Random Matrix Theory, homework 2, due April 7.

Problem 1. The Circular Unitary Ensemble is a log-correlated random field. Let $(e^{\mathrm{i}\theta_k})_{1\leq k\leq N}$ be the eigenvalues of a Haar-distributed matrix in $\mathrm{U}(N)$. The eigenangles have joint probability distribution

$$\mathbb{P}(\mathrm{d}\boldsymbol{\theta}) = \frac{1}{N!}\prod_{1\leq i<j\leq N}|e^{\mathrm{i}\theta_i}-e^{\mathrm{i}\theta_j}|^2\frac{\mathrm{d}\theta_1}{2\pi}\cdots\frac{\mathrm{d}\theta_N}{2\pi}.$$

(i) Prove that $\chi=\sum_{k=1}^N\delta_{\theta_k}$ is a determianntal point process with correlation kernel

$$K(x,y)=K^{(N)}(x,y)=\frac{1}{2\pi}\frac{\sin N\frac{x-y}{2}}{\sin\frac{x-y}{2}}$$

with respect to the Lebesgue measure on $(0,2\pi)$.

(ii) Let $\phi:[0,2\pi)\to\mathbb{R}$ be bounded measurable. Prove that

$$\mathbb{E}\prod_{k=1}^N(1+\phi(\theta_k))=\sum_{n\geq 0}\frac{1}{n!}\int_{(0,2\pi)^n}\prod_{j=1}^n\phi(x_j)\det_{n\times n}K(x_i,x_j)\mathrm{d}x_1\dots\mathrm{d}x_n.$$

You will need to explain why the right hand side converges.

(iii) Read Section 3 in the book *Trace ideals and applications*.

(iv) Let $A\subset[0,2\pi)$ be measurable. On $\mathrm{L}^2(A)$, define $K\phi$ the convolution operator with kernel $K\phi$, where ϕ is bounded measurable:

$$(K\phi)(f)(x)=\int K(x,y)\phi(y)f(y)\mathrm{d}y.$$

Prove that $K\mathbb{1}_A$ is trace-class with spectrum in $[0,1]$. Let $X=\chi(A)$. Show that

$$\log\mathbb{E}(e^{\mathrm{i}\xi X})=\log\det(\mathrm{Id}+K\mathbb{1}_A(e^{\mathrm{i}\xi}-1))=-\sum_{k=1}^\infty\frac{(1-e^{\mathrm{i}\xi})^k}{k}\mathrm{Tr}((K\mathbb{1}_A)^k).$$

(v) The formula $\log\mathbb{E}(e^{\mathrm{i}\xi X})=\sum_{\ell=1}^\infty \mathrm{C}_\ell(X)\frac{(\mathrm{i}\xi)^\ell}{\ell!}$ defines the cumulants $\mathrm{C}_\ell(X)$ of the random variable X. Prove that for any $\ell\geq 3$,

$$\mathrm{C}_\ell(X)=(-1)^\ell(\ell-1)!\,\mathrm{Tr}(K\mathbb{1}_A-(K\mathbb{1}_A)^\ell)+\sum_{j=2}^{\ell-1}\alpha_{j\ell}\;\mathrm{C}_j(X)$$

for some universal constants $\alpha_{j\ell}$.

(vi) Take $A=[0,x)$ $(x\in(0,2\pi))$ in this question and the next one. Prove that

$$\mathrm{C}_2(X)=\int_0^x\mathrm{d}u\int_x^{2\pi}\mathrm{d}v\;|K(u,v)|^2\underset{N\to\infty}{\sim}\pi^{-2}\log N.$$

(vii) Prove that $\mathrm{C}_\ell(X/\sqrt{\log N})$ converges to 0 as $N\to\infty$ for any $\ell\geq 3$. For this you can first prove the trace inequality

$$0\leq\mathrm{Tr}(K\mathbb{1}_A-(K\mathbb{1}_A)^\ell)\leq(\ell-1)\mathrm{Tr}(K\mathbb{1}_A-(K\mathbb{1}_A)^2).$$

Show that $(X-\mathbb{E}\,X)/\sqrt{\log N}$ converges weakly to a Gaussian random variable with variance π^{-2}. Compare this result to the case of N independent uniform points on the circle.

(viii) Consider $X_k=\chi([0,x_k))-Nx_k/(2\pi)$ where $x_k=N^{-\alpha_k}$, $0<\alpha_1<\cdots<\alpha_\ell<1$. Prove a joint central limit theorem for the random variables $X_1,\dots,X_\ell$ as $N\to\infty$. Compare this result to the case of N independent uniform points on the circle.

Problem 2. Loop equations and linear statistics for the Gaussian Unitary Ensemble. Consider the probability distribution of eigenvalues from the Gaussian Unitary Ensemble:

$$\mu(\mathrm{d}\boldsymbol{\lambda}) = \frac{1}{Z_N} \prod_{1\leq k<\ell\leq N} |\lambda_k - \lambda_\ell|^2 e^{-\frac{N}{2}\sum_{k=1}^N \lambda_k^2} \mathrm{d}\lambda_1 \dots \mathrm{d}\lambda_N$$

on the simplex $\lambda_1 < \dots < \lambda_N$. For a smooth $f : \mathbb{R} \to \mathbb{R}$ supported on $(-2+\kappa, 2-\kappa)$ $(\kappa > 0)$ we consider the general linear statistics $S_N(f) = \sum_{k=1}^N f(\lambda_k) - N \int f(s)\varrho(s)\mathrm{d}s$, where $\varrho(s) = (2\pi)^{-1}\sqrt{(4-s^2)_+}$. We want to prove the weak convergence of $S_N(f)$ to a Gaussian random variable for large N, with no need of any normalization.

We are interested in the Fourier transform $Z(u) = \mathbb{E}_\mu(e^{\mathrm{i}uS_N(f)})$. We will need a complex modification of the GUE, namely $\mathrm{d}\mu^u(\boldsymbol{\lambda}) = \frac{e^{\mathrm{i}uS_N(f)}}{Z(u)}\mathrm{d}\mu(\boldsymbol{\lambda})$, assuming that $Z(u) \neq 0$. Let $s_N(z) = \frac{1}{N}\sum_k \frac{1}{z-\lambda_k}$ and $m_{N,u}(z) = \mathbb{E}^{\mu^u}(s_N(z))$. The Stieltjes transform of the semicircle distribution is $m(z) = \int \frac{\varrho(s)}{z-s}\mathrm{d}s = \frac{z-\sqrt{z^2-4}}{2}$, where the square root is chosen so that m is holomorphic on $[-2,2]^{\mathrm{c}}$ and $m(z) \to 0$ as $|z| \to \infty$.

(i) Prove that

$$(m_{N,u}(z) - m(z))^2 - \sqrt{z^2-4}\,(m_{N,u}(z) - m(z)) + \frac{\mathrm{i}u}{N}\int_{\mathbb{R}} \frac{f'(s)}{z-s}\varrho_1^{(N,u)}(s)\mathrm{d}s = -\mathrm{var}_{\mu^u}(s_N(z))\,.$$

This is called the (first) loop equation. To derive it, you may first prove that

$$m_{N,u}(z)^2 + \int_{\mathbb{R}} \frac{-s + \mathrm{i}uN^{-1}f'(s)}{z-s}\varrho_1^{(N,u)}(s)\mathrm{d}s = -\mathrm{var}_{\mu^u}(s_N(z))\,.$$

Hint: integrate by parts or change variables $\lambda_k = y_k + \varepsilon(\mathfrak{Re}/\mathfrak{Im})\frac{1}{z-y_k}$ and note $\partial_{\varepsilon=0}\log Z(u) = 0$.

(ii) Remember the rigidity for Wigner matrices, in particular for GUE: for any $\xi, D > 0$ there exists $C > 0$ such that uniformly in $N \geq 1$ and $k \in [\![1, N]\!]$ we have $\mu\left(|\lambda_k - \gamma_k| > N^{-\frac{2}{3}+\xi}(\hat{k})^{-\frac{1}{3}}\right) \leq CN^{-D}$, where $\int_{-\infty}^{\gamma_k}\varrho(s\mathrm{d}s) = \frac{k}{N}$ and $\hat{k} = \min(k, N+1-k)$. Assume $Z(u) \neq 0$. Prove that

$$|\mu^u|\left(|\lambda_k - \gamma_k| > N^{-\frac{2}{3}+\xi}(\hat{k})^{-\frac{1}{3}}\right) \leq C\frac{N^{-D}}{|Z(u)|},$$

where $|\mu^u|$ is the total variation of the complex measure μ^u. Conclude that uniformly in $z = E+\mathrm{i}\eta$, $-2+\kappa < E < 2-\kappa$, $0 < |\eta| < 1$, we have

$$|\mathrm{var}_{\mu^u}(s_N(z))| = \mathrm{O}\left(\frac{N^{-2+2\xi}}{\eta^2|Z(u)|^2}\right).$$

(iii) Prove that uniformly in $-2+\kappa < E < 2-\kappa$, $N^{-1+\xi} \leq \eta \leq 1$, we have

$$m_{N,u}(z) - m(z) = \frac{1}{\sqrt{z^2-4}}\frac{\mathrm{i}u}{N}\int_{\mathbb{R}}\frac{f'(s)}{z-s}\varrho(s)\mathrm{d}s + \mathrm{O}\left(\frac{N^{-2+3\xi}}{\eta^2|Z(u)|^2}\right).$$

(iv) Let $\chi : \mathbb{R} \to \mathbb{R}^+$ be a smooth function such that $\chi(y) = 1$ for $|y| < 1/2$ and $\chi(y) = 0$ for $|y| > 1$. Prove that for any $\lambda \in \mathbb{R}$, we have

$$f(\lambda) = -\frac{1}{2\pi}\iint_{\mathbb{R}^2}\frac{\mathrm{i}yf''(x)\chi(y) + \mathrm{i}(f(x)+\mathrm{i}yf'(x))\chi'(y)}{x+\mathrm{i}y-\lambda}\mathrm{d}x\mathrm{d}y,$$

where the right hand side converges absolutely. For this, you can reproduce the proof of Cauchy's integral formula based on Green's theorem, considering the quasi-analytic extension $(f(x) + \mathrm{i}yf'(x))\chi(y)$.

(v) Note that $\partial_u \log Z(u) = \mathbb{E}_{\mu^u}(\mathrm{i}S_N(f))$. Conclude that bulk linear statistics converge to a Gaussian random variable.

Exercise 1. Fluctuations for the Ginibre ensemble. Consider the joint distribution of eigenvalues from the Ginibre ensemble,

$$\mathbb{P}(\mathrm{d}\boldsymbol{z}) = \frac{1}{Z_N} \prod_{1\leq i<j\leq N} |z_i - z_j|^2 \prod_{i=1}^{N} e^{-N|z_i|^2} \mathrm{dA}(z_i)$$

where dA is the Lebesgue measure on $\mathbb{C}$. Let $\mathscr{C}$ be a smooth Jordan curve, with interior A, finite length $\ell(\mathscr{C})$, strictly included in the unit disk $\{|z|<1\}$. Let $X_{\mathscr{C}} = \chi(A) - \mathbb{E}(\chi(A))$ where $\chi = \sum_{i=1}^{N} \delta_{z_i}$. By mimicking the method from Problem 1, prove the weak convergence

$$\frac{X_{\mathscr{C}}}{\ell(\mathscr{C})^{1/2} N^{1/4}} \to \mathscr{N}(0, c)$$

as $N \to \infty$, with some c independent of $\mathscr{C}$. What about joint convergence of $(X_{\mathscr{C}_1}, \dots, X_{\mathscr{C}_n})$ where all Jordan curves $\mathscr{C}_1, \dots, \mathscr{C}_n$ satisfy the above assumptions?

Exercise 2. The semicircle law for band matrices. Let H_N be a symmetric matrix with $H_N(i,j)$ a standard Bernoulli random variable when $|i-j| \leq W/2$ or $||i-j|-N| \leq W/2$, 0 otherwise. All entries are independent, up to the symmetry constraint. Assume $1 \ll W \leq N$.

Prove that the empirical spectral measure of $W^{-1/2}H_N$ converges (in probability, say) to the semicircle distribution $\varrho(s) = (2\pi)^{-1}\sqrt{(4-s^2)_+}$.

Open problem 1. In Exercise 1, what happens when the Jordan curve is not smooth and has infinite length? In particular, if $\log \operatorname{var}(X_{\mathscr{C}}) \sim \alpha(\mathscr{C}) \log N$, does $\alpha(\mathscr{C})$ only depend on the Hausdorff dimension of $\mathscr{C}$? Or the Minkowski dimension?

Open problem 2. In Exercise 2, let $u_1, \dots, u_N$ be the L^2-normalized eigenvectors of H_N and $\alpha \in (0,1)$, $D > 0$.

Assume $\alpha < 1/2$. Prove that there exists $\delta > 0$ such that for N greater than some $N_0(\alpha, D)$, with probability at least $1 - N^{-D}$ the following holds: for any $k \in [\![1, N]\!]$, $\|u_k\|_\infty > N^{-1/2+\delta}$.

Assume $\alpha > 1/2$. Prove that for any $\delta > 0$, for N greater than some $N_0(\alpha, D, \delta)$, with probability at least $1 - N^{-D}$ the following holds: for any $k \in [\![1, N]\!]$, $\|u_k\|_\infty < N^{-1/2+\delta}$.