Brownian behaviour of the Riemann zeta function around the critical line

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Abstract

We establish a Brownian extension to Selberg's central limit theorem for the Riemann zeta function. As a result, various functional results on the behaviour of Brownian motion imply analogous results for the statistical behaviour of ζ.

In this paper, we use the convention $f \ll g$ to mean $f = O(g)$; if the implied constant depends upon another variable ε, we shall write $f \ll_\varepsilon g$. $x \wedge y$ denotes the minimum of x and y, and $x^+ = \max(x, 0)$ is the positive part of x. $\log \zeta$ is defined in the usual way (see [16] for example).

1 Introduction

1.1 General behaviour of ζ around the critical line

Selberg's central limit theorem, proven in [16], states that

$$\frac{1}{\sqrt{\log \log T}} \log \zeta(\frac{1}{2} + i\tau) \underset{\text{law}}{\longrightarrow} \mathcal{N}(0, 1) \tag{1.1}$$

as $T \to +\infty$, where here τ is chosen uniformly at random from $[0, T]$. A simpler proof (for $\log |\zeta|$) can be found in [15].

This theorem was the first major result on the statistical behaviour of ζ around the critical line; subsequently, a lot of interest has formed around this problem, which is today a fundamental question in probabilistic number theory. This section is a quick summary of some relevant results; for a more exhaustive survey, see [17].

Another fundamental result in this regard is Montgomery's pair correlation theorem [11] giving some information on the correlation of the positions of consecutive zeroes of ζ; assuming the Riemann Hypothesis, in a certain (very restrictive) sense, it converges to the same limit distribution as the pair correlation for eigenvalues of a random unitary matrix. Extending this to bona fide weak convergence is currently an important open problem, and this phenomenon remains quite poorly understood.

This result suggests the existence of a link between the theory of ζ and that of random matrices, and much work has since been done to expand upon this link. For instance, in [1] [2], Louis-Pierre Arguin et al. showed the so-called FHK conjecture [6] [7] describing the asymptotic behaviour of $\max_{|h| \leq 1} |\zeta(\frac{1}{2} + i\tau + ih)|$. In fact, most of the following theorems in this paper have analoguous results, either proven or conjectured, in the theory of random unitary matrices; a more extensive summary of this connection can be found in the survey [4].

Another possible direction (which is largely related) is to try and directly generalise Selberg's result. Tsang [18] extended Selberg's work in his thesis, showing that

$$\frac{1}{\sqrt{\alpha \log\log T}} \log\zeta(\frac{1}{2} + \frac{1}{(\log T)^{\alpha}} + i\tau) \underset{\text{law}}{\longrightarrow} \mathcal{N}(0,1) \tag{1.2}$$

for $\alpha \in [0,1]$. Recently, there has been more work around finding multidimesional extensions of the theorem ie. to understand the joint behaviour of

$$\log\zeta(\frac{1}{2} + i\tau_1), \ldots, \log\zeta(\frac{1}{2} + i\tau_n) \tag{1.3}$$

for some (possibly complex) random shifts $\tau_1, \tau_2, \ldots, \tau_n$.

The "microscopic" behaviour, when η_i are of order $\frac{1}{\log T}$, is closely linked to the pair correlation problem mentioned above, and no results have yet been shown. However, when τ_i become larger, there has been much work on this problem. In [9], Hughes et al. identified a "macroscopic" scale at which the values of $\log\zeta$ are uncorrelated: namely, where τ_i are of order $\exp((\log T)^{\lambda_i})$ for some $0 < \lambda_1 < \ldots < \lambda_n$. A natural question to ask is thus whether we can identify a "mesoscopic" scale where the τ_i are closer together and non-trivial correlations appear. An answer to this question (in the case of vertical shifts) was provided by Bourgade in [5]:

Theorem 1. *Consider functions $f_1, \ldots, f_n : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that:*

- *f_i are bounded and, if $i \neq j$,*

$$\frac{\log|f_i(t) - f_j(t)|}{\sqrt{\log\log T}} \longrightarrow c_{i,j} \in [0, +\infty] \tag{1.4}$$

Then, for $T > 0$, if τ is a uniform random variable on $[0,T]$,

$$\frac{1}{\sqrt{\log\log T}} \left(\log\zeta(\frac{1}{2} + i\tau + if_1(T)), \ldots, \log\zeta(\frac{1}{2} + i\tau + if_n(T))\right) \tag{1.5}$$

converges in law to a complex centred Gaussian vector $(Y_1, \ldots, Y_n)$, with covariances

$$\mathrm{Cov}(Y_i, Y_j) = \begin{cases} 1 & \text{if } i = j \\ 1 \wedge c_{i,j} & \text{if } i \neq j \end{cases} \tag{1.6}$$

This theorem shows that non-trivial mesoscopic behaviour appears at the scale $f_l(t) = \frac{1}{(\log T)^{\alpha_l}}$, for some $0 \leq \alpha_1, \ldots, \alpha_n \leq 1$. In fact, this theorem can be interpreted as a finite-dimensional convergence of the random function

$$\alpha \in [0,1] \mapsto \log\zeta(\frac{1}{2} + i\tau + \frac{i}{(\log T)^{\alpha}}) \tag{1.7}$$

towards the Gaussian process

$$f(\alpha) = \mathcal{B}_\alpha + \mathcal{D}_{1-\alpha} \tag{1.8}$$

with $\mathcal{B}$ a Brownian motion, and $\mathcal{D}_\alpha$ a process whose coordinates are all independant centred normal variables with variance α. However, this limit process is not measurable: in fact, any Gaussian process with the covariances from (1.6) is necessarily non-measurable (this is shown in [5]). As a result, we cannot hope to have convergence in distribution for this sequence of functions.

Bourgade's methods also allow us to state a similar result for horizontal correlations:

Theorem 2. *Let $0 \leq \alpha_1, \ldots, \alpha_n$: for $T > 0$, if τ is a uniform random variable on $[0, T]$,*

$$\frac{1}{\sqrt{\log\log T}} \left(\log\zeta(\frac{1}{2} + \frac{1}{(\log T)^{\alpha_1}} + i\tau), \ldots, \log\zeta(\frac{1}{2} + \frac{1}{(\log T)^{\alpha_n}} + i\tau) \right) \tag{1.9}$$

converges in law to a complex centred Gaussian vector $(Y_1, \ldots, Y_n)$, with covariances

$$\mathrm{Cov}(Y_i, Y_j) = 1 \wedge \alpha_i \wedge \alpha_j \tag{1.10}$$

Here, we have made the f_i explicit for clarity. We will prove Theorem 2 later on.

1.2 Main theorem

The covariance structure given in Theorem 2 is more stable than that of the vertical case: it follows the same law as Brownian motion. As a result, it now actually seems plausible that our sequence of functions might converge in distribution to Brownian motion; this is the object of the main theorem of this paper.

Theorem 3. *If $T > 1$, define*

$$Z^{(T)} : \begin{array}{ccc} [0,1] & \longrightarrow & \mathbb{C} \\ \alpha & \longmapsto & \dfrac{1}{\sqrt{\log\log T}} \log\zeta(\frac{1}{2} + \dfrac{1}{(\log T)^\alpha} + i\tau) \end{array}$$

where τ is a uniform random variable over $[0, T]$. Then, as $T \to +\infty$, $Z^{(T)}$ converges in law to complex Brownian motion in $\mathcal{C}^0([0,1], \mathbb{C})$.

Otherwise stated, if $F : \mathcal{C}^0 \to \mathbb{R}$ is a continuous bounded functional, then

$$\mathbb{E}[F(Z^{(T)})] \underset{T\to+\infty}{\longrightarrow} \mathbb{E}[F(B)] \tag{1.11}$$

where B is a (complex) Brownian motion.

The theorem is stated for $\alpha \leq 1$, simply because the process doesn't exhibit any interesting behaviour beyond this point: if we wish to take the process with $\alpha \in [0, +\infty[$, the limit process would follow the law of

$$\tilde{\mathcal{B}}_\alpha = \begin{cases} \mathcal{B}_\alpha & \text{if } \alpha \leq 1 \\ \mathcal{B}_1 & \text{if } \alpha \geq 1 \end{cases} \tag{1.12}$$

Nevertheless, even allowing $\alpha \in [0, +\infty[$, we have convergence in distribution of the sequence of functions in $\mathcal{C}^0([0, +\infty[, \mathbb{C})$ (for the topology of uniform convergence). The proof remains largely the same, beyond some minor changes to the proof of Theorem 4.

1.3 Possible generalisations

1.3.1 A dual result in random matrix theory

There is an analogous result to Theorem 1 in the theory of random unitary matrices, also shown in [5] (and the methods there also allow us to prove something analogous to Theorem 2). One might wonder whether we can also generalise this to a functional convergence towards Brownian motion, similar to Theorem 3. More specifically, if U_n is a random $n-$dimensional unitary matrix, distributed according to the uniform Haar measure $\mu_{U(n)}$, the relevant function to consider would be

$$Z^{(T)} : \begin{array}{ccc} [0,1] & \longrightarrow & \mathbb{C} \\ \alpha & \longmapsto & \dfrac{1}{\sqrt{\log T}} \det(\exp(n^{-\alpha}) I_n - U_n) \end{array}$$

It seems quite likely that this should also converge in distribution to Brownian motion.

1.3.2 Other correlation structures for ζ

One might also wonder whether different correlation structures could appear by having our process vary, not horizontally or vertically, but in any other direction. Sadly, this is not the case.

Proposition 1. *For $T > 0$, let $\varepsilon_T : [0,1] \to \mathbb{R}$ be decreasing functions such that $\varepsilon_T(1) \underset{T\to+\infty}{\longrightarrow} 0$; and let $f_T : [0,1] \to \mathbb{R}$ be monotonous functions. Set*

$$Z^{(T)} : \begin{array}{ccc} [0,1] & \longrightarrow & \mathbb{C} \\ \alpha & \longmapsto & \dfrac{1}{\sqrt{|\log \varepsilon_T(1)|}} \log \zeta(\tfrac{1}{2} + \varepsilon_T(\alpha) + i\tau + i f_T(\alpha)) \end{array}$$

Then, if $Z^{(T)}$ converges in distribution in $C^0([0,1])$, it is towards a process with independent increments.

Proof. Assume that $Z^{(T)}$ converges in distribution to a process X. Then, if $\alpha < \beta$,

$$\mathrm{Cov}(X_\alpha X_\beta) = \lim_{T\to+\infty} \frac{\log \varepsilon_T(\alpha)}{\log \varepsilon_T(1)} \wedge \frac{\log |f_T(\alpha) - f_T(\beta)|}{\log \varepsilon_T(1)} = \varphi(\alpha) \wedge K(\alpha, \beta) \tag{1.13}$$

where $\varphi(\alpha) = \limsup_{T\to+\infty} \dfrac{\log \varepsilon_T(\alpha)}{\log \varepsilon_T(1)}$, and $K(\alpha,\beta) = \limsup_{T\to+\infty} \dfrac{\log |f_T(\alpha) - f_T(\beta)|}{\log \varepsilon_T(1)}$. This follows from the methods in [5]. For the same reason,

$$\mathrm{Var}\, X_\alpha = \varphi(\alpha) \tag{1.14}$$

However, since f_T is monotonous for all T, we can also see that

$$K(\alpha, \beta) = K(\alpha, \frac{\alpha+\beta}{2}) \wedge K(\frac{\alpha+\beta}{2}, \beta) \tag{1.15}$$

and so, repeating this, we can construct $\alpha_n < \beta_n$ converging to some $\gamma \in [\alpha, \beta]$ such that $K(\alpha, \beta) = K(\alpha_n, \beta_n)$. Since X is continuous, K is too and so $K(\alpha,\beta) = \mathrm{Var}\, X_\gamma = \varphi(\gamma)$. Thus, $\mathrm{Cov}(X_\alpha, X_\beta) = \varphi(\alpha)$ and X has independent increments. $\square$

The proof also shows that we cannot expect convergence even in broader spaces like Skorokhod space: if the limit process is even remotely reasonable, it is Brownian motion.

2 Some interesting corollaries

Convergence in law to Brownian motion give us a few interesting corollaries regarding the global behaviour of the Riemann zeta function in the critical strip; to the author's knowledge, these corollaries are novel. Similar corollaries hold if we replace $\log|\zeta|$ with $\operatorname{Im}\log\zeta$.

2.1 Applying the reflection principle

Corollary 1 (Reflection principle)**.** *If $T > 1$, define*

$$S_T = \sup_{\sigma \geq 0} \log|\zeta(\frac{1}{2} + \sigma + i\tau)| \tag{2.1}$$

where τ is a uniform random variable over $[0, T]$. Then, as $T \to +\infty$,

$$\frac{1}{\sqrt{\log\log T}} S_T \underset{law}{\longrightarrow} |\mathcal{N}(0,1)|$$

Proof. It follows from Theorem 3 (over $[0, +\infty[$), and the reflection principle for Brownian motion, that

$$\sup_{\sigma \in]0,1]} \frac{1}{\sqrt{\log\log T}} \log|\zeta(\frac{1}{2} + \sigma + i\tau)| \underset{law}{\longrightarrow} |\mathcal{N}(0,1)| \tag{2.2}$$

Furthermore, if $\sigma \geq 1$, $|\zeta(\frac{1}{2} + \sigma + i\tau)| \leq 2$; this proves the corollary.

□

2.2 The arcsine law for ζ

Corollary 2 (Arcsine law)**.** *For $T > 1$, define the probability measure μ_T over $[\frac{1}{\log T}, 1]$ by*

$$d\mu_T(\sigma) = \frac{1}{\sigma \log\log T} d\sigma \tag{2.3}$$

Then, if $\tau \in [0, T]$ is chosen uniformly, the distribution of

$$M_T = \mu_T\{\sigma \in [\frac{1}{\log T}, 1], |\zeta(\frac{1}{2} + \sigma + i\tau)| \geq 1\}$$

converges weakly to an arcsine law ie. $\mathbb{P}(M_T \leq y) \underset{T \to +\infty}{\longrightarrow} \frac{2}{\pi} \arcsin\sqrt{y}$.

Proof. Applying the arcsine law for Brownian motion, we know that

$$\tilde{M_T} = \lambda\{\alpha \in [0,1], \log|\zeta(\frac{1}{2} + \frac{1}{(\log T)^\alpha} + i\tau)| \geq 0\} \tag{2.4}$$

converges weakly to an arcsine law. However, if $\alpha \in [0, 1]$ is chosen uniformly, $\frac{1}{(\log T)^\alpha}$ follows the law μ_T; the corollary follows.

□

2.3 An iterated logarithm law for ζ

Corollary 3 (Law of the iterated logarithm)**.** *If $\alpha > 0$, set*

$$S_t(\alpha) = \sup_{0\leq\beta\leq\alpha} |\log|\zeta(\frac{1}{2} + \frac{1}{(\log T)^\beta} + it)|| \tag{2.5}$$

Then, if $\tau \in [0, T]$ is chosen uniformly at random, there exists $\alpha_T \underset{T\to+\infty}{\longrightarrow} 0$ such that

$$\frac{S_\tau(\alpha_T)}{\sqrt{2\alpha_T \log\log\frac{1}{\alpha_T}\log\log T}} \xrightarrow[T\to+\infty]{\mathbb{P}} 1 \tag{2.6}$$

Furthermore, (2.6) holds for any $\alpha_T \leq \alpha'_T \underset{T\to+\infty}{\longrightarrow} 0$.

This originates from the so-called iterated logarithm law for Brownian motion, which follows from a similar law for random walks [10]. The usual statement is that

$$\limsup_{t\to+\infty} \frac{B_t}{\sqrt{2t\log\log t}} = 1 \text{ a.s.} \tag{2.7}$$

but, noting that $(t \to tB_{\frac{1}{t}})$ is also a Wiener process, an analogous law follows for $t \to 0$.

Proof. From Theorem 3, we see that if $0 < \alpha < 1$

$$\lim_{T\to+\infty} \mathbb{E}\left[\left|\frac{S_\tau(\alpha)}{\sqrt{2\alpha\log\log\frac{1}{\alpha}\log\log T}} - 1\right|\right] = \mathbb{E}\left[\left|\frac{\sup\limits_{0\leq\alpha'\leq\alpha}|B_{\alpha'}|}{\sqrt{2\alpha\log\log\frac{1}{\alpha}}} - 1\right|\right] \tag{2.8}$$
$$\underset{\alpha\to 0}{\longrightarrow} 0$$

Therefore, if $\varepsilon, \varepsilon' > 0$, we can find $\eta > 0$ such that, if $\alpha \leq \eta$ and $T \geq \tilde{T}_0(\alpha, \varepsilon, \varepsilon')$,

$$\mathbb{E}\left[\left|\frac{S_\tau(\alpha)}{\sqrt{2\alpha\log\log\frac{1}{\alpha}\log\log T}} - 1\right|\right] \leq \varepsilon\varepsilon' \tag{2.9}$$

and so, with probability at least $1 - \varepsilon$,

$$\left|\frac{S_\tau(\alpha)}{\sqrt{2\alpha\log\log\frac{1}{\alpha}\log\log T}} - 1\right| \leq \varepsilon' \tag{2.10}$$

The corollary is thus proven. $\square$

2.4 Local time for ζ

Corollary 4. *For $t > 0$, let $L_t^{(T)}$ be the local time of $\operatorname{Re} Z^{(T)}$ ie. the unique function $L_t : \mathbb{R} \to \mathbb{R}_+$ such that, for any function $\varphi \in \mathcal{C}^0(\mathbb{R}, \mathbb{R})$,*

$$\int_0^t \varphi(\operatorname{Re} Z^{(T)}(u))du = \int_{\mathbb{R}} \varphi(v) L_t(v) dv \tag{2.11}$$

Then, $L_t^{(T)}$ converges weakly to the local time $L_t^{(\infty)}$ of Brownian motion, in the following sense: if $\varphi \in C_b^0(\mathbb{R})$ is a bounded continuous function,

$$\mathbb{E}[\langle L_t^{(T)}, \varphi \rangle] \underset{T \to +\infty}{\longrightarrow} \mathbb{E}[\langle L_t^{(\infty)}, \varphi \rangle] \tag{2.12}$$

Proof. Let $\varphi \in \mathcal{C}_b^0(\mathbb{R})$. Then,

$$\begin{aligned} \mathbb{E}\left[\int_{\mathbb{R}} \varphi(v) L_t^{(T)}(v) dv\right] &= \mathbb{E}\left[\int_0^t \varphi \circ \operatorname{Re} Z^{(T)}(u) du\right] \\ &\underset{T \to +\infty}{\longrightarrow} \mathbb{E}\left[\int_0^t \varphi(B_u) du\right] \\ &= \mathbb{E}\left[\int_{\mathbb{R}} \varphi(v) L_t^{(\infty)}(v) dv\right] \end{aligned} \tag{2.13}$$

□

This result grants us insight into the distribution of values of $\log \zeta$. For instance:

Corollary 5. *Let $N > 0, \varepsilon > 0$. Then, there exists $T_0(N, \varepsilon) > 0$ such that, if $T \geq T_0$ and $\tau \in [0, T]$ is uniformly chosen,*

$$\mathbb{P}(\log|\zeta(\frac{1}{2} + \sigma + i\tau)| \text{ changes sign at least } N \text{ times for } \sigma \in [0, 1]) \geq 1 - \varepsilon \tag{2.14}$$

Proof. By monotone convergence, it is sufficient to show that, for any $\varepsilon > 0$, $\eta > 0$, there exists $T_0(\eta, \varepsilon)$ such that, for $T \geq T_0$,

$$\mathbb{P}(\operatorname{Re} Z^{(T)} \text{ changes sign over } [0, \eta]) \geq 1 - \varepsilon \tag{2.15}$$

To show this, consider $\varphi(x) = x^+ \wedge 1$. By Portmanteau's theorem,

$$\liminf_{T \to +\infty} \mathbb{P}(\langle L_\eta^{(T)}, \varphi \rangle > 0) \geq \mathbb{P}(\langle L_\eta^{(\infty)}, \varphi \rangle > 0) = 1 \tag{2.16}$$

and so, for large enough T, $\operatorname{Re} Z^{(T)} > 0$ at some point in $[0, \eta]$ with probability at least $1 - \varepsilon$.

The same argument applied to $\varphi(x) = x^- \wedge 1$ shows (2.15), and the corollary.

□

3 Overview of the proof of Theorem 3

We must prove two points in order to establish convergence in distribution:

- the convergence of finite-dimensional distributions;
- the tightness of our process $Z^{(T)}$.

3.1 Convergence of finite-dimensional distributions

In this section, we prove Theorem 2 using the methods from [5]. In particular, we shall use the following lemma from that paper:

Lemma 1. *Let $a_{p,T}$ be complex numbers indexed by prime p and $T \geq 1$. Assume that:*

1. $\sup_p |a_{p,T}| \underset{T\to\infty}{\longrightarrow} 0$*;*
2. $\sum_p |a_{p,T}|^2 \underset{T\to\infty}{\longrightarrow} a^2$ *for some $a \geq 0$;*
3. *there exists (m_T) such that $\log m_T = o(\log T)$ and*

$$\sum_{p>m_T} |a_{p,T}|^2 (1 + \frac{p}{T}) \underset{T\to\infty}{\longrightarrow} 0 \tag{3.1}$$

Then, if $\tau \in [0, T]$ is a uniform random variable, $\sum_p a_{p,T} p^{-i\tau}$ converges in distribution to $\mathcal{N}(0, a^2)$.

In order to make use of this, we shall replace $\log \zeta$ with a related Dirichlet series. Indeed, it is known that, if $\sigma \geq \frac{1}{2}$,

$$\log \zeta(\sigma + i\tau) - \sum_{p \leq T} \frac{1}{p^{\sigma+i\tau}} \tag{3.2}$$

is bounded in $\mathcal{L}^2$, and thus converges in distribution to 0 once divided by $\sqrt{\log\log T}$. Here, we shall take

$$\sigma_i = \frac{1}{2} + \frac{1}{(\log T)^{\alpha_i}} \text{ for } 1 \leq i \leq n \tag{3.3}$$

Using the Cramér–Wold method, in order to show Theorem 2, it is thus sufficient to show that for all $\mu_1, \dots, \mu_n \in \mathbb{C}$,

$$\frac{1}{\sqrt{\log\log T}} \sum_{l=1}^{n} \mu_l \log \zeta(\sigma_l + i\tau) \xrightarrow{\text{law}} \mathcal{N}(0, a^2) \tag{3.4}$$

$$\text{ie. } \frac{1}{\sqrt{\log\log T}} \sum_{l=1}^{n} \mu_l \sum_{p \leq T} \frac{1}{p^{\sigma_l + i\tau}} \xrightarrow{\text{law}} \mathcal{N}(0, a^2)$$

where $a^2 = \sum_{1 \leq i,j \leq n} \mu_i \mu_j (\alpha_i \wedge \alpha_j)$. Now, setting

$$a_{p,T} = \frac{\mathbb{1}_{p\leq T}}{\sqrt{\log\log T}} \sum_{l=1}^{n} \frac{1}{p^{\sigma_l}} \tag{3.5}$$

we simply need to check that the prerequisites of Lemma 1 hold. (1.) is clearly true; (3.) holds if we set $m_T = T^{\frac{1}{\log\log T}}$. Finally, in order to check (2.), we just need to show

$$\sum_{p\leq T} \frac{1}{p^{\sigma_i+\sigma_j}} \sim (\alpha_i \wedge \alpha_j) \log\log T \text{ for any } 1 \leq i,j \leq n \tag{3.6}$$

This is shown in Lemma 3.3 of [5], and we skip the proof here: it is similar to the proof of Lemma 3 later on. All prerequisites having been checked, Theorem 2 is therefore proven.

3.2 The tightness criterion

Before proving the tightness of $Z^{(T)}$, let us state a criterion which will be crucial to the proof. It is a modified version of a statement by Prokhorov [14, Theorem 2.1], which itself is adapted from a criterion by Kolmogorov for the continuity of stochastic processes.

Theorem 4 (Kolmogorov tightness criterion)**.** *Let $\{Z^{(T)}, T \geq 0\}$ be a sequence of stochastic processes on $\mathcal{C}([0,1],\mathbb{R})$. Assume that, for some $T_0 \geq 0$,*

- *$\{Z^{(T)}(x_0), T \geq T_0\}$ is tight for some x_0;*
- *if $\varepsilon > 0$, there exist events A_ε^T of probability at least $1-\varepsilon$, and constants $A \geq 0, B > 1$ such that for $T \geq T_0$ and for all $0 \leq a, b \leq 1$,*

$$\mathbb{E}[|Z^{(T)}(a) - Z^{(T)}(b)|^A \mathbb{1}_{A_\varepsilon^T}] \ll_\varepsilon |a-b|^B \tag{3.7}$$

Then, the sequence is tight.

The original statement of this theorem does not include $\mathbb{1}_{A_\varepsilon^T}$; this is a relatively minor change, but we nevertheless include a proof for completeness.

Proof. Setting $\varepsilon > 0$, we must show that for large enough T, $Z^{(T)}$ stays in a compact subset of $\mathcal{C}^0([0,1])$ with probability at least $1-\varepsilon$. Replacing ε by 2ε, we may replace $Z^{(T)}$ by $Z^{(T)}\mathbb{1}_{A_\varepsilon^T}$, and so our main hypothesis becomes

$$\forall x,y \in [0,1], \mathbb{E}[|Z^{(T)}(x) - Z^{(T)}(y)|^A] \leq C|x-y|^B \tag{3.8}$$

If $0 < \gamma < 1$, we are going to bound the γ-Hölder norm of $Z^{(T)}$:

$$\|\varphi\|_\gamma = |\varphi(x_0)| + \sup_{0\leq x,y\leq 1} \frac{|\varphi(x)-\varphi(y)|}{|x-y|^\gamma} \tag{3.9}$$

In fact, it is sufficient to bound its restriction to dyadic intervals

$$\|\varphi\|_\gamma^{\mathcal{D}} = |\varphi(x_0)| + \sup_{\substack{n\geq 1 \\ 0\leq k\leq 2^n-1}} \frac{|\varphi(\frac{k+1}{2^n}) - \varphi(\frac{k}{2^n})|}{(\frac{1}{2^n})^\gamma} \tag{3.10}$$

since $\|\varphi\|_\gamma \leq 2(1-2^{-\gamma})\|\varphi\|_\gamma^{\mathcal{D}}$. Accordingly, if $n > 0$, $0 \leq k < 2^n$, and $T > T_0, M > 0$,

$$\begin{aligned} \mathbb{P}(|Z^{(T)}(\frac{k+1}{2^n}) - Z^{(T)}(\frac{k}{2^n})| > \frac{M}{2^{\gamma n}}) &\leq \frac{2^{A\gamma n}}{M^A}\mathbb{E}[|Z^{(T)}(\frac{k+1}{2^n}) - Z^{(T)}(\frac{k}{2^n})|^A] \\ &\leq \frac{1}{M^A}2^{(A\gamma - B)n} \end{aligned} \tag{3.11}$$

Furthermore, we can find $M' > 0$ such that $\mathbb{P}(|Z^{(T)}(x_0)| > M') < \frac{\varepsilon}{2}$. Thus, if we take $\gamma < \dfrac{B-1}{A}$, we may sum over all dyadic numbers:

$$\begin{aligned} \mathbb{P}(\|Z^{(T)}\|_\gamma^{\mathcal{D}} > M + M') &\leq \frac{\varepsilon}{2} + \sum_{k,n} \mathbb{P}(|Z^{(T)}(\frac{k+1}{2^n}) - Z^{(T)}(\frac{k}{2^n})| > \frac{M}{2^{\gamma n}}) \\ &\leq \frac{1}{M^A}\frac{2^{1+A\gamma-B}}{1-2^{1+A\gamma-B}} + \frac{\varepsilon}{2} \end{aligned} \tag{3.12}$$

For large enough M, this is at most ε. Thus, we have shown that $\|Z^{(T)}\|_\gamma$ is bounded by a certain constant with probability at least $1-\varepsilon$: since the unit ball for $\|\cdot\|_\gamma$ is compact, this concludes our proof. $\square$

If we want to extend our result to $\alpha \in [0, +\infty[$, the proof above is not quite sufficient because the unit ball for $\|\cdot\|_\gamma$ is no longer compact. We need some criteria to ensure well-behavedness at infinity, and the following conditions are sufficient:

- almost surely, $Z^{(T)}(\alpha)$ converges to a (random) limit l as $\alpha \to \infty$
- for $T > 0$ and $\varepsilon > 0$, $|Z^{(T)}(\alpha) - l|\mathbb{1}_{A_\varepsilon^T} \leq a(\varepsilon, T) \underset{\alpha\to+\infty}{\longrightarrow} 0$ for some **deterministic** $a(\varepsilon, T)$. This means that $|Z^{(T)}(\alpha) - l|\mathbb{1}_{A_\varepsilon^T}$ converges to 0 uniformly in the underlying random seed ω.

These criteria, and $\|\cdot\|_\gamma$-boundedness, are enough to guarantee tightness (by a more general Arzelà-Ascoli theorem). In our specific case regarding the Riemann zeta function,

- the first criterion is clearly fulfilled;
- equation (5.18) later on guarantees that the second condition is verified.

This means that we effectively have convergence in law over $[0, +\infty[$.

3.3 Structure of the proof of tightness

Theorem 4 is quite useful for our purposes. It allows us to transform a tightness problem, which would require some pretty strong uniform controls on $\log\zeta$, into a moments calculation for which we have much better tools. In our case, we shall take $A = 4$ and $B = 2$, owing to the roughly $\frac{1}{2}$-Holderian behaviour of Brownian motion.

Are the prerequisites of Theorem 4 verified by the process $Z^{(T)}$? The first one clearly is (taking for instance $x_0 = 0$) but the second one is much less obvious, and the purpose of the rest of this paper will be to prove that it holds. Specifically, from now on, we will set $\varepsilon > 0$, and $0 \leq a < b \leq 1$: for large enough $T > 0$, our aim is to construct an adequate A_ε^T (independent of a, b) such that

$$\mathbb{E}[|Z^{(T)}(a) - Z^{(T)}(b)|^4 \mathbb{1}_{A_\varepsilon^T}] \leq C_\varepsilon |a-b|^2 \tag{3.13}$$

We will also set

$$\sigma_1 = \frac{1}{2} + \frac{1}{(\log T)^a} \text{ and } \sigma_2 = \frac{1}{2} + \frac{1}{(\log T)^b} \tag{3.14}$$

so that (3.13) becomes

$$\mathbb{E}[|\log\zeta(\sigma_1 + i\tau) - \log\zeta(\sigma_2 + i\tau)|^4 \mathbb{1}_{A_\varepsilon^T}] \leq C_\varepsilon (b-a)^2 \tag{3.15}$$

In order to show this, we will proceed by approximating $\log\zeta$ by a well-chosen Dirichlet sum. This allows us to effectively compute moments by expanding out the sum. Specifically, we will write, for $x > 1$,

$$\frac{\zeta'}{\zeta}(s) = -\sum_{n \leq x^3} \frac{\Lambda_x(n)}{n^s} + e_x(s) \tag{3.16}$$

$$\text{where } \Lambda_x(n) = \begin{cases} \Lambda(n) & \text{if } n \leq x \\ \Lambda(n) \dfrac{\log^2 \frac{x^3}{n} - 2\log^2 \frac{x^2}{n}}{\log^2 n} & \text{if } x \leq n \leq x^2 \\ \Lambda(n) \dfrac{\log^2 \frac{x^3}{n}}{\log^2 n} & \text{if } x^2 \leq n \leq x^3 \end{cases}.$$

Here, we will be taking $x = T^{\frac{1}{20}}$, although all of the following results are valid for $x = T^c$ with small enough c. As a result,

$$\mathbb{E}[|Z^{(T)}(a) - Z^{(T)}(b)|^4 \mathbb{1}_{A_\varepsilon^T}] \leq \frac{32}{(\log\log T)^2} \mathbb{E}\left[\left|\sum_{n \leq x^3} \frac{\Lambda_x(n)}{n^{\sigma_1 + i\tau}\log n} - \sum_{n \leq x^3} \frac{\Lambda_x(n)}{n^{\sigma_2 + i\tau}\log n}\right|^4\right] + \frac{32}{(\log\log T)^2} \mathbb{E}[\left|\int_{\sigma_2}^{\sigma_1} e_x(\sigma + i\tau) d\sigma\right|^4 \mathbb{1}_{A_\varepsilon^T}] \tag{3.17}$$

The first term can be bounded quite effectively; this will be done in Section 4. To bound the second term, we will rely upon methods developed by Selberg in his original paper on the CLT [16]; this will be done in Section 5. In particular, the specific choice of Λ_x was made in order to be able to apply these methods.

With both terms bounded, we will mostly have proven our main theorem. However, setting $\sigma_c = \dfrac{1}{2} + \dfrac{40\log\frac{1}{\varepsilon}}{\log T}$, it turns out that this method breaks down when $\sigma_2 \leq \sigma_c$. This case is not too complicated, and is handled separately at the end of the paper.

4 Moments of Dirichlet sums

As announced, in this section we will show the following proposition.

Proposition 2. *Here and for the rest of this paper, we will take $x = T^{\frac{1}{20}}$. Then,*

$$\mathbb{E}\left[\left|\sum_{n\leq x^3}\frac{\Lambda_x(n)}{n^{\sigma_1+i\tau}\log n}-\sum_{n\leq x^3}\frac{\Lambda_x(n)}{n^{\sigma_2+i\tau}\log n}\right|^4\right]\ll(b-a)^2(\log\log T)^2$$

In order to prove Proposition 2, we shall use the two following technical lemmas:

Lemma 2. *Let $\varphi:\mathbb{N}\to\mathbb{R}_+$ satisfy the following conditions:*

- *$\varphi(n)=0$ if n is not a p^k for some prime p and $k>0$;*
- *if p is prime and $i\geq 1$, $\varphi(p^i)\leq\varphi(p)$*

Then, setting $x=T^{\frac{1}{20}}$,

$$\mathbb{E}\left[\left|\sum_{n\leq x^3}\frac{\varphi(n)}{n^{\frac{1}{2}+i\tau}}\right|^4\right]\ll\left(\sum_{p\leq x^3\ prime}\frac{\varphi(p)^2}{p}\right)^2 \tag{4.1}$$

with the implied constant being independent of φ.

Lemma 3. *Let $0\leq\alpha\leq\beta\leq 1$, and let η,η' be functions of T such that $\log\eta\sim-\alpha\log\log T$ and $\log\eta'\sim-\beta\log\log T$. Then,*

$$\sum_{p\leq x^3\ prime}\frac{1}{p^{1+\eta}}-\frac{1}{p^{1+\eta'}}\ll(\beta-\alpha)\log\log T \tag{4.2}$$

These lemmas are very similar to existing results in the literature, but those are not quite sufficient for our purposes due to the dependence on σ,σ'.

Assuming these lemmas, we may set

$$\varphi(n)=\frac{\Lambda_x(n)}{\log n}(n^{-(\sigma_1-\frac{1}{2})}-n^{-(\sigma_2-\frac{1}{2})}) \tag{4.3}$$

and, applying Lemma 2,

$$\mathbb{E}\left[\left|\sum_{n\leq x^3}\frac{\Lambda_x(n)}{n^{\sigma_1+i\tau}\log n}-\sum_{n\leq x^3}\frac{\Lambda_x(n)}{n^{\sigma_2+i\tau}\log n}\right|^4\right]\ll\left(\sum_{p\leq x^3\text{ prime}}(p^{-\sigma_1}-p^{-\sigma_2})^2\right)^2 \tag{4.4}$$

$$=\left(\sum_{p\leq x^3}\left(\frac{1}{p^{2\sigma_1}}-\frac{1}{p^{\sigma_1+\sigma_2}}\right)-\left(\frac{1}{p^{\sigma_1+\sigma_2}}-\frac{1}{p^{2\sigma_2}}\right)\right)^2$$

Applying Lemma 3, this shows Proposition 2.

Proof of Lemma 2. We may write

$$\begin{aligned}\mathbb{E}\left[\left|\sum_{n\le x^3}\frac{\varphi(n)}{n^{\frac{1}{2}+i\tau}}\right|^4\right] &= \mathbb{E}\left[\left|\sum_{m,n\le x^3}\frac{\varphi(m)\varphi(n)}{\sqrt{mn}}e^{-i\tau\log(mn)}\right|^2\right]\\ &= \mathbb{E}\left[\left|\sum_{l\ge 1}e^{-i\tau\log l}\sum_{\substack{m,n\le x^3\\ mn=l}}\frac{\varphi(m)\varphi(n)}{\sqrt{mn}}\right|^2\right]\end{aligned} \tag{4.5}$$

in order to apply the following identity by Montgomery-Vaughan.

Lemma 4. *Let $\lambda_1,\dots,\lambda_N\in\mathbb{R}$, $\alpha_1,\dots,\alpha_N$ and set $\delta=\min_{i,j\le N}|\lambda_i-\lambda_j|$. Then,*

$$\int_0^T\left|\sum_{k=1}^N\alpha_k e^{i\lambda_k t}\right|^2 dt=(T+O(\delta^{-1}))\sum_{k=1}^N|\alpha_k|^2 \tag{4.6}$$

A proof of this can be found in [12]. In this case, we obtain

$$\begin{aligned}\mathbb{E}\left[\left|\sum_{n\le x^3}\frac{\varphi(n)}{n^{\frac{1}{2}+i\tau}}\right|^4\right] &\ll \sum_{l\ge 1}\frac{1}{l}\left|\sum_{\substack{m,n\le x^3\\ mn=l}}\varphi(m)\varphi(n)\right|^2\\ &\overset{\text{def}}{=}\sum_{l\ge 1}\frac{1}{l}\Phi_x(l)^2\\ &\le \sum_{p<q\text{ prime}}\sum_{1\le i,j}\frac{\Phi_x(p^iq^j)^2}{p^iq^j}+\sum_{i\ge 2}\sum_{p\text{ prime}}\frac{\Phi_x(p^i)^2}{p^i}\end{aligned} \tag{4.7}$$

We separate these two terms in order to effectively bound Φ_x in each case. To bound the first term, expanding out Φ_x,

$$\begin{aligned}\sum_{p<q\text{ prime}}\sum_{1\le i,j}\frac{\Phi_x(p^iq^j)^2}{p^iq^j} &\le \sum_{p,q\le x^3\text{ prime}}\sum_{1\le i,j}\frac{\varphi(p^i)^2\varphi(q^j)^2}{p^iq^j}\\ &\le 2\sum_{p,q\le x^3\text{ prime}}\frac{\varphi(p)^2\varphi(q)^2}{pq}\\ &\le 2\left(\sum_{p\le x^3\text{ prime}}\frac{\varphi(p)^2}{p}\right)^2\end{aligned} \tag{4.8}$$

We handle the second term in the same manner:

$$\begin{aligned}\sum_{i\geq 2}\sum_{p\text{ prime}}\frac{\Phi_x(p^i)^2}{p^i} &= \sum_{i\geq 2}\sum_{p\text{ prime}}\sum_{k+l=i,k'+l'=i}\frac{\varphi(p^k)\varphi(p^{k'})\varphi(p^l)\varphi(p^{l'})}{p^i}\\ &\leq \sum_{p\text{ prime}}\varphi(p)^4\sum_{i\geq 2}\frac{i^2}{p^i}\\ &\ll \sum_{p\text{ prime}}\frac{\varphi(p)^4}{p^2}\leq\left(\sum_{p\leq x^3\text{ prime}}\frac{\varphi(p)^2}{p}\right)^2\end{aligned}\tag{4.9}$$

This concludes the proof of Lemma 2. □

We now just have to prove our other technical result.

Proof of Lemma 3. First, note that by taking the derivative of $\alpha \mapsto \frac{1}{(\log T)^\alpha}$, we obtain

$$\eta - \eta' \ll \eta(\beta-\alpha)\log\log T \tag{4.10}$$

We shall be using equation (4.10) throughout the rest of this paper.

Let us rewrite:

$$\begin{aligned}\sum_{p\leq x^3}\frac{1}{p^{1+\eta}} &= \sum_{n\leq x^3}\frac{\pi(n)-\pi(n-1)}{n^{1+\eta}}\\ &= \sum_{n\leq x^3}\pi(n)\Big(\frac{1}{n^{1+\eta}}-\frac{1}{(n+1)^{1+\eta}}\Big)+\frac{\pi(N)}{N^{1+\eta}}\end{aligned}\tag{4.11}$$

where $N = \lfloor x^3 \rfloor$, and π denotes the prime-counting function. We now wish to approximate this sum by its associated integral. Specifically, setting

$$I_x(\eta) = \int_1^{x^3}\pi(u)\Big(\frac{1}{u^{1+\eta}}-\frac{1}{(u+1)^{1+\eta}}\Big)du \tag{4.12}$$

we will split up our problem:

$$\sum_{p\leq x^3}\frac{1}{p^{1+\eta}}-\sum_{p\leq x^3}\frac{1}{p^{1+\eta'}} = \left(I_x(\eta')-\sum_{p\leq x^3}\frac{1}{p^{1+\eta'}}\right)-\left(I_x(\eta)-\sum_{p\leq x^3}\frac{1}{p^{1+\eta}}\right)+(I_x(\eta)-I_x(\eta')) \tag{4.13}$$

Now,

$$
\begin{aligned}
&\frac{d}{d\eta}\left(I_x(\eta) - \sum_{p\le x^3}\frac{1}{p^{1+\eta}}\right)\\
&\quad = (1+\eta)\left(\int_1^{x^3}\pi(u)\Big(-\frac{\log u}{u^{1+\eta}} + \frac{\log\lfloor u\rfloor}{\lfloor u\rfloor^{1+\eta}} + \frac{\log(u+1)}{(u+1)^{1+\eta}} - \frac{\log\lfloor u+1\rfloor}{\lfloor u+1\rfloor^{1+\eta}}\Big)du - \frac{\pi(N)\log N}{N^{1+\eta}}\right)\\
&\quad \ll \int_1^{x^3}\pi(u)\frac{\log u}{u^{3+\eta}}du + \frac{\pi(N)\log N}{N^{1+\eta}} \ll 1
\end{aligned}
\tag{4.14}
$$

since $\pi(u) \sim \dfrac{u}{\log u}$. As a result,

$$
\left(I_x(\eta) - \sum_{p\le x^3}\frac{1}{p^{1+\eta}}\right) - \left(I_x(\eta') - \sum_{p\le x^3}\frac{1}{p^{1+\eta'}}\right) \ll (\eta-\eta') \ll (\beta-\alpha)\log\log T \tag{4.15}
$$

by (4.10). As a result, we now just need to control $I_x(\eta) - I_x(\eta')$.

$$
\begin{aligned}
I_x(\eta) - I_x(\eta') &\ll \int_2^{x^3}\frac{u}{\log u}\left(\frac{1}{u^{1+\eta}} - \frac{1}{(u+1)^{1+\eta}} - \frac{1}{u^{1+\eta'}} + \frac{1}{(u+1)^{1+\eta'}}\right)du\\
&= \int_2^{x^3}\left(\frac{\eta}{u^{1+\eta}\log u} - \frac{\eta'}{u^{1+\eta'}\log u}\right)du + \int_2^{x^3}\varepsilon(u,\eta) - \varepsilon(u,\eta')du
\end{aligned}
\tag{4.16}
$$

with $\varepsilon(u,\eta) = \dfrac{1}{u^{\eta}\log u} - \dfrac{u}{(u+1)^{1+\eta}\log u} - \dfrac{\eta}{u^{1+\eta}\log u}$. Since $\dfrac{d}{d\eta}\varepsilon(u,\eta) \ll \dfrac{1}{u^{2+\eta}}$,

$$
\int_2^{x^3}(\varepsilon(u,\eta) - \varepsilon(u,\eta'))du \ll \eta - \eta' \ll (\beta-\alpha)\log\log T \tag{4.17}
$$

In order to bound the main integral, we may now set $v = \eta\log u$ (resp. $v = \eta'\log u$):

$$
\begin{aligned}
\int_2^{x^3}\left(\frac{\eta}{u^{1+\eta}\log u} - \frac{\eta'}{u^{1+\eta'}\log u}\right)du &= \eta\int_{\eta\log 2}^{3\eta\log x}\frac{dv}{ve^v} - \eta'\int_{\eta'\log 2}^{3\eta'\log x}\frac{dv}{ve^v}\\
&= (\eta-\eta')\int_{\eta\log 2}^{3\eta'\log x}\frac{dv}{ve^v} + \eta\int_{3\eta'\log x}^{3\eta\log x}\frac{dv}{ve^v} - \eta'\int_{\eta'\log 2}^{\eta\log 2}\frac{dv}{ve^v}\\
&= I_1 + I_2 - I_3
\end{aligned}
\tag{4.18}
$$

It is quite clear that $I_2 \ll I_3$; meanwhile,

$$
I_3 = \eta'\int_{\eta'\log 2}^{\eta\log 2}\frac{1}{v}(1+O(1))dv \ll \eta'\log\frac{\eta}{\eta'} + (\eta-\eta') \ll (\beta-\alpha)\log\log T \tag{4.19}
$$

Finally,

$$I_1 \ll (\eta - \eta') \left(\int_{\eta \log 2}^{1} \frac{dv}{v} + \int_{1}^{3\eta' \log x} e^{-v} dv \right) \tag{4.20}$$
$$\ll \eta(1 + \log \eta)(\beta - \alpha) \log \log T \ll (\beta - \alpha)(\log \log T)$$

which concludes the proof.

□

5 The contribution of zeta zeroes

As a reminder, we have set $\sigma_c = \frac{1}{2} + \frac{40 \log \frac{1}{\varepsilon}}{\log T}$. We still have two points to handle in order to apply Theorem 4:

- we must bound the second term in (3.17);
- we must handle the case $\sigma_2 \leq \sigma_c$.

In the interests of legibility, we will define (and work with) $\eta_1 = \sigma_1 - \frac{1}{2}$, $\eta_2 = \sigma_2 - \frac{1}{2}$, $\eta_c = \sigma_c - \frac{1}{2}$, etc.

5.1 Bounding the error e_x

Recall that we set

$$e_x(s) = \log \zeta(s) + \sum_{n \leq x^3} \frac{\Lambda_x(n)}{n^s} \tag{5.1}$$

We wish to show that

$$\mathbb{E}[\left| \int_{\sigma_2}^{\sigma_1} e_x(\sigma + i\tau) d\sigma \right|^4 \mathbb{1}_{A_\varepsilon^T}] \ll_\varepsilon (b-a)^2 (\log \log T)^2 \tag{5.2}$$

when $\sigma_c \leq \sigma_2 \leq \sigma_1$. Our main tool for showing this will be the following identity from [16, equation (4.9)]

Lemma 5. *Let $t \geq 2$ and $2 \leq x \leq t^2$.*
Furthermore, set

$$\sigma_{x,t} = \frac{1}{2} + 2 \max_{\rho}(\beta, \frac{2}{\log x})$$

where $\rho = \frac{1}{2} + \beta + i\gamma$ ranges over all zeroes of ζ such that $|t - \gamma| \leq \frac{x^{3|\beta|}}{\log x}$. Then, if $\sigma \geq \sigma_{x,t}$,

$$e_x(\sigma+it) \ll x^{-\frac{1}{2}(\sigma-\frac{1}{2})}\left(\left|\sum_{n\le x^3}\frac{\Lambda_x(n)}{n^{\sigma_{x,t}+it}}\right|+\log t\right) \tag{5.3}$$

Applying this lemma, if $\sigma_{x,t} \le \sigma_2 \le \sigma_1$ and $T \le t \le 2T$,

$$\left|\int_{\sigma_2}^{\sigma_1} e_x(\sigma+it)d\sigma\right| \ll \left(\left|\sum_{n\le x^3}\frac{\Lambda_x(n)}{n^{\sigma_{x,t}+it}}\right|+\log T\right)^4 \min(\frac{x^{-\frac{\eta_1}{2}}}{\log x},(\sigma_1-\sigma_2)x^{-\frac{\eta_2}{2}}) \tag{5.4}$$

This is a good start, but do we go from the condition $\sigma_2 \ge \sigma_{x,t}$ to $\sigma_2 \ge \sigma_c$? For this, we need to show that $\sigma_{x,t}$ is usually smaller than σ_c: we will then cut out the region where $\sigma_{x,t} \ge \sigma_c$ by choosing A_ε^T adequately. This is the object of the following lemma.

Lemma 6. *Set*

$$Y_\varepsilon = \bigcup_\rho [\gamma - \varepsilon\frac{x^{4|\beta|}}{\log T}, \gamma + \varepsilon\frac{x^{4|\beta|}}{\log T}] \tag{5.5}$$

where $\rho = \frac{1}{2}+\beta+i\gamma$ ranges over non-trivial zeroes of ζ, then:

- *the measure of $Y_\varepsilon \cap [0,T]$ is $O(\varepsilon T)$;*
- *if $t \in [0,T]\setminus Y_\varepsilon$, $\sigma_{x,t} \le \sigma_c$.*

Proof. We may ignore the zeroes to the left of the critical axis, since ζ has reflectional symmetry with regards to the critical axis.

Set, for $\eta \ge 0$

$$\mathcal{N}(\eta,T) = \#\{\rho = \frac{1}{2}+\beta+i\gamma \text{ such that } \zeta(\rho)=0, \beta\ge\eta, 0\le\gamma\le T\} \tag{5.6}$$

It is known (see [16]) that $\mathcal{N}(\eta,T) \ll T\log T\exp(-\frac{1}{4}\eta\log T)$. Thus, setting $N = \mathcal{N}(0,T)$, we may label $\frac{1}{2}+\beta_1 \ge \frac{1}{2}+\beta_2 \ge \ldots \ge \frac{1}{2}+\beta_N$ the abcissae of zeroes of ζ with ordinates in $[0,T]$. Now:

$$\begin{aligned}
|Y_\varepsilon \cap [0,T]| &\le \sum_{i=1}^{N-1} 2\varepsilon\frac{x^{4\beta_i}}{\log T} \\
&= \frac{2}{5}\varepsilon\sum_{i=1}^{N-1}\mathcal{N}(\frac{1}{2}+\beta_i,T)\int_{\beta_{i+1}}^{\beta_i} x^{4\beta}d\beta + 2\varepsilon N\frac{x^{4\beta_N}}{\log T} \\
&\le \varepsilon\int_0^1 x^{4\beta}\mathcal{N}(\frac{1}{2}+\beta,T)d\beta + 2\varepsilon N\frac{x^{4\beta_N}}{\log T}
\end{aligned} \tag{5.7}$$

Since $\beta_N \ll \frac{1}{\log T}$ and $N \ll T\log T$, $2\varepsilon N\frac{x^{4\beta_N}}{\log T} = O(\varepsilon T)$. Meanwhile,

$$\int_0^1 x^{4\beta}\mathcal{N}(\frac{1}{2}+\beta,T)d\beta \ll T\log T\int_0^1 T^{-\frac{\beta}{20}}d\beta \\ \ll T \tag{5.8}$$

This shows the first part of Lemma 6. For the second point, simply note that, if $t \in [0,T]\backslash Y_\varepsilon$, and $\rho = \frac{1}{2}+\beta+i\gamma$ is a zero of ζ such that $|t-\gamma| \leq \frac{x^{3\beta}}{\log x}$,

$$\varepsilon\frac{x^{4\beta}}{\log x} \leq |t-\gamma| \leq \frac{x^{3\beta}}{\log x} \text{ so } x^\beta \leq \frac{1}{\varepsilon}, \text{ and } \beta \leq \frac{\log\frac{1}{\varepsilon}}{\log x} \tag{5.9}$$

Taking the maximum over all applicable ρ, we see that $\sigma_{x,t} \leq \dfrac{1}{2}+\dfrac{40\log\frac{1}{\varepsilon}}{\log T}$. The lemma is thus shown.

□

This means that we can set $A_\varepsilon^T = (\tau \notin Y_\varepsilon)$ and apply Lemma 5 to tackle (5.2):

$$\mathbb{E}[\left|\int_{\sigma_2}^{\sigma_1} e_x(\sigma+i\tau)d\sigma\right|^4 \mathbb{1}_{A_\varepsilon^T}] \ll \left(\mathbb{E}\left[\left|\sum_{n\leq x^3}\frac{\Lambda_x(n)}{n^{\sigma_{x,\tau}+i\tau}}\right|^4\right]+(\log T)^4\right)\min(\frac{x^{-2\eta_1}}{(\log x)^4},(\eta_1-\eta_2)^4x^{-2\eta_2}) \tag{5.10}$$

To conclude, we will use the following proposition, whose proof will be given shortly:

Proposition 3.

$$\mathbb{E}\left[\left|\sum_{n\leq x^3}\frac{\Lambda_x(n)}{n^{\sigma_{x,\tau}+i\tau}}\right|^4\mathbb{1}_{t\notin X_\varepsilon}\right] \ll_\varepsilon (\log T)^4$$

Assuming Proposition 3,

- if $b-a \leq \dfrac{1}{\log\log T}$, then $\dfrac{\eta_1}{\eta_2} \ll 1$ and, applying (4.10),

$$\mathbb{E}[\int_{\sigma_2}^{\sigma_1}|e_x(\sigma+i\tau)|^4 d\sigma \mathbb{1}_{A_\varepsilon^T}] \ll_\varepsilon (\log T)^4\eta_1^4(b-a)^4(\log\log T)^4x^{-2\eta_2} \\ \ll (b-a)^4(\eta_2\log T)^4e^{-\frac{2}{13}\eta_2\log T}(\log\log T)^4 \\ \ll (b-a)^2(\log\log T)^2 \tag{5.11}$$

- if $b-a \geq \dfrac{1}{\log\log T}$, then:

$$\mathbb{E}[\int_{\sigma_2}^{\sigma_1}|e_x(\sigma+i\tau)|^4 d\sigma \mathbb{1}_{A_\varepsilon^T}] \ll_\varepsilon \left(\frac{\log T}{\log x}\right)^4 x^{-2\eta_2} \\ \ll 1 \ll (b-a)^2(\log\log T)^2 \tag{5.12}$$

We therefore just need to show Proposition 3.

Proof. Set $\eta_0 = \dfrac{1}{\log T}$: then,

$$\mathbb{E}\left[\left|\sum_{n\le x^3}\frac{\Lambda_x(n)}{n^{\sigma_{x,\tau}+i\tau}}\right|^4 \mathbb{1}_{A_\varepsilon^T}\right] \le 8\mathbb{E}\left[\left|\sum_{n\le x^3}\frac{\Lambda_x(n)}{n^{\frac{1}{2}+\eta_0+i\tau}}\right|^4\right] + 8\mathbb{E}\left[\left|\sum_{n\le x^3}\frac{\Lambda_x(n)}{n^{\frac{1}{2}+i\tau}}\left(n^{-\eta_0}-n^{-\eta_{x,\tau}}\right)\right|^4 \mathbb{1}_{A_\varepsilon^T}\right] \tag{5.13}$$

However, applying Lemma 2 with $\varphi(n) = \Lambda_x(n)n^{-\eta_0}$:

$$\begin{aligned}\mathbb{E}\left[\left|\sum_{n\le x^3}\frac{\Lambda_x(n)}{n^{\frac{1}{2}+\eta_0+i\tau}}\right|^4\right] &\ll \left(\sum_{p\le x^3 \text{ prime}}\frac{(\log p)^2}{p^{1+2\eta_0}}\right)^2 \\ &\ll \left(\int_2^A \frac{(\log t)^{1-2\eta_0}}{t^{1+2\eta_0}}dt\right)^2\end{aligned} \tag{5.14}$$

for some $A \sim \dfrac{x^3}{3\log x}$. Setting $u = \eta_0 \log t$ and changing variables,

$$\begin{aligned}\int_2^A \frac{(\log t)^{1-2\eta_0}}{t^{1+2\eta_0}}dt &= \frac{1}{\eta_0^{2-2\eta_0}}\int_{\eta_0\log 2}^{\eta_0\log A}\frac{u^{1-2\eta_0}}{e^{2u}}du \\ &\ll \eta_0^{-2}\end{aligned} \tag{5.15}$$

Meanwhile, if we look at the second term in (5.13),

$$\begin{aligned}\mathbb{E}\left[\left|\sum_{n\le x^3}\frac{\Lambda_x(n)}{n^{\frac{1}{2}+i\tau}}\left(n^{-\eta_0}-n^{-\eta_{x,\tau}}\right)\right|^4 \mathbb{1}_{A_\varepsilon^T}\right] &\ll \mathbb{E}\left[\left|\sum_{n\le x^3}\frac{\Lambda_x(n)}{n^{\frac{1}{2}+\eta_0+i\tau}}\left((\eta_{x,\tau}-\eta_0)\log n\right)\right|^4 \mathbb{1}_{A_\varepsilon^T}\right] \\ &\ll \left(\log\frac{20}{\varepsilon}\right)^4 \mathbb{E}\left[\left|\sum_{n\le x^3}\frac{\Lambda_x(n)}{n^{\frac{1}{2}+\eta_0+i\tau}}\right|^4\right] \ll_\varepsilon \eta_0^{-4}\end{aligned}$$

Thus,

$$\mathbb{E}\left[\left|\sum_{n\le x^3}\frac{\Lambda_x(n)}{n^{\sigma_{x,\tau}+i\tau}}\right|^4 \mathbb{1}_{A_\varepsilon^T}\right] \ll_\varepsilon \eta_0^{-4} = (\log T)^4$$

□

5.2 Towards the critical line

As mentioned earlier, the case $\sigma_2 \leq \sigma_c$ needs to be handled separately. Specifically, it still remains to be shown that:

Proposition 4. *If $\sigma_2 \leq \sigma_1 \leq \sigma_c$,*

$$\mathbb{E}[|\log \zeta(\sigma_1 + i\tau) - \log \zeta(\sigma_2 + i\tau)|^4 \mathbb{1}_{A_\varepsilon^T}] \ll_\varepsilon (b-a)^2 (\log \log T)^2 \tag{5.16}$$

The case where $\sigma_2 \leq \sigma_c \leq \sigma_1$ follows easily, applying the "triangular inequality" $|a+b|^4 \ll |a|^4 + |b|^4$.

Also note that the argument below also applies in the more general case where, potentially, $a, b > 1$ (the case where $b-a$ is much larger than $\log \log T$ is easily handled). This allows us, as mentioned in Section 3.2, to apply the tightness criterion over $[0, +\infty[$.

Proof. Note that, if $\sigma \leq \sigma_c$, and $T \leq t \leq 2T$,

$$\begin{aligned} |\frac{\zeta'}{\zeta}(\sigma + it) - \frac{\zeta'}{\zeta}(\sigma_c + it)| &= |\sum_{\rho = \beta + i\gamma} \frac{1}{\eta - \beta + i(t-\gamma)} - \frac{1}{\eta_c - \beta + i(t-\gamma)} + O(\log T)| \\ &\leq \sum_\rho \frac{\eta_c - \eta}{|\eta - \beta + i(t-\gamma)||\eta_c - \beta + i(t-\gamma)|} + O(\log T) \\ &\leq (\eta_c - \eta) \sum_\rho \frac{1}{(t-\gamma)^2} + O(\log T) \end{aligned} \tag{5.17}$$

Now,

$$\mathbb{E}\left[\sum_\rho \frac{1}{(t-\gamma)^2} \mathbb{1}_{t \notin Y_\varepsilon}\right] \ll_\varepsilon (\log T)^2$$

As a result, we can increase the size of Y_ε in such a way that the measure of $Y_\varepsilon \cap [T, 2T]$ remains $O(\varepsilon T)$, and if $t \in [T, 2T] \setminus Y_\varepsilon$,

$$\sum_\rho \frac{1}{(t-\gamma)^2} \ll_\varepsilon (\log T)^2 \text{ and so } |\frac{\zeta'}{\zeta}(\sigma + it) - \frac{\zeta'}{\zeta}(\sigma_c + it)| \ll_\varepsilon \log T \tag{5.18}$$

Consequently,

$$\begin{aligned} \mathbb{E}[|\log \zeta(\sigma_1 + i\tau) - \log \zeta(\sigma_2 + i\tau)|^4 \mathbb{1}_{A_\varepsilon^T}] &\ll \mathbb{E}\left[\left|\int_{\sigma_2}^{\sigma_1} \frac{\zeta'}{\zeta}(\sigma + i\tau) d\sigma\right|^4 \mathbb{1}_{A_\varepsilon^T}\right] \\ &\ll_\varepsilon (\sigma_1 - \sigma_2)^4 ((\log T)^4 + \mathbb{E}[|\frac{\zeta'}{\zeta}(\sigma_c + i\tau)|^4 \mathbb{1}_{A_\varepsilon^T}]) \\ &\ll_\varepsilon (\eta_1 \log T)^4 (b-a)^4 (\log \log T)^4 \\ &\ll_\varepsilon (b-a)^2 (\log \log T)^2 \end{aligned} \tag{5.19}$$

given that $b - a \ll_{\varepsilon} \dfrac{1}{\log\log T}$.

$\square$

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