Stochastic analysis, homework 1.

Exercise 1. Let $X_i, i \geq 1$, be iid random variables, $X_i \geq 0, E(X_i) = 1$. Prove that if $Y_n = \prod_1^n X_k$, $\mathcal{F}_n = \sigma(X_k, k \leq n)$, $(Y_n)_{n\geq 0}$ is a $(\mathcal{F}_n)$-martingale.

Prove that if $\mathbb{P}(X_1 = 1) < 1$, Y_n converges to 0 almost surely.

Exercise 2. Let $(X_n, n \geq 0)$ be a non-negative supermartingale. Show the following maximal ineqality: for $a > 0$,

$$a\mathbb{P}\left(\sup_{[\![0,n]\!]} X_k > a\right) \leq \mathbb{E}(X_0).$$

Exercise 3. Let $X_0 > 0$, and at time $n+1$ you get $\epsilon_n Y_n$ where Y_n was your stake at time n, the ϵ_n's are iid and $\mathbb{P}(\epsilon_n = 1) = p = 1 - \mathbb{P}(\epsilon_n = -1)$, $p \in (1/2, 1)$: what you own at time $n+1$ is

$$X_{n+1} = X_n + \epsilon_{n+1} Y_n,$$

where $Y_n \in \mathcal{F}_n$, $0 \leq Y_n \leq X_n$, $\mathcal{F}_n = \sigma(\epsilon_1, \dots, \epsilon_n)$. The game lasts at some finite time $T \in \mathbb{N}^*$.

You want to maximize the expected return $\mathbb{E}\left(\log \frac{X_n}{X_0}\right)$, by finding the good strategy, i.e. what suitable $\mathcal{F}_n$-measurable function Y_n to choose. Prove that for some $\lambda > 0$ explicit in terms of p, $((\log X_n) - n\lambda, n \geq 0)$ is a $(\mathcal{F}_n)$-supermartingale, so that

$$\mathbb{E}\left(\log \frac{X_n}{X_0}\right) \leq n\lambda.$$

Find a strategy such that equality occurs in the above equation.

Exercise 4. Let $(S_n)_{n\geq 0}$ be a $(\mathcal{F}_n)$-martingale and τ a stopping time with finite expectation. Assume that there is a $c > 0$ such that, for all n, $\mathbb{E}(|S_{n+1} - S_n| \mid \mathcal{F}_n) < c$.

Prove that $(S_{\tau\wedge n})_{n\geq 0}$ is a uniformly bounded martingale, and that $\mathbb{E}(S_\tau) = \mathbb{E}(S_0)$.

Consider now the random walk $S_n = \sum_k^n X_k$, the X_k's being iid, $\mathbb{P}(X_1 = 1) = \mathbb{P}(X_1 = -1) = 1/2$. For some $a \in \mathbb{N}^*$, let $\tau = \inf\{n \mid S_n = -a\}$. Prove that

$$\mathbb{E}(\tau) = \infty.$$

Exercise 5. As previously, consider the random walk $S_n = \sum_k^n X_k$, the X_k's being iid, $\mathbb{P}(X_1 = 1) = \mathbb{P}(X_1 = -1) = 1/2$, $\mathcal{F}_n = \sigma(X_i, 0 \leq i \leq n)$.

Prove that $(S_n^2 - n, n \geq 0)$ is a $(\mathcal{F}_n)-$martingale. Let τ be a bounded stopping time. Prove that $\mathbb{E}(S_\tau^2) = \mathbb{E}(\tau)$.

Take now $\tau = \inf\{n \mid S_n \in \{-a, b\}\}$, where $a, b \in \mathbb{N}^*$. Prove that $\mathbb{E}(S_\tau) = 0$ and $\mathbb{E}(S_\tau^2) = \mathbb{E}(\tau)$. What is $\mathbb{P}(S_\tau = -a)$? What is $\mathbb{E}(\tau)$? Get the last result of the previous exercise by justifying the limit $b \to \infty$.

Exercise 6. Let $X_n, n \geq 0$, be iid complex random variables such that $\mathbb{E}(X_1) = 0, 0 < \mathbb{E}(|X_1|^2) < \infty$. For some parameter $\alpha > 0$, let

$$S_n = \sum_{k=1}^n \frac{X_k}{k^\alpha}.$$

Prove that if $\alpha > 1/2$, S_n converges almost surely. What if $0 < \alpha \leq 1/2$?