**Stochastic calculus, homework 11, due December 12th.**

**Exercise 1.** Let $B$ be a Brownian motion. For $n \geq 0$, let

$$H_n(x,y) = (\partial_\alpha)^n \mid_{\alpha=0} e^{\alpha x - \frac{\alpha^2}{2} y}.$$

Prove that for $n = 1, 2, 3$, $H_n(B_t, t)$ is a martingale. Prove it for any $n$.

**Exercise 2.**

(i) Read the lecture notes and prove in a fully rigorous way (ssuming Theorem 5.9) the following fact: the dynamics of the Ehrenfest urn converge to the Ornstein Uhlenbeck process for a large number of particles.

(ii) Assume we start with $10^{25}$ particles in the right side and none on the other side. How long does it take to reach equilibrium if we transfer one particle every $\varepsilon$ second? At equilibrium, what are the typical fluctuations for the number of particles in the first urn?

**Exercise 3.** Consider the general equation

$$\mathrm{d}X_t = (c(t) + d(t)X_t)\mathrm{d}t + (e(t) + f(t)X_t)\mathrm{d}B_t,\ X_0 = 0.$$

where $c, d, e, f$ are deterministic. We try to find a solution of type $X = X^{(1)}X^{(2)}$ where

$$\mathrm{d}X_t^{(1)} = d(t)X_t^{(1)}\mathrm{d}t + f(t)X_t^{(1)}\mathrm{d}B_t,\ X_0^{(1)} = 1,$$
$$\mathrm{d}X_t^{(2)} = a(t)\mathrm{d}t + b(t)\mathrm{d}B_t,\ X_0^{(2)} = X_0,$$

and $a, b$ are stochastic processes to be chosen.

(i) Prove that $X_t^{(1)} = e^{\int_0^t f(s)\mathrm{d}B_s - \frac{1}{2}\int_0^t f(s)^2 \mathrm{d}s + \int_0^t d(s)\mathrm{d}s}$ is a solution.

(ii) Identify necessary formulas for $a$ and $b$.

(iii) Conclude a general formula for the solution of the initial equation.

**Exercise 4.** For a given Brownian motion $B$, let $X$ be a solution of

$$\mathrm{d}X_t = \sigma(X_t)\mathrm{d}B_t + b(X_t)\mathrm{d}t,\ X_0 = x,$$

and $X^{(n)}$ be a solution of

$$\mathrm{d}X_t = \sigma^{(n)}(X_t)\mathrm{d}B_t + b^{(n)}(X_t)\mathrm{d}t,\ X_0 = x,$$

where all functions are Lipschitz with the same absolute constant independent of $n$. Assume pointwise convergence of $\sigma^{(n)}$ to $\sigma$, and of $b^{(n)}$ to $b$. Prove that for any $t > 0$, as $n \to \infty$,

$$\mathbb{E}\left(\sup_{[0,t]} |X_s - X_s^{(n)}|^2\right) \to 0.$$

**Exercise 5.** Let $B^1$ and $B^2$ be independent Brownian motions, defined on the same probability space. Let

$$X_t = e^{B_t^1}\int_0^t e^{-B_s^1}\mathrm{d}B_s^2,\ Z_t = \sinh B_t^1.$$

Prove that both processes have the same distribution.