Stochastic calculus, homework 6, due November 7th.

Exercise 1. Let B be a Brownian motion. Prove that almost surely we have $\sup_{s\geq 0} B_s = +\infty$ and $\inf_{s\geq 0} B_s = -\infty$.

Exercise 2. Let B be a Brownian motion and $T_a = \inf\{s \geq 0 : B_s = a\}$. Prove (with no calculation) that for any $b > a > 0$ the random variable $T_b - T_a$ is independent of T_a.

Simulate a Brownian motion (based on the method suggested by Donsker's theorem, with a thousand time steps between times 0 and 1) and the corresponding process $(T_a)_{a\geq 0}$. Give a printed copy of the code with the homework (no matter which language) and a samples of both curves.

Exercise 3. Let M and N be two martingales bounded in L^2, and define

$$\langle M, N\rangle = \frac{1}{2}\left(\langle M+N\rangle - \langle N\rangle - \langle M\rangle\right).$$

Prove that $MN - \langle M, N\rangle$ is a martingale.

Exercise 4. Let B be a Brownian motion starting at $x > 0$, and $T_0 = \inf\{s \geq 0 : B_s = 0\}$. That is the distribution of $\sup_{t\leq T_0} B_t$?

Hint: at some point in class we studied maxima of positive martingales converging to 0.

Exercise 5: *bonus.* Let f be a continuous function on $\mathbb{R}$.

(1) Prove that if $(f(B_t), t \geq 0)$ is a $(\mathcal{F}_t)_{t\geq 0}$ martingale, then f is affine.
(2) Suppose that $(f(B_t), t \geq 0)$ is a $(\mathcal{F}_t)_{t\geq 0}$ submartingale: prove that f has no proper local maximum. Hint: for $c > 0$, use the stopping times $T = T_c \wedge T_{-1}$ and $S = \inf\{t \geq T : B_t = -1 \textit{ or } c+\epsilon \textit{ or } c-\epsilon\}$.
(3) Suppose that $(f(B_t), t \geq 0)$ is a $(\mathcal{F}_t)_{t\geq 0}$ submartingale: prove that f is convex.