Stochastic calculus, homework 7, due November 20th.

Below B, $B^{(1)}$ and $B^{(2)}$ are standard, independent, Brownian motions.

Exercise 1. Apply Itô's forula to write the following processes as the sum of a local martingale and a finite variation process.

(i) $X_t = B_t^2$;
(ii) $X_t = e^{B_t} + t^3$;
(iii) $X_t = B_t^3 - 3tB_t$;
(iv) $X_t = \cosh(B_t)$;
(v) $X_t = (B_t^2 + 1)^{-1}$;
(vi) $X_t = (B_t^{(1)})^2 + (B_t^{(2)})^2$.

Exercise 2. Let $X_t = e^{\int_0^t u(s)\mathrm{d}s}$ and $Y_t = Y_0 + \int_0^t v(s)e^{-\int_0^s u(r)\mathrm{d}r}\mathrm{d}B_s$ where u and v are deterministic functions. Let $Z_t = X_t Y_t$. Prove that

$$\mathrm{d}Z_t = u(t)Z_t\mathrm{d}t + v(t)\mathrm{d}B_t.$$

Exercise 3. For any $t \in [0,1)$, let $Z_t = \frac{1}{\sqrt{1-t}}e^{-\frac{B_t^2}{2(1-t)}}$.

(i) Prove that $(Z_t)_{0\leq t<1}$ is a martingale which converges almost surely to 0 as $t \to 1$.
(ii) Calculate $\mathbb{E}(Z_t)$.
(iii) Write Z as $Z_t = e^{\int_0^t X_s\mathrm{d}B_s - \frac{1}{2}\int_0^t X_s^2\mathrm{d}s}$ for some explicit process X.

Exercise 4. Let $A_t = \int_0^t e^{B_s+\nu s}\mathrm{d}s$ and assume the process S satisfies $\mathrm{d}S_t = S_t(r\mathrm{d}t + \sigma\mathrm{d}B_t)$. For a smooth function f, prove that $f(t, S_t, A_t)$ is local martingale if and only if f satisfies an explicit partial differential equation.

Exercise 5. Assume $\mathrm{d}X_t = \mathrm{d}B_t + \frac{1}{X_t}\mathrm{d}t$. Show that X_t^{-1} is a local martingale. Is it a martingale?