Random Matrix Theory, homework 3, due December 22nd.

Exercise 1. Eigenvalues dynamics for covariance matrices. Let $M = M(t)$ be a $N \times N$ random matrix with $(M_{i,j}(t))_{t\geq 0}$ a real-valued, standard Brownian motion, and denote $H(t) = M(t)^{\mathrm{T}} M(t)$. Assume that $M(0)$ is such that $H(0)$ has N distinct eigenvalues $\lambda_1(0) > \cdots > \lambda_N(0) > 0$. Prove that the process $(\boldsymbol{\lambda}(t))_{t\geq 0}$ satisfies

$$\mathrm{d}\lambda_i(t) = 2\sqrt{\lambda_i(t)}\mathrm{d}B_i(t) + N\mathrm{d}t + \sum_{k\neq i} \frac{\lambda_i(t) + \lambda_k(t)}{\lambda_i(t) - \lambda_k(t)}\mathrm{d}t, \quad 1 \leq i \leq N,$$

where the B_i's are independent, standard Brownian motions. As a first step you can provide a formal derivation of the above result. As a second step you can prove that almost surely, a process satisfying this SDE has non-colliding particles.

Exercise 2. Non-collision for arbitrary driving noise. Consider the stochastic differential equation

$$\mathrm{d}x_i(t) = \frac{\mathrm{d}b_i(t)}{\sqrt{2N}} + \frac{1}{2N}\sum_{j\neq i} \frac{1}{x_i(t) - x_j(t)}\mathrm{d}t, \quad 1 \leq |i| \leq N,$$

where $0 < x_1(0) < \cdots < x_N(0)$, $x_{-i}(0) = -x_i(0)$, $(i \geq 1)$, $(b_i)_{1\leq i\leq N}$ is a collection of continuous martingales, and $b_{-i}(t) = -b_i(t)$, $(i \geq 1)$. Note that the drift above includes a repulsive term between x_i and x_{-i}, equal to $\frac{1}{2N}\frac{1}{2x_i}$.

Assume that $\frac{\mathrm{d}\langle b_i\rangle_t}{\mathrm{d}t} \leq 1$ for any $i \geq 1, t \geq 0$. Prove existence and strong uniqueness for the above stochastic differential equation.

Exercise 3. Andreiev's identity, Kostlan's theorem.

(i) On a measured space $(E, \mathcal{E}, \mu)$, prove that for any functions $(\phi_i, \psi_i)_{i=1}^N \in L_2(\mu)^{2N}$,

$$\frac{1}{N!}\int_{E^N} \det(\phi_i(\lambda_j))\ \det(\psi_i(\lambda_j))\ \mu(\mathrm{d}\lambda_1)\dots\mu(\mathrm{d}\lambda_N) = \det(f_{i,j}) \quad \text{where } f_{i,j} = \int_E \phi_i(\lambda)\psi_j(\lambda)\mu(\mathrm{d}\lambda).$$

(ii) Let $E = \mathbb{C}$, m the Lebesgue measure on E, $g \in L_2(\mu)$, with $\mu(\mathrm{d}\lambda) = \frac{N}{\pi}e^{-N|\lambda|^2}m(\mathrm{d}\lambda)$, and $\{\lambda_1, \dots, \lambda_N\}$ be the eigenvalues from the complex Ginibre ensemble (with the standard normalization for a limiting circular law on $\mathbb{D}$). Prove that

$$\mathbb{E}\left(\prod_{k=1}^N g(\lambda_k)\right) = N^{\frac{N(N-1)}{2}}\det(f_{i,j})_{i,j=1}^N \quad \text{where } f_{i,j} = \frac{1}{(j-1)!}\int \lambda^{i-1}\bar{\lambda}^{j-1}g(\lambda)\mu(\mathrm{d}\lambda).$$

(iii) Prove that the set $N\{|\lambda_1|^2, \dots, |\lambda_N|^2\}$ is distributed as $\{\gamma_1, \dots, \gamma_N\}$, a set of (unordered) independent Gamma variables of parameters $1, 2, \dots, N$.

Exercise 4. Moment matching. Let X be a real-valued random variable with mean zero, variance 1. To simplify, we assume that X takes values in $[-10, 10]$. Prove that for any $\gamma < 10^{-10}$ there exists a random variable Y_γ, valued in $[-100, 100]$, such that

$$Z_\gamma := \sqrt{1-\gamma}Y_\gamma + \sqrt{\gamma}G$$

and X have the same moments of order $1, 2, 3$, and

$$|\,\mathbb{E}[Z_\gamma^4 - X^4]| \leq 10^{10}\gamma.$$

Here G is a standard Gaussian random variable, independent of Y_γ.