll \frac{1}{T} + \frac{1}{\varepsilon^2 T} \int_T^{2T} \big(|LM(\sigma_0 + \mathrm{i}(t+1)) - 1|^2 + |LM(\sigma_0 + \mathrm{i}t) - 1||(L'M + LM')(\sigma_0 + \mathrm{i}t)|\big) \mathrm{d}t.$$

Applying the Cauchy-Schwarz inequality and Lemma 4.2 this is

$$\ll \frac{1}{\varepsilon^2 (\log T)^{100}} + \frac{1}{\varepsilon^2 (\log T)^{50}} \Big(\frac{1}{T} \int_T^{2T} \big(|L'M|^2 + |LM'|^2\big)(\sigma_0 + \mathrm{i}t) \mathrm{d}t\Big)^{\frac{1}{2}}.$$

We use the Cauchy-Schwarz inequality once again to bound that term by

$$\ll \Big(\frac{1}{T} \int_T^{2T} (|L|^4 + |L'|^4)(\sigma_0 + \mathrm{i}t) \mathrm{d}t\Big)^{\frac{1}{4}} \Big(\frac{1}{T} \int_T^{2T} (|M|^4 + |M'|^4)(\sigma_0 + \mathrm{i}t) \mathrm{d}t\Big)^{\frac{1}{4}},$$

Since M^2, M'^2 are Dirichlet polynomials of length $X^{2\nu} \ll_\eta T^\eta$ for all $\eta > 0$ and coefficients $O(\log^2 T)$ the mean value theorem for Dirichlet polynomials, Lemma 3.3 above, gives us that the second term is $\ll \log^2 T$.

Since we need only estimates and not asymptotics for the fourth momentum of L, L', we can use the method of proof for Theorem D in [4] to show that for $1/2 \le \sigma \le 1$ we have (uniformly in σ):

$$\int_T^{2T} |L(\sigma + it)|^4 dt \ll T \log^4 T \tag{26}$$

$$\int_T^{2T} |L'(\sigma + it)|^4 dt \ll T \log^8 T \tag{27}$$

For L the proof of Theorem D in [4] translates directly using the Approximate Functional Equation for L, (32) below, since everything is done by direct majorization which works the same for L as for ζ, while for L' we use its Approximate Functional Equation, (40) below, and we majorize all extra $\log n, \log s$ terms by $\log T$ and the method of proof in [4] applies then too. This completes the proof. □

Note: we believe that following carefully the methods of Conrey, [3], one can actually prove asymptotics for the fourth momentum of L, L' using the respective Approximate Functional Equations below.

4.3. **Step 3.** The last stage in our proof involves connecting $\log |M(\sigma_0 + \mathrm{i}t)|$ (for most t) with (close relatives) of the Dirichlet polynomials over primes $P_j(t)$. For $0 \le j \le K-2$, define the Dirichlet polynomials

$$\mathcal{P}_j(t) = \sum_{n \in J_j} \frac{\Lambda(n)\chi(n)}{n^{\sigma_0 + \mathrm{i}t} \log n}, \qquad \text{and} \qquad \widetilde{P}_j(t) = \sum_{p \in J_j} \frac{\chi(p)}{p^{\sigma_0 + \mathrm{i}t}}. \tag{28}$$

Note that $P_j(t)$ is simply the real part of $\widetilde{P}_j(t)$, and the difference between $\mathcal{P}_j$ and $\widetilde{P}_j$ is only in the prime powers; estimating the contribution of prime cubes and larger powers trivially we see that

$$Q(t) = \sum_{j=0}^{K-2}(\mathcal{P}_j(t) - \widetilde{P}_j(t)) = \frac{1}{2}\sum_{p\le\sqrt{X}}\frac{\chi(p^2)}{p^{2\sigma_0+2\mathrm{i}t}} + \mathrm{O}(1). \tag{29}$$

Our goal is to show that for most t one has $\max_{|t-u|\le 1}|M(\sigma_0+\mathrm{i}u) - \exp(-\sum_{j=0}^{K-2}\mathcal{P}_j(u))|$ is small, and we begin with the following preliminary lemma.

Lemma 4.5. *With notation as above,*

$$\mathbb{P}\Big(\max_{|t-u|\le 1}|Q(u)| \ge \log\log\log T\Big) = \mathrm{o}(1),$$

and

$$\mathbb{P}\Big(\max_{|t-u|\le 1}\max_{0\le j\le K-2}|\widetilde{P}_j(u)| \ge 10K^{-\frac{1}{2}}\log\log T\Big) = \mathrm{o}(1).$$

The proof is identical to the proof of Lemma 4.4 in the main paper [2] since everything is done by majorization and $|\chi(n)|\le 1$.

We are ready to connect $M(\sigma_0+\mathrm{i}t)$ with $\exp(-\sum_{j=0}^{K-3}\mathcal{P}_j(t))$ for most values of t.

Lemma 4.6. *We have*

$$\mathbb{P}\Big(\max_{|t-u|\le 1}\Big|M(\sigma_0+\mathrm{i}u) - \exp\Big(-\sum_{j=0}^{K-2}\mathcal{P}_j(u)\Big)\Big| > (\log T)^{-2}\Big) = \mathrm{o}(1)\ .$$

Proof. Recalling that $\nu = 100K\log\log T$, we define the truncated exponential

$$\mathcal{M}(t) = \sum_{k\le\nu}\frac{(-1)^k}{k!}\Big(\sum_{j=0}^{K-2}\mathcal{P}_j(t)\Big)^k. \tag{30}$$

By Lemma 4.5, we know that with probability $1+\mathrm{o}(1)$ (in t) one has

$$\max_{|t-u|\le 1}\Big|\sum_{j=0}^{K-2}\mathcal{P}_j(u)\Big| \le \max_{|t-u|\le 1}\Big(|Q(u)| + \sum_{j=0}^{K-2}|\widetilde{P}_j(u)|\Big) \le 10K\log\log T.$$

For such a typical t, one has

$$\max_{|u-t|\le 1}\Big|\mathcal{M}(u) - \exp\Big(-\sum_{j=0}^{K-2}\mathcal{P}_j(u)\Big)\Big| \le \sum_{k>\nu}\frac{1}{k!}(10K\log\log T)^k \ll (\log T)^{-100}.$$

Therefore, the lemma would follow once we establish that

$$\mathbb{P}\Big(\max_{|t-u|\le 1}|M(\sigma_0+\mathrm{i}u) - \mathcal{M}(u)| > (\log T)^{-3}\Big) = \mathrm{o}(1)\ . \tag{31}$$

Since by definition

$$\exp(-\sum_{p|n\implies p\le X}\frac{\Lambda(n)\chi(n)}{n^{\sigma_0+\mathrm{i}t}\log n}) = \Pi_{p\le X}(1-\frac{\chi(p)}{p^{\sigma_0+it}})$$

the quantities $M(\sigma_0+\mathrm{i}u)$ and $\mathcal{M}(u)$ are almost identical, differing only in a small number of terms. More precisely, if we write $\mathcal{M}(u) = \sum_n b(n)n^{-\sigma_0-\mathrm{i}u}$, it follows that (i) $|b(n)|\le 1$

always, (ii) $b(n) = 0$ unless $n \le X^\nu$ is composed only of primes p below X with $(p,q) = 1$, and (iii) $b(n) = \mu(n)a(n)\chi(n)$ unless $\Omega(n) > \nu$, or if there is a prime $p \le X, (p,q) = 1$ such that $p^k | n$ with $p^k > X$. Therefore, an application of Lemma 3.3 gives

$$\mathbb{E}[|M(\sigma_0 + \mathrm{i}t) - \mathcal{M}(t)|^2] \ll \sum_{\substack{p|n \implies p \le X \\ \Omega(n) > \nu}} \frac{1}{n} + \Big(\sum_{\substack{p \le X \\ p^k > X}} \frac{1}{p^k} \Big) \Big(\sum_{p|n \implies p \le X} \frac{1}{n} \Big).$$

The second term above is $\ll (\log X)/\sqrt{X} \ll (\log T)^{-100}$. Since $e^{(\Omega(n) - \nu)/2}$ is ≥ 1 when $\Omega(n) > \nu$, and is positive for all other n, we may bound the first term above by

$$e^{-\nu/2} \sum_{p|n \implies p \le X} \frac{e^{\Omega(n)/2}}{n} \ll (\log T)^{-50K} \prod_{p \le X} \Big(1 + \sum_{j=1}^{\infty} \frac{e^{j/2}}{p^j} \Big) \ll (\log T)^{-50}.$$

We conclude that

$$\mathbb{E}[|M(\sigma_0 + \mathrm{i}t) - \mathcal{M}(t)|^2] \ll (\log T)^{-50}.$$

A simple application of Lemma 3.3 also shows that $\mathbb{E}[|M'(\sigma_0 + \mathrm{i}t)|^2]$ and $\mathbb{E}[|\mathcal{M}'(t)|^2]$ are $\ll (\log T)^3$. The estimate (31) follows as in Lemmas 4.5 and 4.6 by a successive application of the Sobolev inequality, Chebyshev's inequality and the Cauchy-Schwarz inequality, proving the lemma. □

4.4. **Finishing the proof of Proposition 3.1.** From Lemma 4.1 we obtain for any $V \ge 2$

$$\mathbb{P}(\max_{|t-u| \le 1} |L(\tfrac{1}{2} + \mathrm{i}u)| \ge V) \ge \mathbb{P}(\max_{|t-u| \le \frac{1}{4}} |L(\sigma_0 + \mathrm{i}u)| \ge 2V) + \mathrm{o}(1).$$

By Lemma 4.4 this quantity is

$$\ge \mathbb{P}(\max_{|t-u| \le \frac{1}{4}} |M(\sigma_0 + \mathrm{i}u)|^{-1} \ge 4V) + \mathrm{o}(1),$$

and by Lemma 4.6 the above is

$$\ge \mathbb{P}\Big(\max_{|t-u| \le \frac{1}{4}} \sum_{j=0}^{K-2} \mathrm{Re}\ \mathcal{P}_j(u) \ge \log(8V) \Big) + \mathrm{o}(1).$$

Invoking Lemma 4.5, we may replace $\mathrm{Re}\mathcal{P}_j(u)$ by $P_j(u)$ with the appropriate error, and also discard the terms with $j = 0$ and $j = K-2$: thus, the quantity above is

$$\ge \mathbb{P}\Big(\max_{|t-u| \le \frac{1}{4}} \sum_{j=1}^{K-3} P_j(u) \ge \log(8V) + \log\log\log T + 20K^{-\frac{1}{2}} \log\log T \Big) + \mathrm{o}(1).$$

Taking $V = (\log T)^{1-2\varepsilon}$, the proposition follows.

Proof of Selberg Lemma for L Functions: Prop 4.3

Lemma 4.7. *Approximate Functional Equation for the L Function*

For $s=\sigma+it$, A a positive constant, $0<\sigma<1$, $t>100$, $2\pi xy=t$, $x>A, y>A$, χ a primitive character of modulus q, and $L=L_\chi$, we have

$$L(s)=\sum_{n\le qx}\frac{\chi(n)}{n^s}+c(s)\sum_{n\le y}\frac{\overline{\chi}(n)}{n^{1-s}}+O_{A,q}(x^{-\sigma}\log t+t^{\frac12-\sigma}y^{\sigma-1}+y^{\frac12}x^{-\frac12-\sigma}) \tag{32}$$

where

$$c(s)=\chi(-1)G_\chi(1)e^{\pi i(1-s)/2}\Gamma(1-s)\frac{q^{-s}}{(2\pi)^{1-s}}$$
$$|c(s)|=\left(\frac{qt}{2\pi}\right)^{1/2-\sigma}\Big(1+O(1/t)\Big)$$

and $G_\chi(m)$ is the Gauss sum associated with χ,

$$G_\chi(m)=\sum_{n=1}^{q}\chi(n)e^{2\pi inm/q}$$

Proof. Recall that χ primitive implies that $G_\chi(m)=\overline{\chi}(m)G_\chi(1)$. Hence

$$\sum_{n=1}^{q}\chi(n)e^{-2\pi imn/q}=\chi(-1)\sum_{n=1}^{q}\chi(-n)e^{-2\pi imn/q}=\chi(-1)\overline{\chi}(m)G_\chi(1) \tag{33}$$

Let N much larger than t. From (17) we have

$$L(s)=\sum_{n\le qN}\frac{\chi(n)}{n^s}+O(N^{-\sigma})=q^{-s}\sum_{h=1}^{q}\chi(h)\sum_{n\le N}\frac{1}{((n+h)/q)^s}+O(N^{-\sigma})$$

Now for $0<a\le 1$, Lemma 4.10 from [11] shows that

$$\sum_{x<n\le N}\frac{1}{(n+a)^s}=\sum_{\frac{t}{2\pi(N+a)}-\frac12<m\le\frac{t}{2\pi(x+a)}+\frac12}\int_x^N\frac{e^{2\pi imu}}{(u+a)^s}du+O(x^{-\sigma}\log(t/x+2)) \tag{34}$$

Since $\frac{t}{2\pi(N+a)}$ is small, the first integral is for $m=0$ hence it is

$$\frac{(N+a)^{1-s}-(x+a)^{1-s}}{1-s}$$

But

$$\sum_{h=1}^{q}\chi(h)\frac{(N+h/q)^{1-s}-(x+h/q)^{1-s}}{1-s}=O(x^{-\sigma})$$

since

$$\frac{(x+a)^{1-s}-x^{1-s}}{1-s}=O(x^{-\sigma}),\text{and}\sum_{h=1}^{q}\chi(h)=0$$

Now for $0<\sigma<1$ and t positive,

$$\int_{-a}^{\infty}\frac{e^{2\pi imu}}{(u+a)^s}du=e^{-2\pi ima}\int_0^\infty\frac{e^{2\pi imv}}{v^s}dv=e^{-2\pi ima}\Gamma(1-s)\frac{e^{\pi i(1-s)/2}}{(2\pi m)^{1-s}} \tag{35}$$

which is well known and easy to prove with residues.

Since $m \geq 1$, $m - \frac{t}{2\pi u} \geq \frac{3}{4}, u \geq N$ so

$$\int_N^\infty e^{i(2\pi mu - t\log(u+a))}du = O(1/m)$$

by the first derivative test for integrals (Lemma 4.2 in [11]) hence the upper tail of (35) satisfies:

$$\int_N^\infty \frac{e^{2\pi imu}}{(u+a)^s}du = O(N^{-\sigma}/m) \tag{36}$$

We have

$$\int_{-a}^x \frac{e^{2\pi imu}}{(u+a)^s}du = [\frac{(u+a)^{1-s}}{1-s}e^{2\pi imu}]_{-a}^x - \frac{2\pi im}{1-s}\int_{-a}^x (u+a)^{1-s}e^{2\pi imu}du \tag{37}$$

We now recall that in the sum of integrals in (34) we have $m \leq \frac{t}{2\pi(x+a)} + \frac{1}{2}$

If we restrict only to $m \leq \frac{t}{2\pi(x+a)} - \frac{1}{2}$ we can again estimate the integral above by the first derivative test for integrals since $t/(2\pi(x+a) - m \geq 1/2$ so the lower tail of (35) satisfies:

$$\int_{-a}^x \frac{e^{2\pi imu}}{(u+a)^s}du = O(\frac{x^{1-\sigma}}{t}) + O(\frac{mx^{1-\sigma}}{t(m - t/(2\pi(x+a))})) \tag{38}$$

For the unique $\frac{t}{2\pi(x+a)} - \frac{1}{2} < m \leq \frac{t}{2\pi(x+a)} + \frac{1}{2}$ so $m \approx y$ we can use the second derivative test for integrals (Lemma 4.5 in [11]) and obtain a lower tail error

$$O(\frac{x^{1-\sigma}}{t}) + O(\frac{yx^{1-\sigma}}{t}(t/x^2)^{-1/2}) = O(x^{1-\sigma}t^{-1/2}) = O(t^{1/2-\sigma}y^{\sigma-1})$$

Adding all the errors (which are $\approx y$ in number) and remembering that N is very large so the upper tail errors are smaller than the lower tail ones, while $t \approx xy$, we have:

$$\sum_{x<n\leq N} \frac{1}{(n+a)^s} = \Gamma(1-s)\frac{e^{\pi i(1-s)/2}}{(2\pi)^{1-s}} \sum_{1\leq m\leq \frac{t}{2\pi(x+a)}+\frac{1}{2}} \frac{e^{-2\pi ima}}{m^{1-s}} + O(x^{-\sigma}\log t + t^{1/2-\sigma}y^{\sigma-1}) \tag{39}$$

Using that

$$|\Gamma(1-s)\frac{e^{\pi i(1-s)/2}}{(2\pi)^{1-s}}| \approx t^{1/2-\sigma}$$

we can replace $\sum_{1\leq m\leq \frac{t}{2\pi(x+a)}+\frac{1}{2}} \frac{e^{-2\pi ima}}{m^{1-s}}$ by $\sum_{1\leq m\leq y} \frac{e^{-2\pi ima}}{m^{1-s}}$ with an error of

$$O((t/x^2)t^{1/2-\sigma}y^{\sigma-1}) = O(y^{1/2}x^{-1/2-\sigma})$$

since there are clearly at most $\approx t/x^2$ extra terms in the second sum, each with $m \approx y$

But now taking $a = h/q, h = 1, \ldots, q$ in the relation (39) where we replace all summations to be up to y as above and summing on h, relation (33) gives us precisely equation (32), so the main part of Lemma 4.7 is proved, while remembering that $|G_\chi(1)| = \sqrt{q}$ gives the claimed asymptotic for $|c(s)|$

□

Lemma 4.8. *Approximate Functional Equation for the Derivative of the L Function*

For large $T > 0$ and $s = \sigma + it, t \approx T, 0 < \sigma < 1$, $2\pi xy = t, x \approx \sqrt{T}, y \approx \sqrt{T}$, χ a primitive character of modulus q, and $L = L_\chi$, we have

$$L'(s) = \sum_{n \le qx} -\frac{\chi(n)\log n}{n^s} - c(s)\log\frac{qs}{2\pi i}\sum_{n\le y}\frac{\overline{\chi}(n)}{n^{1-s}} + c(s)\sum_{n\le y}\frac{\overline{\chi}(n)\log n}{n^{1-s}} + O(T^{-\sigma/2}\log^2 T) \tag{40}$$

Proof. (32) gives us that

$$L(s) = \sum_{n\le qx}\frac{\chi(n)}{n^s} + c(s)\sum_{n\le y}\frac{\overline{\chi}(n)}{n^{1-s}} + f(s)$$

where $f(s)$ is analytic and satisfies $f(s) = O(T^{-\sigma/2}\log T)$ by our choices for t, x, y. Since $L'(s) = \frac{\partial L}{\partial\sigma}(\sigma + it)$ we can keep t, x, y constant (so the sums do not change since their length depends only on t, x, y) and differentiate each sum term by term, noting that $(\log c(s))' = \frac{c'(s)}{c(s)} = -\log\frac{qs}{2\pi i} + O(\frac{1}{T})$ by the Stirling approximation, so we obtain the main three terms above.

Applying Cauchy to a small circle around s of radius $\frac{1}{\log T}$ we get that

$$f'(s) = O(T^{-\sigma/2}\log^2 T\int_0^{2\pi} T^{-\frac{\cos\theta}{2\log T}}d\theta) = O(T^{-\sigma/2}\log^2 T)$$

□

Let's now prove our result (Proposition 4.3) following the method of Selberg with the simplifications allowed by our assumption that $h, k \le T^\varepsilon$

For any fixed $\varepsilon > 0$ and $1 \le h, k \le T^\varepsilon$, $(h, q) = (k, q) = 1$ and $1/2 < \sigma \le 1$, we have

$$\int_T^{2T}\chi(h)\bar{\chi}(k)\Big(\frac{h}{k}\Big)^{\mathrm{it}}|L(\sigma + \mathrm{it})|^2\,\mathrm{dt} =$$

$$\int_T^{2T}(L_{0,q}(2\sigma)\Big(\frac{(h,k)^2}{hk}\Big)^{\sigma} + \Big(\frac{qt}{2\pi}\Big)^{1-2\sigma}L_{0,q}(2-2\sigma)\Big(\frac{(h,k)^2}{hk}\Big)^{1-\sigma}\mathrm{dt} + \mathrm{O}(T^{1-\sigma/2+4\varepsilon})$$

Proof. Wlog we assume $(h, k) = 1$ also since if $h = dh_1, k = dk_1, h/k = h_1/k_1, \chi(h)\bar{\chi}(k) = \chi(h_1)\bar{\chi}(k_1), \frac{d^2}{hk} = \frac{1}{h_1k_1}$ so the result for (h, k) follows from the one for (h_1, k_1).

Let $\tau = \sqrt{\frac{t}{2\pi}}$. We will apply relation (32) for $x = \tau\sqrt{\frac{h}{k}}, y = \tau\sqrt{\frac{k}{h}}$ and for $x = \tau\sqrt{\frac{k}{h}}, y = \tau\sqrt{\frac{h}{k}}$ and we notice that in both cases the error in (32) is $O(T^{-\sigma/2+\varepsilon})$. We use (32) directly with $x = \tau\sqrt{\frac{h}{k}}$ and conjugate with $x = \tau\sqrt{\frac{k}{h}}$, so

$$L(s) = \sum_{n\le q\tau\sqrt{\frac{h}{k}}}\frac{\chi(n)}{n^s} + c(s)\sum_{n\le\tau\sqrt{\frac{k}{h}}}\frac{\overline{\chi}(n)}{n^{1-s}} + O(T^{-\sigma/2+\varepsilon}) \tag{41}$$

$$\overline{L(s)} = \sum_{n\le q\tau\sqrt{\frac{k}{h}}} \frac{\overline{\chi}(n)}{n^{\sigma-it}} + \overline{c(s)} \sum_{n\le \tau\sqrt{\frac{h}{k}}} \frac{\chi(n)}{n^{1-\sigma+it}} + O(T^{-\sigma/2+\varepsilon}) \tag{42}$$

Using the standard majorization (Lemma 1 in [9]): for $0\le\sigma\le 1, R\ge 2$

$$\sum_{1\le n<m\le R} \frac{1}{(mn)^\sigma \log\frac{m}{n}} = O(R^{2-2\sigma}\log^2 R) \tag{43}$$

we immediately see that (here $R = O(T^{1/2+\epsilon}), \sigma > 1/2$)

$$(\int_T^{2T} |\sum_{n\le q\tau\sqrt{\frac{h}{k}}} \frac{\chi(n)}{n^s}|dt)^2 \le T\int_T^{2T} |\sum_{n\le q\tau\sqrt{\frac{h}{k}}} \frac{\chi(n)}{n^s}|^2 dt = O(T^2\log T) \tag{44}$$

and similarly for the other expression with $x = \tau\sqrt{\frac{k}{h}}$

Similarly using $|c(s)| = O(T^{1/2-\sigma})$ we get

$$(\int_T^{2T} |c(s) \sum_{n\le \tau\sqrt{\frac{k}{h}}} \frac{\overline{\chi}(n)}{n^{1-s}}|dt)^2 \le T^{2-2\sigma}\int_T^{2T} |\sum_{n\le \tau\sqrt{\frac{k}{h}}} \frac{\overline{\chi}(n)}{n^{1-s}}|^2 dt = O(T^2\log T) \tag{45}$$

So using the expansion $|L(s)|^2 = L(s)\overline{L(s)}$ given by (41) and (42) respectively and $|\chi(h)\overline{\chi}(k)\left(\frac{h}{k}\right)^{\text{it}}| = 1$ we get that all terms that have $O(T^{-\sigma/2+\varepsilon})$ as at least one factor are $O(T^{1-\sigma/2+2\varepsilon})$

Next we deal with

$$P = \int_T^{2T} \chi(h)\overline{\chi}(k)\left(\frac{h}{k}\right)^{\text{it}} (\sum_{n\le q\tau\sqrt{\frac{h}{k}}} \frac{\chi(n)}{n^{\sigma+it}})(\sum_{m\le q\tau\sqrt{\frac{k}{h}}} \frac{\overline{\chi}(m)}{m^{\sigma-it}})dt$$

We note that each off-diagonal term where $hm \ne kn$ is

$$O(\frac{1}{(mn)^\sigma|\log\frac{mh}{nk}|}) = O(T^{2\sigma\varepsilon}\frac{1}{(mhnk)^\sigma|\log\frac{mh}{nk}|})$$

So renaming $mh = M, nk = N << \sqrt{Thk} << T^{1/2+\varepsilon}$ and noting that each M, N is uniquely determined by m, n respectively, another application of (43) shows that the sum of the off-diagonal terms is $O(T^{(1/2+\varepsilon)(2-2\sigma)+2\sigma\varepsilon}\log^2 T) = O(T^{1-\sigma+3\varepsilon})$

Since $(h,k) = 1$, the diagonal part $hm = kn$ is parametrized by $m = kM, n = hM, M \le q\left(\frac{t}{2\pi hk}\right)^{1/2}$ and of course the crucial $(M,q) = 1$ due to the fact that $\chi(m) = \chi(n) = 0$ if $(M,q)\ne 1$. Hence we get that

$$P = \int_T^{2T} \sum_{1\le M\le q\left(\frac{t}{2\pi hk}\right)^{1/2}} \frac{\chi_{0,q}(M)(hk)^{-\sigma}}{M^{2\sigma}}dt + O(T^{1-\sigma+3\varepsilon}) \tag{46}$$

But using the method of proof for the equation (4) and noting that for χ_0 we have $\sum_{h=1}^{q}\chi_{0,q}(h)=\phi(q)$ we obtain that for any $R\geq 1,\sigma\neq 1,1/4\leq\sigma\leq 2$ we have

$$L_{0,q}(\sigma)=\sum_{n\leq qR}\frac{\chi_{0,q}(n)}{n^{\sigma}}-\frac{\phi(q)R^{1-\sigma}}{1-\sigma}+\mathrm{O}(R^{-\sigma}). \tag{47}$$

Substituting the above with $R=\left(\frac{t}{2\pi hk}\right)^{1/2},\sigma\to 2\sigma$ in (46) we get

$$P=\int_{T}^{2T}((hk)^{-\sigma}L_{0,q}(2\sigma)-(hk)^{-1/2}\frac{\phi(q)\left(\frac{t}{2\pi}\right)^{1/2-\sigma}}{1-2\sigma}dt+O(T^{1-\sigma/2}+T^{1-\sigma+3\varepsilon}) \tag{48}$$

Since $|c(s)|=\left(\frac{qt}{2\pi}\right)^{1/2-\sigma}\Big(1+O(1/t)\Big)$ it is clear that in dealing with the second main term

$$P_1=\int_{T}^{2T}|c(s)|^2\chi(h)\overline{\chi}(k)\left(\frac{h}{k}\right)^{it}(\sum_{n\leq\tau\sqrt{\frac{h}{k}}}\frac{\chi(n)}{n^{1-\sigma+it}})(\sum_{m\leq\tau\sqrt{\frac{k}{h}}}\frac{\overline{\chi}(m)}{m^{1-\sigma-it}})dt$$

we can replace $|c(s)|$ with $\left(\frac{qt}{2\pi}\right)^{1/2-\sigma}$ and majorize the error trivially using (44)

For the off-diagonal terms we need to use the second Mean Value Theorem for integrals applied to the real and imaginary part respectively and note that for $\lambda\neq 0$ real and $1/2\leq\sigma\leq 1$ we have

$$\int_{T}^{2T}t^{1-2\sigma}e^{i\lambda t}dt=O(\frac{T^{1-2\sigma}}{|\lambda|}) \tag{49}$$

and then as above for P (with $\sigma\to 1-\sigma$) we get that the sum of the off diagonal terms is

$$O(T^{1-2\sigma}T^{\sigma+3\varepsilon})=O(T^{1-\sigma+3\varepsilon})$$

The diagonal main term is parametrized again by

$$m=kM,n=hM,M\leq\left(\frac{t}{2\pi hk}\right)^{1/2},(M,q)=1$$

and we get that

$$P_1=\int_{T}^{2T}\left(\frac{qt}{2\pi}\right)^{1-2\sigma}\sum_{1\leq M\leq\left(\frac{t}{2\pi hk}\right)^{1/2}}\frac{\chi_{0,q}(M)(hk)^{\sigma-1}}{M^{2-2\sigma}}dt+O(T^{1-\sigma+3\varepsilon}) \tag{50}$$

We apply again (47) but this time with $2-2\sigma$ and $R=\frac{1}{q}\left(\frac{t}{2\pi hk}\right)^{1/2}$ and we notice that the error now is $O(T^{1-2\sigma}T^{\sigma-1/2})=O(T^{1/2-\sigma})$, while in the oscillating term $\frac{\phi(q)R^{2\sigma-1}}{2\sigma-1}$ the new q term from R precisely cancels with the $q^{1-2\sigma}$ from $c(s)$, hk again appear at power $\sigma-1-\sigma+1/2=1/2$, $\frac{t}{2\pi}$ also appears again at power $1-2\sigma+\sigma-1/2=1/2-\sigma$ and the denominator $2\sigma-1$ is precisely the negative of $1-2\sigma$ from P so the two oscillating terms cancel out! Hence:

$$P+P_1=\int_T^{2T}(L_{0,q}(2\sigma)(hk)^{-\sigma}+\Big(\frac{qt}{2\pi}\Big)^{1-2\sigma}L_{0,q}(2-2\sigma)(hk)^{\sigma-1})\mathrm{d}t+\mathrm{O}(T^{1-\sigma+3\varepsilon}) \tag{51}$$

It remains to deal with the cross product of the two main terms which is more delicate since it involves the argument of $c(s)$; since up to exchanging h with k, one is the conjugate of the other, we need to deal with only one and will majorize everything by absolute values once we estimate the integral of $\overline{c(s)}(h/mnk)^{it}$. We use again that for $T\le t\le 2T$ we have $c(\sigma+it)=\alpha(\sigma+it)\Big(\frac{qt}{2\pi}\Big)^{1/2-\sigma}\Big(1+O(1/T))\Big)$, $|\alpha(s)|=1$ and estimate the error integral from the $O(1/T)$ using (44) as before.

From Stirling's approximation it follows precisely as 4.12.3 of [11], that up to the arguments of $\chi(-1),G_\chi(1)$ we have that $\alpha(\sigma+it)=\Big(\frac{tq}{2\pi e}\Big)^{-it}$.

If $r=\frac{2\pi mnk}{qh}$, applying the first derivative and the second derivative test respectively we get that for any $T\le T_1\le 2T, r\le T_1$ we have

$$\int_{T_1}^{2T}\Big(\frac{tqh}{2\pi emnk}\Big)^{it}dt=\int_{T_1}^{2T}\Big(\frac{t}{er}\Big)^{it}dt=O(\min(\frac{1}{\log(T_1/r)},\sqrt{T})) \tag{52}$$

Since a pair (n,m) appears in the product

$$(\sum_{n\le q\tau\sqrt{\frac{h}{k}}}\frac{\chi(n)}{n^{\sigma+it}})(\sum_{m\le\tau\sqrt{\frac{h}{k}}}\frac{\chi(m)}{m^{1-\sigma+it}}) \tag{53}$$

only if $\frac{2\pi n^2k}{hq^2}\le t, \frac{2\pi m^2k}{h}\le t$ we first assume $n/q>m$ and call $T_2=\frac{2\pi n^2k}{hq^2}>\frac{2\pi m^2k}{h}$ so $T_2/r=\frac{n}{mq}>1$

If $T_2\le T$ then clearly $T/r\ge\frac{n}{mq}$ and the corresponding (n,m) appears in (53) for all $t\in[T,2T]$ hence (52) with $T_1=T$ gives the bound

$$\int_T^{2T}\Big(\frac{tqh}{2\pi emnk}\Big)^{it}dt=O\Big(\frac{1}{\log(T/r)}\Big)=O\Big(\frac{1}{\log(\frac{n}{mq})}\Big)$$

If $T<T_2\le 2T$ we again apply (52) with $T_1=T_2$ and get the same bound $O\Big(\frac{1}{\log(\frac{n}{mq})}\Big)$ as above.

If $T_2>2T$ then clearly the (n,m) term doesn't appear in our integral as the corresponding $t=T_2$ is too large.

Introducing (either applying the first derivative test to the complete function or using the second MVT for integrals for the real and imaginary parts separately) the monotonic term

$\left(\frac{qt}{2\pi}\right)^{1/2-\sigma}$ in the oscillating integral we get (for $n > mq$) the bound

$$\int_T^{2T} \left(\frac{qt}{2\pi}\right)^{1/2-\sigma} \left(\frac{tqh}{2\pi emnk}\right)^{it} dt = O\left(\frac{T^{1/2-\sigma}}{\log(\frac{n}{mq})}\right) \tag{54}$$

Similarly for $n < mq$ we get the bound

$$\int_T^{2T} \left(\frac{qt}{2\pi}\right)^{1/2-\sigma} \left(\frac{tqh}{2\pi emnk}\right)^{it} dt = O\left(\frac{T^{1/2-\sigma}}{\log(\frac{mq}{n})}\right) \tag{55}$$

For $n = mq$ we use the square root bound from (52) hence we get

$$\int_T^{2T} \left(\frac{qt}{2\pi}\right)^{1/2-\sigma} \left(\frac{th}{2\pi em^2k}\right)^{it} dt = O(T^{1-\sigma}) \tag{56}$$

Putting all together we get that the cross term is

$$O(\sum_{n/q \neq m << T^{1/2+\varepsilon}} n^{-\sigma} m^{\sigma-1} \left(\frac{T^{1/2-\sigma}}{\log(|\frac{mq}{n})|}\right) + \sum_{m << T^{1/2+\varepsilon}} m^{-1} T^{1-\sigma})$$

Since $m^{\sigma-1} = m^{-\sigma} m^{2\sigma-1} << T^{\sigma-1/2+\varepsilon} m^{-\sigma}$ applying (43) (with $mq \to m$) we get

$$\sum_{n \neq mq << T^{1/2+\varepsilon}} n^{-\sigma} m^{\sigma-1} \left(\frac{T^{1/2-\sigma}}{\log(|\frac{mq}{n})|}\right) << T^{1-\sigma+3\varepsilon} \log^2 T << T^{1-\sigma+4\varepsilon}$$

while

$$\sum_{m << T^{1/2+\varepsilon}} m^{-1} T^{1-\sigma}) << T^{1-\sigma} \log T << T^{1-\sigma+\varepsilon}$$

so the cross terms are $<< T^{1-\sigma+4\varepsilon}$

Hence looking at all the errors obtained in the various terms above, we see that they are all at most $O(T^{1-\sigma/2+4\varepsilon})$ and the proposition is finally proved! □

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