Stochastic calculus, practice for the midterm exam

These are just examples of typical exercises for the midterm. You may also be asked to state important theorems and give some short proofs of results seen in class. Lecture notes will not be allowed.

Exercise 1. Let $(X_i)_{i\geq 1}$ be i.i.d. uniform on $[0,1]$. What is the limit of

$$\frac{X_1+\cdots+X_n}{X_1^2+\cdots+X_n^2}$$

as $n\to\infty$?

Exercise 2. Let Y_1, Y_2, Y_3 be independent $\mathscr{N}(0,1)$ random variables. Let

$$X_1=Y_1+Y_3,\ X_2=Y_2+4Y_3,\ X_3=2Y_1-2Y_2+xY_3.$$

for some real number x.

(i) Explain why $\mathbf{X}=(X_1,X_2,X_3)$ is a Gausssian vector.
(ii) What is the covariance matrix for $\mathbf{X}$?
(iii) For what values of x are X_1 and X_3 independent?

Exercise 3. Let X be a real Gaussian random variable with mean 0 and variance 7. Calculate

(i) $\mathbb{E}(e^{\lambda X})$ for any $\lambda\in\mathbb{C}$;
(ii) $\mathbb{E}(X^7-3X^2+12X-4)$.

Exercise 4. Let $X_1,X_2,\dots$ be independent random variables and $\mathbb{P}(X_j=2)=1-\mathbb{P}(X_j=-1)=\frac{1}{3}$ for any j. Let $S_n=X_1+\cdots+X_n$ and $\mathcal{F}_n=\sigma(X_1,\dots,X_n)$.

(i) Calculate $\mathbb{E}(S_n)$, $\mathbb{E}(S_n^2)$, $\mathbb{E}(S_n^3)$.
(ii) If $m<n$, calculate $\mathbb{E}(S_n\mid\mathcal{F}_m)$, $\mathbb{E}(S_n^2\mid\mathcal{F}_m)$, $\mathbb{E}(S_n^3\mid\mathcal{F}_m)$.
(iii) If $m<n$, calculate $\mathbb{E}(X_m\mid S_n)$.

Exercise 5. Let $1/2<q<1$ and $X_1,X_2,\dots$ be independent random variables and and $\mathbb{P}(X_j=1)=1-\mathbb{P}(X_j=-1)=q$ for any j. Let $S_n=X_1+\cdots+X_n$ and $\mathcal{F}_n=\sigma(X_1,\dots,X_n)$.

(i) Is S a $(\mathcal{F}_n)_{n\geq 0}$-martingale, submartingale, supermartingale?
(ii) Find r such that S_n-rn is a $(\mathcal{F}_n)_{n\geq 0}$-martingale.
(iii) Let $\theta=(1-q)/q$ and $M_n=\theta^{S_n}$. Prove that M a $(\mathcal{F}_n)_{n\geq 0}$-martingale.

Exercise 6. Let B be a standard Brownian motion, $\sigma>0$ and $M_t=e^{\sigma B_t-\frac{\sigma^2}{2}t}$. Show that M is a martingale (with respect to which filtration?).

Exercise 7. Let B be a standard Brownian motion. Compute the following.

(1) $\mathbb{E}\left(B_t^2\mid\mathcal{F}_s\right)$
(2) $\mathbb{E}\left(B_t^3\mid\mathcal{F}_s\right)$
(3) $\mathbb{E}\left(B_t^4\mid\mathcal{F}_s\right)$
(4) $\mathbb{E}\left(e^{4B_t+12}\mid\mathcal{F}_s\right)$

Exercise 8. Let B be a standard Brownian motion and $M_t=\max_{0\leq s\leq t}B_s$.

(1) Explain why M_t has the same distribution as $\sqrt{t}M_1$.
(2) What is the density of M_t?

Exercise 9. Let B be a standard Brownian motion and $\lambda > 0$. What is $\mathbb{E}(e^{-\lambda B_t})$? What is $\mathbb{E}(e^{-\lambda B_t^2})$?

Exercise 10. Let B be a Brownian motion and $\mathcal{F}_t = \sigma(B_s, s \leq t)$.
(i) Show that $(B_t^2 - t)_{t\geq 0}$ is a $(\mathcal{F}_t)_{t\geq 0}$-martingale.
(ii) Is there a deterministic function $(f_t)_{t\geq 0}$ such that $(e^{(B_t^2-t)-f_t})_{t\geq 0}$ is a $(\mathcal{F}_t)_{t\geq 0}$-martingale?

Exercise 11. Let $(X_i)_{i\geq 1}$ be a sequence of i.i.d. random variables with mean 0 and finite variance $\mathbb{E}(X_j^2) = \sigma^2 > 0$. Let $S_n = X_1 + \cdots + X_n$.
(i) State Donsker's theorem.
(ii) What is $\lim_{n\to\infty} \mathbb{E}\left(\frac{|S_n|}{\sqrt{n}}\right)$?

Exercise 12. Define continuous uniformly integrable martingales. State the stopping time theorem for uniformly integrable continuous martingales.

Exercise 13. Let $X_n, n \geq 0$, be independent random variables. Assume that $\mathbb{E}(X_j) = 0$ and there exists a $\alpha > 0$ such that $\mathbb{E}(|X_j|^2) = j^{-\alpha}$ for any $j \geq 1$. Let $S_n = \sum_{k=1}^n X_k$. For which values of α does S converge almost surely?