Stochastic analysis, homework 2.

In all the following, B is a Brownian motion.

Exercise 1

1 Calculate $\mathbb{E}\left(B_s B_t^2\right)$, $\mathbb{E}\left(B_t \mid \mathcal{F}_s\right)$, $\mathbb{E}\left(B_t \mid B_s\right)$, for $t > s > 0$.

2 What is $\mathbb{E}\left(B_s^2 B_t^2\right)$, still for $t > s$?

3 What is the law of $B_t + B_s$? Same question for $\lambda_1 B_{t_1} + \cdots + \lambda_k B_{t_k}$ $(0 < t_1 < \cdots < t_k)$? What is the law of $\int_0^1 B_s \mathrm{d}s$?

Exercise 2 Convergence types.

1 Study the convergence in probability of $\frac{\log(1+B_t^2)}{\log t}$ as $t \to \infty$.

2 What about the almost sure convergence of $\frac{\log(1+B_t^2)}{\log t}$ as $t \to \infty$?

Exercise 3 Martingales from Brownian motion. Amongst the following processes, which ones are $\mathcal{F}$-martingales, where $\mathcal{F}$ is the natural filtration of $(B_s, s \geq 0)$? $B_t^2 - t$, $B_t^3 - 3\int_0^t B_s \mathrm{d}s$, $B_t^3 - 3tB_t$, $tB_t - \int_0^t B_s \mathrm{d}s$.

Exercise 4 Scaling and equalities in law.

1 Let $T_a = \inf\{t \mid B_t = a\}$ and $S_1 = \sup\{B_s, s \leq 1\}$. Prove that $T_a \overset{\text{law}}{=} a^2 T_1$. Prove that $T_1 \overset{\text{law}}{=} 1/S_1^2$.

2 Let $g_t = \sup\{s \leq t \mid B_s = 0\}$ and $d_t = \inf\{s \geq t \mid B_s = 0\}$. Prove that g is not a stopping time, and that d is a stopping time. Prove that $g_t \overset{\text{law}}{=} t g_1$, $d_t \overset{\text{law}}{=} t d_1$, $g_t \overset{\text{law}}{=} \frac{t}{d_1} \overset{\text{law}}{=} \frac{1}{d_{1/t}}$.

Exercise 5 A convergence in law. Prove that as $t \to \infty$, $\left(\int_0^t e^{B_s} \mathrm{d}s\right)^{1/\sqrt{t}}$ converges in law towards $e^{|\mathcal{N}|}$, where $\mathcal{N}$ is a standard Gaussian random variable.

Exercise 6 The zeros of Brownian motion. Let $\mathcal{Z} = \{t \geq 0 \mid B_t = 0\}$.

1 Prove that $\mathcal{Z}$ is almost surely a closed and unbounded set, with no isolated points.

2 Prove that $\mathcal{Z}$ is almost surely uncountable.