Stochastic calculus, homework 4, due October 17th.

Exercise 1. Let B be a Brownian motion and f a continuous bounded function. Prove that for any $u \in (0,t)$, we have

$$\mathbb{E}(f(B_t)) = \mathbb{E}(f(B_{t-u} + G\sqrt{u}))$$

where G is a standard Gaussian random variable independent of B_{t-u}.

Exercise 2. Let B and $\widetilde{B}$ be two independent Brownian motions, $\mathcal{F}_s = \sigma(B_u, 0 \leq u \leq s)$, and h a continuous bounded function. Prove that for any $s < t$

$$\mathbb{E}\left(\int_0^t h(r, B_r)\mathrm{d}r \mid \mathcal{F}_s\right) = \int_0^s h(r, B_r)\mathrm{d}r + \mathbb{E}\left(\int_0^{t-s} h(s+u, B_s + \widetilde{B}_u)\mathrm{d}u\right),$$

where the first expectation is integration with respect to B, and the second one with respect to $\widetilde{B}$ only.

Exercise 3. Let B be a Brownian motion and $S_t = e^{\mu t + \sigma B_t}$. Compute the expectation and variance of $\int_0^t S_u \mathrm{d}u$ and $\int_0^t \log S_u \mathrm{d}u$. Are these random variables Gaussian?

Exercise 4. Let $T_a = \inf\{t \mid B_t = a\}$ and $S_1 = \sup\{B_s, s \leq 1\}$. Prove that $T_a \stackrel{\text{law}}{=} a^2 T_1$. Prove that $T_1 \stackrel{\text{law}}{=} 1/S_1^2$. What is the density of the random variable T_1?

Hint: for the first and second questions you can use invariance by scaling of the Brownian motion. For the third question, you can use a statement from the reflection principle.

Exercise 5: *bonus.* Prove that as $t \to \infty$, $\left(\int_0^t e^{B_s}\mathrm{d}s\right)^{1/\sqrt{t}}$ converges in law towards $e^{|\mathcal{N}|}$, where $\mathcal{N}$ is a standard Gaussian random variable.

Hint: the random variable $|\mathcal{N}|$ appears in the reflection principle.