Stochastic analysis, homework 5.

Exercise 1 For a given Brownian motion B, let X be a solution of

$$\mathrm{d}X_t = \sigma(X_t)\mathrm{d}B_t + b(X_t)\mathrm{d}t,\ X_0 = x,$$

and $X^{(n)}$ be a solution of

$$\mathrm{d}X_t = \sigma^{(n)}(X_t)\mathrm{d}B_t + b^{(n)}(X_t)\mathrm{d}t,\ X_0 = x,$$

where all functions are Lipschitz with the same absolute constant independent of n. Assume pointwise convergence of $\sigma^{(n)}$ to σ, and of $b^{(n)}$ to b. Prove that for any $t > 0$, as $n \to \infty$,

$$\mathbb{E}\left(\sup_{[0,t]} |X_s - X_s^{(n)}|^2\right) \to 0.$$

Exercise 2 Let B be a Brownian motion, $a > 0$, $\gamma \geq 0$, and $T_{a,\gamma} = \inf\{t \geq 0 \mid B_t + \gamma t = a\}$. Prove that the density of $T_{a,\gamma}$ with respect to the Lebesgue measure on $\mathbb{R}_+$ is

$$\frac{a}{\sqrt{2\pi t^3}} e^{\frac{-(a-\gamma t)^2}{2t}}.$$

Exercise 3 Let B be a Brownian motion, $a > 0$, $\gamma \in \mathbb{R}$, and $S_{a,\gamma} = \inf\{t \geq 0 \mid |B_t + \gamma t| = a\}$. Are $S_{a,\gamma}$ and $B_{S_{a,\gamma}} + \gamma S_{a,\gamma}$ independent under the Wiener measure?

Exercise 4 Let X and Y be independent Brownian motions.

1 Assume $X_0 = Y_0 = 0$, and note $T_a = \inf\{t \geq 0 \mid X_t = a\}$ for $a > 0$. Prove that T_a has the same law as $a^2/\mathcal{N}^2$, where $\mathcal{N}$ is a standard normal variable.

2 Prove that Y_{T_a} has the same law as $a\,C$, where the Cauchy random variable C is defined through its density with respect to the Lebesgue measure,

$$\frac{1}{\pi(1+x^2)}.$$

3 Let $(X_0, Y_0) = (\epsilon, 0)$, where $0 < \epsilon < 1$. Note $Z_t = X_t + \mathrm{i}Y_t$. Justify that the winding number

$$\theta_t = \frac{1}{2\pi} \arg Z_t$$

can be properly defined, continuously from $\theta_0 = 0$. Let $T^{(\epsilon)} = \inf\{t \geq 0 \mid |Z_t| = 1\}$. Prove that

$$\frac{\theta_{T^{(\epsilon)}}}{\log \epsilon}$$

is distributed as $\frac{1}{2\pi}C$, C being a Cauchy random variable.

4 Let $(X_0, Y_0) \neq (0, 0)$ and define as previously $Z_t = X_t + \mathrm{i}Y_t$ and $\arg Z_t$ continuously from $\arg Z_0 \in [0, 2\pi)$. Prove that, as $t \to \infty$,

$$\frac{2 \arg Z_t}{\log t} \xrightarrow{\text{law}} C.$$