Stochastic calculus, final exam

Lecture notes are not be allowed. Below, B always means a standard Brownian motion.

Exercise 1. Write each of the following process, what is the drift, and what is the volatility? In other words, write the corresponding Ito formula.

1) B_t^2
2) $\cos(t) + e^{B_t}$
3) $B_t^3 - 3tB_t$
4) $B_t^2 \widetilde{B}_t$ where $\widetilde{B}$ is a Brownian motion independent of B
5) $e^{-B_t} \int_0^t \sinh(B_s)\mathrm{d}B_s$

Exercise 2. Is $(B_{2t} - B_t)_{t\geq 0}$ a Brownian motion? Is it a Gaussian process?

Exercise 3. Let $X_t = \int_0^t f(s)\mathrm{d}B_s$ where f is an adapted process such that $\mathbb{E}(f(s)^2) \leq Cs^p$ for all $s \geq 0$, for some constants $C, p > 0$. Show that

$$\mathbb{E}(|X_t|) \leq \left(\frac{C}{p+1}\right)^{1/2} t^{\frac{p+1}{2}}.$$

Hint: first calculate $\mathbb{E}(X_t^2)$.

Exercise 4. Let $\sigma > 0$. Is $(e^{\sigma B_t - \frac{\sigma^2}{2}t})_{t\geq 0}$ a Gaussian process? Is it a martingale? Is it a uniformly integrable martingale?

Exercise 5. Let $\tau = \inf\{t \geq 0 \mid B_t = 1\}$. Prove that $\mathbb{E}(e^{-\tau/2}) = \frac{1}{e}$.

Exercise 6. Assume $X_t = \int_0^t a(s)\mathrm{d}s + \int_0^t b(s)\mathrm{d}B_s$, where a and b are deterministic measurable and bounded functions. Give sufficient conditions on a and b such that X is a Brownian motion.

Exercise 7. Let $\mu \in \mathbb{R}$, $\sigma > 0$ and let B be a standard Brownian motion. Let also X be the strong solution of the stochastic differential equation $\mathrm{d}X_t = \mu X_t \mathrm{d}t + \sigma X_t \mathrm{d}B_t$, and $X_0 > 0$.

Find the stochastic differential equation satisfied by $Y_t = 1/X_t$. For which values of μ and σ are X and Y simultaneously submartingales?

Exercise 8. Let a, b, c, z be constants. What is the stochastic differential equation satisfied by

$$X_t = e^{(a-c^2/2)t+cB_t}\left(z + b\int_0^t e^{-(a-c^2/2)s-cB_s}\mathrm{d}s\right)?$$

Exercise 9. Using the Feynman-Kac formula, calculate

$$\mathbb{E}\left(e^{-\sigma\int_t^T B_s \mathrm{d}s} \mid B_t = x\right).$$

Exercise 10. What stochastic differential equation does $(e^{-t}B_{e^{2t}})_{t\geq 0}$ satisfy? What is the name of this process?

Exercise 11. Let X and Y be independent Brownian motions.

1) Assume $X_0 = Y_0 = 0$, and note $T_a = \inf\{t \geq 0 \mid X_t = a\}$ for $a > 0$. Prove that T_a has the same law as $a^2/\mathcal{N}^2$, where $\mathcal{N}$ is a standard normal variable.

2) Prove that Y_{T_a} has the same law as $a\,C$, where the Cauchy random variable C is defined through its density with respect to the Lebesgue measure,

$$\frac{1}{\pi(1+x^2)}.$$

3) Let $(X_0, Y_0) = (\epsilon, 0)$, where $0 < \epsilon < 1$. Note $Z_t = X_t + \mathrm{i}Y_t$. Justify that the winding number

$$\theta_t = \frac{1}{2\pi}\arg Z_t$$

can be properly defined, continuously from $\theta_0 = 0$. Let $T^{(\epsilon)} = \inf\{t \geq 0 \mid |Z_t| = 1\}$. Prove that

$$\frac{\theta_{T^{(\epsilon)}}}{\log \epsilon}$$

is distributed as $\frac{1}{2\pi}C$, C being a Cauchy random variable.

4) Let $(X_0, Y_0) \neq (0, 0)$ and define as previously $Z_t = X_t + \mathrm{i}Y_t$ and $\arg Z_t$ continuously from $\arg Z_0 \in [0, 2\pi)$. Prove that, as $t \to \infty$,

$$\frac{2\arg Z_t}{\log t} \xrightarrow{\text{law}} C.$$

Exercise 12. Let a be a given deterministic function. Calculate

$$\mathbb{E}\left(e^{\int_0^t a(s)B_s^2 \mathrm{d}s}\right).$$