Extreme Eigenvalues of Random Matrices

Satya N. Majumdar

Laboratoire de Physique Théorique et Modèles Statistiques,CNRS,
Université Paris-Sud, France

June 15, 2009

Plan

Plan:

- The problem & its origin

Plan

Plan:

- The problem & its origin

- Top eigenvalue $\lambda_{\max}$ of a Gaussian random matrix and its large deviations

Plan

Plan:

- The problem & its origin
- Top eigenvalue $\lambda_{\max}$ of a Gaussian random matrix and its large deviations
- Prob. distr. of $\lambda_{\max}$ $\Longleftrightarrow$ Coulomb gas with a wall

Plan

Plan:

- The problem & its origin
- Top eigenvalue $\lambda_{\max}$ of a Gaussian random matrix and its large deviations
- Prob. distr. of $\lambda_{\max}$ $\Longleftrightarrow$ Coulomb gas with a wall
- Extension to Wishart matrices (Laguerre ensemble)

Plan

Plan:

- The problem & its origin
- Top eigenvalue $\lambda_{\max}$ of a Gaussian random matrix and its large deviations
- Prob. distr. of $\lambda_{\max}$ $\Longleftrightarrow$ Coulomb gas with a wall
- Extension to Wishart matrices (Laguerre ensemble)
- Summary and Conclusions

A Trivial Problem

DIAGONAL MATRIX

$$\begin{bmatrix} x_{11} & & & & 0 \\ & x_{22} & & & \\ & & x_{33} & & \\ & & & \ddots & \\ 0 & & & & x_{NN} \end{bmatrix}$$

$\Pr[x_{ii} = x] = (2\pi)^{-1/2} \exp[-x^2/2]$ → GAUSSIAN

N Eigenvalues: $\lambda_i = x_{ii}$ → Independent

A Trivial Problem

DIAGONAL MATRIX

$$\begin{bmatrix} x_{11} & & & & \\ & x_{22} & & 0 & \\ & & x_{33} & & \\ & 0 & & \ddots & \\ & & & & x_{NN} \end{bmatrix}$$

$\Pr[x_{ii} = x]$

$= (2\pi)^{-1/2} \exp[-x^2/2]$

$\downarrow$

GAUSSIAN

N Eigenvalues: $\lambda_i = x_{ii} \longrightarrow$ Independent

- $P_N = \mathrm{Prob}[\lambda_1 \leq 0,\ \lambda_2 \leq 0,\ \ldots,\ \lambda_N \leq 0] = 2^{-N} = \exp[-(\ln 2)\, N]$

A Nontrivial Problem

REAL SYMMETRIC MATRIX (NxN)

$$X = \begin{vmatrix} x_{11} & x_{12} & \cdots & x_{1N} \\ x_{21} & x_{22} & \cdots & x_{2N} \\ \vdots & \vdots & & \vdots \\ x_{N1} & \cdots & \cdots & x_{NN} \end{vmatrix}$$

GAUSSIAN ↓

$\Pr[X] \propto \exp[-\frac{1}{2}\mathrm{Tr}(X^2)]$

N eigenvalues : $\lambda_1, \lambda_2, \ldots, \lambda_N$

→ strongly correlated

A Nontrivial Problem

REAL SYMMETRIC MATRIX (NxN)

$$X = \begin{vmatrix} x_{11} & x_{12} & \cdots & x_{1N} \\ x_{21} & x_{22} & \cdots & x_{2N} \\ \vdots & \vdots & & \vdots \\ x_{N1} & \cdots & \cdots & x_{NN} \end{vmatrix}$$

GAUSSIAN → $\Pr[X] \propto \exp[-\frac{1}{2}\mathrm{Tr}(X^2)]$

N eigenvalues : $\lambda_1, \lambda_2, \ldots, \lambda_N$ → strongly correlated

- $P_N = \mathrm{Prob}[\lambda_1 \le 0,\ \lambda_2 \le 0,\ \ldots,\ \lambda_N \le 0] = \mathrm{Prob}[\lambda_{\max} \le 0] = ?$

A Nontrivial Problem

REAL SYMMETRIC MATRIX (NxN)

$$X = \begin{vmatrix} x_{11} & x_{12} & \cdots & x_{1N} \\ x_{21} & x_{22} & \cdots & x_{2N} \\ \vdots & \vdots & & \vdots \\ x_{N1} & \cdots & \cdots & x_{NN} \end{vmatrix}$$

GAUSSIAN → $\Pr[X] \propto \exp[-\frac{1}{2}\mathrm{Tr}(X^2)]$

N eigenvalues : $\lambda_1, \lambda_2, \ldots, \lambda_N$ → strongly correlated

- $P_N = \mathrm{Prob}[\lambda_1 \le 0,\ \lambda_2 \le 0,\ \ldots,\ \lambda_N \le 0] = \mathrm{Prob}[\lambda_{\max} \le 0] = ?$

[R.M. May, Nature, 238, 413 (1972)——Ecosystems]

[Cavagna et. al. 2000, Fyodorov 2004, —— Glassy systems]

[Susskind 2003, Douglas et. al. 2004, Aazami & Easther 2006— String theory]......

Results for P_N:

- $P_N = \mathrm{Prob}[\lambda_1 \le 0,\ \lambda_2 \le 0,\ \ldots,\ \lambda_N \le 0] = ?$

Results for P_N:

- $P_N = \mathrm{Prob}[\lambda_1 \le 0,\ \lambda_2 \le 0,\ \ldots,\ \lambda_N \le 0] = ?$
- $N = 1$: $P_1 = 1/2 = 0.5$ (trivially)

Results for P_N:

- $P_N = \mathrm{Prob}[\lambda_1 \le 0,\ \lambda_2 \le 0,\ \ldots,\ \lambda_N \le 0] = ?$
- $N = 1$: $P_1 = 1/2 = 0.5$ (trivially)
- $N = 2$: $P_2 = \frac{2-\sqrt{2}}{4} = 0.146447..$

Results for P_N:

- $P_N = \mathrm{Prob}[\lambda_1 \le 0,\ \lambda_2 \le 0,\ \ldots,\ \lambda_N \le 0] = ?$
- $N = 1$: $P_1 = 1/2 = 0.5$ (trivially)
- $N = 2$: $P_2 = \frac{2-\sqrt{2}}{4} = 0.146447..$
- $N = 3$: $P_3 = \frac{\pi - 2\sqrt{2}}{4\pi} = 0.0249209..$
  (Beltrani 2007, Dedieu & Malajovich, 2007)

Results for P_N:

- $P_N = \mathrm{Prob}[\lambda_1 \le 0,\ \lambda_2 \le 0,\ \ldots,\ \lambda_N \le 0] = ?$
- $N = 1$: $P_1 = 1/2 = 0.5$ (trivially)
- $N = 2$: $P_2 = \frac{2-\sqrt{2}}{4} = 0.146447..$
- $N = 3$: $P_3 = \frac{\pi-2\sqrt{2}}{4\pi} = 0.0249209..$
  (Beltrani 2007, Dedieu & Malajovich, 2007)

Question: How does P_N decay for large N, i.e., $P_N \to ?$ as $N \to \infty$

Results for P_N:

- $P_N = \mathrm{Prob}[\lambda_1 \leq 0,\ \lambda_2 \leq 0,\ \ldots,\ \lambda_N \leq 0] = ?$
- $N = 1$: $P_1 = 1/2 = 0.5$ (trivially)
- $N = 2$: $P_2 = \frac{2-\sqrt{2}}{4} = 0.146447..$
- $N = 3$: $P_3 = \frac{\pi - 2\sqrt{2}}{4\pi} = 0.0249209..$
(Beltrani 2007, Dedieu & Malajovich, 2007)

Question: How does P_N decay for large N, i.e., $P_N \to ?$ as $N \to \infty$

- Based on numerics, Aazami & Easther (2006) predicted for large N:

$$\boxed{P_N \sim \exp[-\theta N^2]} \text{ with } \theta_{\mathrm{num}} \approx 0.27$$

$\to$ very small probability $\to$ RARE EVENT

Results for P_N:

- $P_N = \mathrm{Prob}[\lambda_1 \leq 0,\ \lambda_2 \leq 0,\ \ldots,\ \lambda_N \leq 0] = ?$
- $N = 1$: $P_1 = 1/2 = 0.5$ (trivially)
- $N = 2$: $P_2 = \frac{2-\sqrt{2}}{4} = 0.146447..$
- $N = 3$: $P_3 = \frac{\pi-2\sqrt{2}}{4\pi} = 0.0249209..$
  (Beltrani 2007, Dedieu & Malajovich, 2007)

Question: How does P_N decay for large N, i.e., $P_N \to ?$ as $N \to \infty$

- Based on numerics, Aazami & Easther (2006) predicted for large N:

$$\boxed{P_N \sim \exp[-\theta N^2]} \text{ with } \theta_{\mathrm{num}} \approx 0.27$$

$\to$ very small probability $\to$ RARE EVENT

- Exact result: $\boxed{\theta = \frac{1}{4}\ln(3) = 0.274653..}$ (Dean and S.M., 2006)

Results for P_N:

- $P_N = \mathrm{Prob}[\lambda_1 \leq 0,\ \lambda_2 \leq 0,\ \ldots,\ \lambda_N \leq 0] = ?$
- $N = 1$: $P_1 = 1/2 = 0.5$ (trivially)
- $N = 2$: $P_2 = \frac{2-\sqrt{2}}{4} = 0.146447..$
- $N = 3$: $P_3 = \frac{\pi - 2\sqrt{2}}{4\pi} = 0.0249209..$ (Beltrani 2007, Dedieu & Malajovich, 2007)

Question: How does P_N decay for large N, i.e., $P_N \to ?$ as $N \to \infty$

- Based on numerics, Aazami & Easther (2006) predicted for large N:

$$\boxed{P_N \sim \exp[-\theta N^2]} \text{ with } \theta_{\mathrm{num}} \approx 0.27$$

$\rightarrow$ very small probability $\rightarrow$ RARE EVENT

- Exact result: $\boxed{\theta = \frac{1}{4}\ln(3) = 0.274653..}$ (Dean and S.M., 2006)

More generally, for $\beta = 1$ (GOE), $\beta = 2$ (GUE) and $\beta = 4$ (GSE)

$$\boxed{P_N \sim \exp[-\beta\theta N^2]} \text{ for large } N$$

Complex Landscapes

A particle moving in a N-dimensional landscape: $V(y_1, y_2, \ldots, y_N)$

$$\frac{dy_i}{dt} = -\nabla_{y_i} V$$

Spin glasses, Structural glasses, Supercooled liquids, String landscapes

Complex Landscapes

A particle moving in a N-dimensional landscape: $V(y_1, y_2, \ldots, y_N)$

$$\frac{dy_i}{dt} = -\nabla_{y_i} V$$

Spin glasses, Structural glasses, Supercooled liquids, String landscapes

- Stationary point (fixed pt.) y^*: $\nabla V = 0$ at $y = y^*$

Complex Landscapes

A particle moving in a N-dimensional landscape: $V(y_1, y_2, \ldots, y_N)$

$$\frac{dy_i}{dt} = -\nabla_{y_i} V$$

Spin glasses, Structural glasses, Supercooled liquids, String landscapes

- Stationary point (fixed pt.) y^*: $\nabla V = 0$ at $y = y^*$
- Near a stationary point: $V(\{y_i\}) \approx \sum_{i,j} H_{i,j}(y_i - y_i^*)(y_j - y_j^*)$

Hessian matrix: $H_{i,j} \equiv \left[\frac{\partial^2 V}{\partial y_i \partial y_j}\right]$

Complex Landscapes

A particle moving in a N-dimensional landscape: $V(y_1, y_2, \ldots, y_N)$

$$\frac{dy_i}{dt} = -\nabla_{y_i} V$$

Spin glasses, Structural glasses, Supercooled liquids, String landscapes

- Stationary point (fixed pt.) y^*: $\nabla V = 0$ at $y = y^*$

- Near a stationary point: $V(\{y_i\}) \approx \sum_{i,j} H_{i,j}(y_i - y_i^*)(y_j - y_j^*)$

$$\text{Hessian matrix: } H_{i,j} \equiv \left[\frac{\partial^2 V}{\partial y_i \partial y_j}\right]$$

- Eigenvalues of the Hessian (stability) matrix determines the nature of the stationary point:

$\Longrightarrow$ Local Minimum, Local Maximum & Saddles

Eigenvalues of the Hessian Matrix

Examples:

- $N = 1$-dimensional surface: Hessian matrix $H = \frac{\partial^2 V}{\partial y^2}$

  If $\frac{\partial^2 V}{\partial y^2} < 0 \rightarrow$ Local Maximum; if $\frac{\partial^2 V}{\partial y^2} > 0 \rightarrow$ Local Minimum

Eigenvalues of the Hessian Matrix

Examples:

- $N = 1$-dimensional surface: Hessian matrix $H = \frac{\partial^2 V}{\partial y^2}$

  If $\frac{\partial^2 V}{\partial y^2} < 0 \rightarrow$ Local Maximum; if $\frac{\partial^2 V}{\partial y^2} > 0 \rightarrow$ Local Minimum

- $N = 2$-dimensional surface: Hessian matrix $H \equiv \begin{pmatrix} \frac{\partial^2 V}{\partial y_1^2} & \frac{\partial^2 V}{\partial y_1 \partial y_2} \\ \frac{\partial^2 V}{\partial y_2 \partial y_1} & \frac{\partial^2 V}{\partial y_2^2} \end{pmatrix}$

Eigenvalues of the Hessian Matrix

Examples:

- $N = 1$-dimensional surface: Hessian matrix $H = \frac{\partial^2 V}{\partial y^2}$

  If $\frac{\partial^2 V}{\partial y^2} < 0 \rightarrow$ Local Maximum; if $\frac{\partial^2 V}{\partial y^2} > 0 \rightarrow$ Local Minimum

- $N = 2$-dimensional surface: Hessian matrix $H \equiv \begin{pmatrix} \frac{\partial^2 V}{\partial y_1^2} & \frac{\partial^2 V}{\partial y_1 \partial y_2} \\ \frac{\partial^2 V}{\partial y_2 \partial y_1} & \frac{\partial^2 V}{\partial y_2^2} \end{pmatrix}$

  Two real eigenvalues: (λ_1, λ_2)

  If $\lambda_1 < 0$ and $\lambda_2 < 0 \rightarrow$ Local Maximum

  If $\lambda_1 > 0$ and $\lambda_2 > 0 \rightarrow$ Local Minimum

  $$\left.\begin{matrix} \lambda_1 < 0, & \lambda_2 > 0 \\ \lambda_1 > 0, & \lambda_2 < 0 \end{matrix}\right\} \quad \rightarrow \text{Saddle}$$

Fraction of Local Maximum/Minimum in Random Hessian Model

- Random $(N \times N)$ Hessian Model: $H \equiv \left[\frac{\partial^2 V}{\partial y_i \partial y_j}\right]$

  $H \to (N \times N)$ real symmetric random matrix

Fraction of Local Maximum/Minimum in Random Hessian Model

- Random $(N \times N)$ Hessian Model: $H \equiv \left[\frac{\partial^2 V}{\partial y_i \partial y_j}\right]$

  $H \to (N \times N)$ real symmetric random matrix

- If the entries are Gaussian: $H \to \text{(GOE)}$

  [Cavagna et. al. (2000), ..., Aazami & Easther (2006)]

Fraction of Local Maximum/Minimum in Random Hessian Model

- Random $(N \times N)$ Hessian Model: $H \equiv \left[\frac{\partial^2 V}{\partial y_i \partial y_j}\right]$

  $H \to (N \times N)$ real symmetric random matrix

- If the entries are Gaussian: $H \to$ (GOE)

  [Cavagna et. al. (2000), ..., Aazami & Easther (2006)]

- In the random Hessian model

  $P_N =$ Fraction of Local Maxima/minima

  $= \text{Prob}[\lambda_1 < 0, \lambda_2 < 0, \ldots, \lambda_N < 0]$

  $= \text{Prob}[\lambda_{\max} < 0]$

  $\xrightarrow[N \to \infty]{} \exp[-\theta N^2] \to$ very small

Fraction of Local Maximum/Minimum in Random Hessian Model

- Random $(N \times N)$ Hessian Model: $H \equiv \left[\frac{\partial^2 V}{\partial y_i \partial y_j}\right]$

  $H \to (N \times N)$ real symmetric random matrix

- If the entries are Gaussian: $H \to$ (GOE)

  [Cavagna et. al. (2000), ..., Aazami & Easther (2006)]

- In the random Hessian model

  $P_N =$ Fraction of Local Maxima/minima

  $$= \text{Prob}[\lambda_1 < 0, \lambda_2 < 0, \ldots, \lambda_N < 0]$$

  $$= \text{Prob}[\lambda_{\max} < 0]$$

  $$\xrightarrow[N \to \infty]{} \exp[-\theta N^2] \to \text{very small}$$

- $\longrightarrow$ Most of the stationary points $\to$ Saddles

Gaussian Random Matrices

- $(N \times N)$ Gaussian random matrix: $X \equiv [x_{i,j}]$

Gaussian Random Matrices

- $(N \times N)$ Gaussian random matrix: $X \equiv [x_{i,j}]$
- Ensembles: Orthogonal (GOE), Unitary (GUE) or Symplectic (GSE)
- $\mathrm{Prob}[x_{i,j}] \propto \exp\left[-\frac{\beta}{2}\,\mathrm{Tr}(X, X)\right]$

  where the Dyson index $\beta = 1$ (GOE), $\beta = 2$ (GUE) or $\beta = 4$ (GSE)

Gaussian Random Matrices

- $(N \times N)$ Gaussian random matrix: $X \equiv [x_{i,j}]$
- Ensembles: Orthogonal (GOE), Unitary (GUE) or Symplectic (GSE)
- $\mathrm{Prob}[x_{i,j}] \propto \exp\left[-\frac{\beta}{2}\,\mathrm{Tr}(X,X)\right]$

  where the Dyson index $\beta = 1$ (GOE), $\beta = 2$ (GUE) or $\beta = 4$ (GSE)
- N real eigenvalues $\{\lambda_1, \lambda_2, \ldots, \lambda_N\} \to$ correlated random variables

Gaussian Random Matrices

- $(N \times N)$ Gaussian random matrix: $X \equiv [x_{i,j}]$
- Ensembles: Orthogonal (GOE), Unitary (GUE) or Symplectic (GSE)
- $\mathrm{Prob}[x_{i,j}] \propto \exp\left[-\frac{\beta}{2}\,\mathrm{Tr}(X,X)\right]$

  where the Dyson index $\beta = 1$ (GOE), $\beta = 2$ (GUE) or $\beta = 4$ (GSE)
- N real eigenvalues $\{\lambda_1, \lambda_2, \ldots, \lambda_N\} \rightarrow$ correlated random variables
- Joint distribution of eigenvalues (Wigner, 1951)

$$P(\lambda_1, \lambda_2, \ldots, \lambda_N) = \frac{1}{Z_N} \exp\left[-\frac{\beta}{2} \sum_{i=1}^{N} \lambda_i^2\right] \prod_{j<k} |\lambda_j - \lambda_k|^\beta$$

Gaussian Random Matrices

- $(N \times N)$ Gaussian random matrix: $X \equiv [x_{i,j}]$
- Ensembles: Orthogonal (GOE), Unitary (GUE) or Symplectic (GSE)
- $\mathrm{Prob}[x_{i,j}] \propto \exp\left[-\frac{\beta}{2}\,\mathrm{Tr}(X,X)\right]$

  where the Dyson index $\beta = 1$ (GOE), $\beta = 2$ (GUE) or $\beta = 4$ (GSE)
- N real eigenvalues $\{\lambda_1, \lambda_2, \ldots, \lambda_N\} \to$ correlated random variables
- Joint distribution of eigenvalues (Wigner, 1951)

$$\boxed{P(\lambda_1, \lambda_2, \ldots, \lambda_N) = \frac{1}{Z_N} \exp\left[-\frac{\beta}{2}\sum_{i=1}^{N} \lambda_i^2\right] \prod_{j<k} |\lambda_j - \lambda_k|^{\beta}}$$

- where Z_N = Partition Function

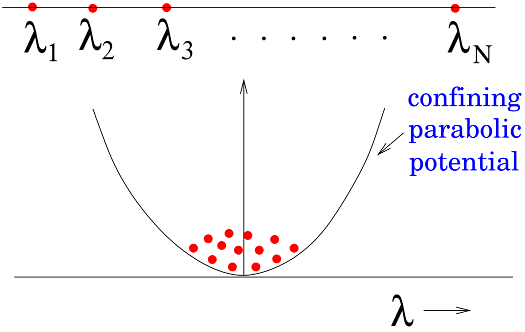
$$= \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} \{\prod_i d\lambda_i\} \exp\left[-\frac{\beta}{2}\sum_{i=1}^{N} \lambda_i^2\right] \prod_{j<k} |\lambda_j - \lambda_k|^{\beta}$$

Coulomb Gas interpretation

- $Z_N = \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} \{\prod_i d\lambda_i\} \exp\left[-\frac{\beta}{2}\left\{\sum_{i=1}^{N} \lambda_i^2 - \sum_{j\neq k} \log|\lambda_j - \lambda_k|\right\}\right]$

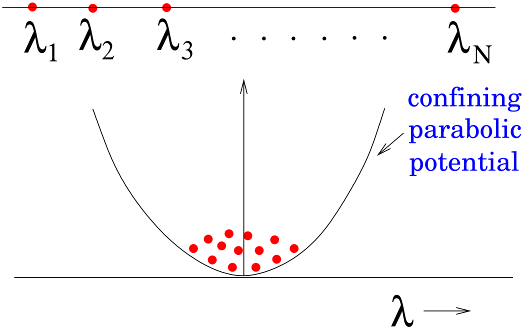
Coulomb Gas interpretation

- $Z_N = \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} \{\prod_i d\lambda_i\} \exp\left[-\frac{\beta}{2}\left\{\sum_{i=1}^{N} \lambda_i^2 - \sum_{j\neq k} \log|\lambda_j - \lambda_k|\right\}\right]$
- 2-d Coulomb gas confined to a line (Dyson) with $\beta \to$ inverse temp.



Coulomb Gas interpretation

- $Z_N = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \{\prod_i d\lambda_i\} \exp\left[-\frac{\beta}{2}\left\{\sum_{i=1}^{N} \lambda_i^2 - \sum_{j\neq k} \log|\lambda_j - \lambda_k|\right\}\right]$
- 2-d Coulomb gas confined to a line (Dyson) with $\beta \to$ inverse temp.



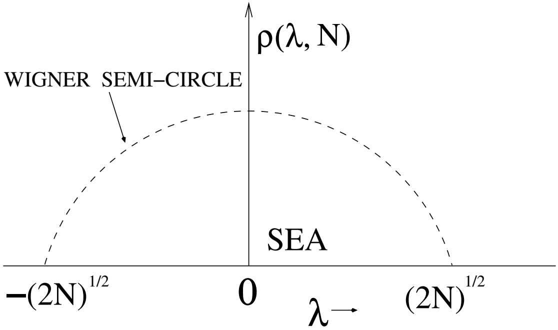
- Balance of energy $\longrightarrow N\,\lambda^2 \sim N^2$
- Typical eigenvalue: $\lambda_{\rm typ} \sim \sqrt{N}$ for large N

Spectral Density: Wigner's Semicircle Law

- Av. density of states: $\rho(\lambda, N) = \langle \frac{1}{N} \sum_{i=1}^{N} \delta(\lambda - \lambda_i) \rangle$

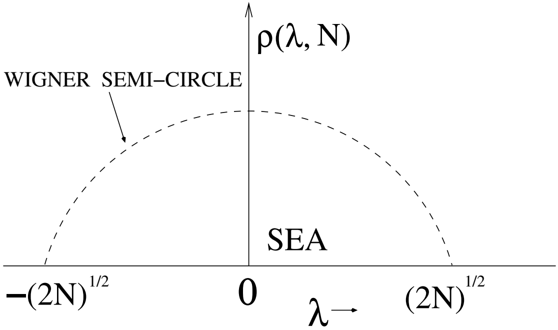
Spectral Density: Wigner's Semicircle Law

- Av. density of states: $\rho(\lambda, N) = \langle \frac{1}{N} \sum_{i=1}^{N} \delta(\lambda - \lambda_i) \rangle$
- Wigner's Semi-circle: $\rho(\lambda, N) \xrightarrow[N\to\infty]{} \sqrt{\frac{2}{N\pi^2}} \left[1 - \frac{\lambda^2}{2N}\right]^{1/2}$



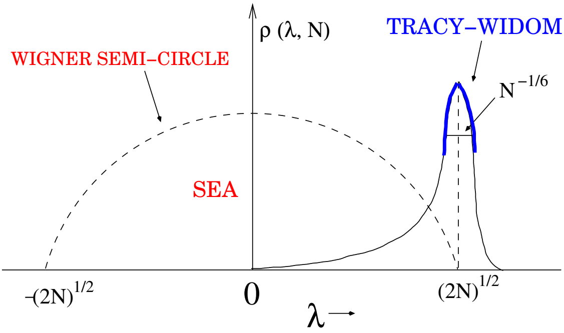
Spectral Density: Wigner's Semicircle Law

- Av. density of states: $\rho(\lambda, N) = \langle \frac{1}{N} \sum_{i=1}^{N} \delta(\lambda - \lambda_i) \rangle$
- Wigner's Semi-circle: $\rho(\lambda, N) \xrightarrow[N\to\infty]{} \sqrt{\frac{2}{N\pi^2}} \left[1 - \frac{\lambda^2}{2N}\right]^{1/2}$

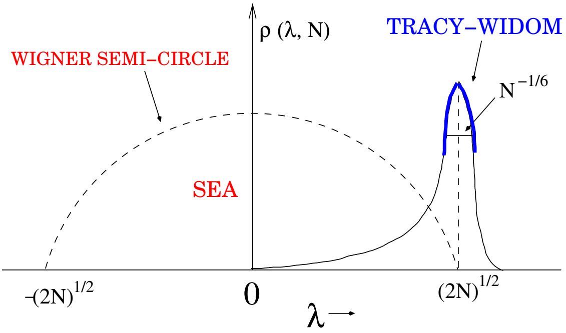


- $\langle \lambda_{\max} \rangle = \sqrt{2N}$ for large N.
- $\lambda_{\max}$ fluctuates from one sample to another. $\text{Prob}[\lambda_{\max}, N] = ?$

Tracy-Widom distribution for $\lambda_{\max}$

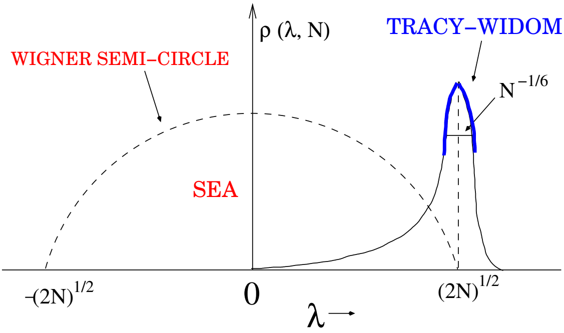


Tracy-Widom distribution for $\lambda_{\max}$



- $\langle \lambda_{\max} \rangle = \sqrt{2N}$; typical fluctuation: $|\lambda_{\max} - \sqrt{2N}| \sim N^{-1/6}$ (small)

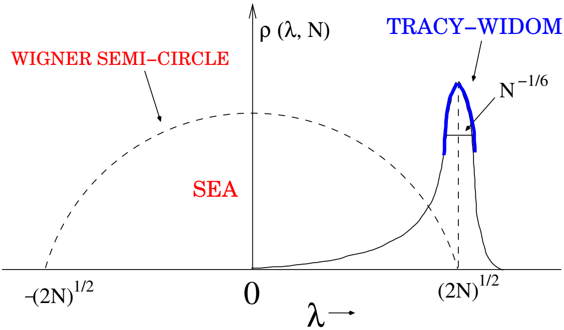
Tracy-Widom distribution for $\lambda_{\max}$



- $\langle \lambda_{\max} \rangle = \sqrt{2N}$; typical fluctuation: $|\lambda_{\max} - \sqrt{2N}| \sim N^{-1/6}$ (small)
- typical fluctuations are distributed via Tracy-Widom (1994):
- cumulative distribution:

$$\mathrm{Prob}[\lambda_{\max} \leq t, N] \rightarrow F_\beta\left(\sqrt{2}N^{1/6}(t - \sqrt{2N})\right)$$

Tracy-Widom distribution for $\lambda_{\max}$

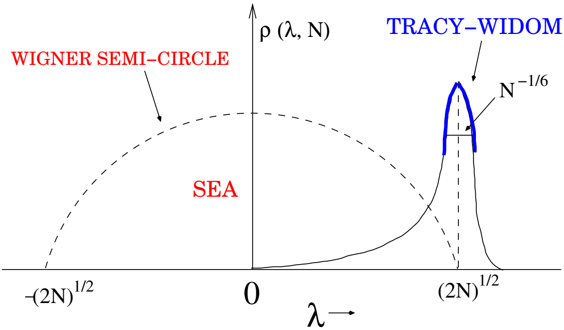


- $\langle \lambda_{\max} \rangle = \sqrt{2N}$; typical fluctuation: $|\lambda_{\max} - \sqrt{2N}| \sim N^{-1/6}$ (small)
- typical fluctuations are distributed via Tracy-Widom (1994):
- cumulative distribution:

$$\mathrm{Prob}[\lambda_{\max} \le t, N] \to F_\beta\left(\sqrt{2}N^{1/6}(t - \sqrt{2N})\right)$$

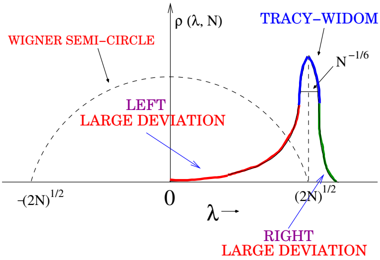
- Prob. density (pdf): $f_\beta(z) = dF_\beta(z)/dz$

Tracy-Widom distribution for $\lambda_{\max}$

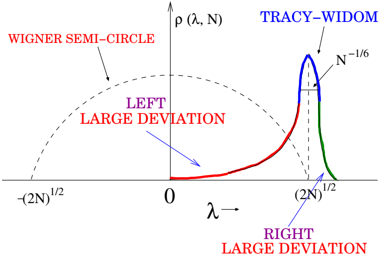


- $\langle \lambda_{\max} \rangle = \sqrt{2N}$; typical fluctuation: $|\lambda_{\max} - \sqrt{2N}| \sim N^{-1/6}$ (small)
- typical fluctuations are distributed via Tracy-Widom (1994):
- cumulative distribution:
$$\mathrm{Prob}[\lambda_{\max} \le t, N] \to F_\beta\left(\sqrt{2}N^{1/6}(t - \sqrt{2N})\right)$$
- Prob. density (pdf): $f_\beta(z) = dF_\beta(z)/dz$
- $F_\beta(z) \to$ solution of Painlevé equation

Probability of Large Deviations of $\lambda_{\max}$:

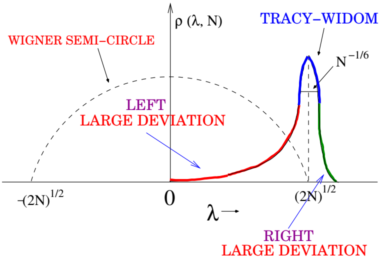


Probability of Large Deviations of $\lambda_{\max}$:



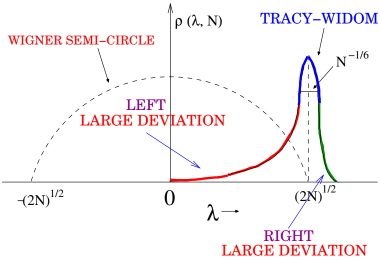
- Tracy-Widom law $\mathrm{Prob}[\lambda_{\max} \leq t, N] \to F_\beta\left(\sqrt{2}N^{1/6}(t - \sqrt{2N})\right)$ describes the prob. of typical (small) fluctuations of $\sim O(N^{-1/6})$ around the mean $\sqrt{2N}$, i.e., when $|\lambda_{\max} - \sqrt{2N}| \sim N^{-1/6}$

Probability of Large Deviations of $\lambda_{\max}$:



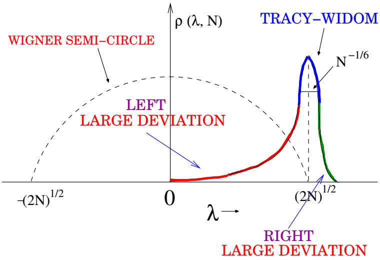
- Tracy-Widom law $\mathrm{Prob}[\lambda_{\max} \le t, N] \to F_\beta\left(\sqrt{2}N^{1/6}(t - \sqrt{2N})\right)$ describes the prob. of typical (small) fluctuations of $\sim O(N^{-1/6})$ around the mean $\sqrt{2N}$, i.e., when $|\lambda_{\max} - \sqrt{2N}| \sim N^{-1/6}$
- Q: How to describe the prob. of large (atypical) fluctuations?

Probability of Large Deviations of $\lambda_{\max}$:



- Tracy-Widom law $\mathrm{Prob}[\lambda_{\max} \le t, N] \to F_\beta\left(\sqrt{2}N^{1/6}(t-\sqrt{2N})\right)$ describes the prob. of typical (small) fluctuations of $\sim O(N^{-1/6})$ around the mean $\sqrt{2N}$, i.e., when $|\lambda_{\max}-\sqrt{2N}| \sim N^{-1/6}$
- Q: How to describe the prob. of large (atypical) fluctuations?
- Ex: $P_N = \mathrm{Prob}[\lambda_{\max} \le 0,\ N] = \mathrm{Prob}[\lambda_1 \le 0, \lambda_2 \le 0, \ldots, \lambda_N \le 0]$

Probability of Large Deviations of $\lambda_{\max}$:



- Tracy-Widom law $\mathrm{Prob}[\lambda_{\max} \le t, N] \to F_\beta\left(\sqrt{2}N^{1/6}(t - \sqrt{2N})\right)$ describes the prob. of typical (small) fluctuations of $\sim O(N^{-1/6})$ around the mean $\sqrt{2N}$, i.e., when $|\lambda_{\max} - \sqrt{2N}| \sim N^{-1/6}$
- Q: How to describe the prob. of large (atypical) fluctuations?
- Ex: $P_N = \mathrm{Prob}[\lambda_{\max} \le 0,\ N] = \mathrm{Prob}[\lambda_1 \le 0, \lambda_2 \le 0, \ldots, \lambda_N \le 0]$

  $\longrightarrow$ $t = 0$, i.e., fluctuation$=t - \sqrt{2N} = -\sqrt{2N} \to$ large

Exact Left Large Deviation Function

- For large $\sim O(\sqrt{N})$ negative deviation: $\sqrt{2N} - t \sim O(\sqrt{N})$

$$\mathrm{Prob}[\lambda_{\max} \le t, N] \sim \exp\left[-\beta\, N^2\, \Phi_-\left(\frac{\sqrt{2N}-t}{\sqrt{N}}\right)\right]$$

Exact Left Large Deviation Function

- For large $\sim O(\sqrt{N})$ negative deviation: $\sqrt{2N} - t \sim O(\sqrt{N})$

$$\mathrm{Prob}[\lambda_{\max} \le t, N] \sim \exp\left[-\beta\, N^2\, \Phi_-\left(\frac{\sqrt{2N}-t}{\sqrt{N}}\right)\right]$$

- where $\Phi_-(y) \to$ left large deviation function:

$$\begin{aligned}\Phi_-(y = z + \sqrt{2}) &= \frac{1}{108}\Big[36z^2 - z^4 + (15z + z^3)\sqrt{z^2+6} \\ &+ 27\left(\ln(18) - 2\ln(-z + \sqrt{6+z^2})\right)\Big]\end{aligned}$$

(D.Dean & S.M., 2006, 2008)

Exact Left Large Deviation Function

- For large $\sim O(\sqrt{N})$ negative deviation: $\sqrt{2N} - t \sim O(\sqrt{N})$

$$\mathrm{Prob}[\lambda_{\max} \le t, N] \sim \exp\left[-\beta\, N^2\, \Phi_-\left(\frac{\sqrt{2N}-t}{\sqrt{N}}\right)\right]$$

- where $\Phi_-(y) \to$ left large deviation function:

$$\begin{aligned} \Phi_-(y = z + \sqrt{2}) &= \frac{1}{108}\Big[36z^2 - z^4 + (15z + z^3)\sqrt{z^2+6} \\ &+ 27\left(\ln(18) - 2\ln(-z + \sqrt{6+z^2})\right)\Big] \end{aligned}$$

(D.Dean & S.M., 2006, 2008)

- In particular, with $t = 0$,

$$P_N = \mathrm{Prob}[\lambda_{\max} \le 0, N] \sim \exp[-\beta\,\theta\, N^2]$$

$$\boxed{\theta = \Phi_-(\sqrt{2}) = \frac{1}{4}\ln(3) = 0.274653\ldots}$$

Exact Right Large Deviation Function

- For large $\sim O(\sqrt{N})$ positive deviation: $t - \sqrt{2N} \sim O(\sqrt{N})$

$$\mathrm{Prob}[\lambda_{\max} = t, N] \sim \exp\left[-\beta\, N\, \Phi_+\left(\frac{t-\sqrt{2N}}{\sqrt{N}}\right)\right]$$

Exact Right Large Deviation Function

- For large $\sim O(\sqrt{N})$ positive deviation: $t - \sqrt{2N} \sim O(\sqrt{N})$

$$\mathrm{Prob}[\lambda_{\max} = t, N] \sim \exp\left[-\beta\, N\, \Phi_+\left(\frac{t-\sqrt{2N}}{\sqrt{N}}\right)\right]$$

- where $\Phi_+(y) \to$ right large deviation function:

$$\Phi_+(y = z - \sqrt{2}) = \frac{z^2 - 1}{2} - \ln(z\sqrt{2}) + \frac{1}{4z^2}\, G(2/z^2)$$

where $G(z) = {}_3F_2[\{1, 1, 3/2\}, 2, 3, z] \to$ Hypergeometric function

(S.M. & Vergassola, 2009)

Exact Right Large Deviation Function

- For large $\sim O(\sqrt{N})$ positive deviation: $t - \sqrt{2N} \sim O(\sqrt{N})$

$$\mathrm{Prob}[\lambda_{\max} = t, N] \sim \exp\left[-\beta\, N\, \Phi_+\left(\frac{t-\sqrt{2N}}{\sqrt{N}}\right)\right]$$

- where $\Phi_+(y) \to$ right large deviation function:

$$\Phi_+(y = z - \sqrt{2}) = \frac{z^2 - 1}{2} - \ln(z\sqrt{2}) + \frac{1}{4z^2}\, G(2/z^2)$$

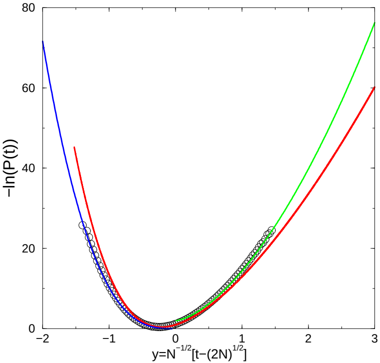
where $G(z) =_3 F_2[\{1, 1, 3/2\}, 2, 3, z] \to$ Hypergeometric function

(S.M. & Vergassola, 2009)

- In particular,

$$\begin{aligned} \Phi_+(y) &\approx \frac{2^{7/4}}{3}\, y^{3/2} \quad \text{as} \quad y \to 0 \\ &\approx y^2/2 \quad \text{as} \quad y \to \infty \end{aligned}$$

Comparison with Simulations:



$N \times N$ real Gaussian matrix ($\beta = 1$): $N = 10$ (S.M. & Vergassola, 2009)

circles $\longrightarrow$ simulation points

red line $\longrightarrow$ Tracy-Widom

blue line $\longrightarrow$ left large deviation function ($\times N^2$)

green line $\longrightarrow$ right large deviation function ($\times N$).

Left Large Deviation: Sketch of the Method

$$\mathrm{Prob}[\lambda_{\max} \le t,\ N] = \frac{Z_N(t)}{Z_N(\infty)}$$

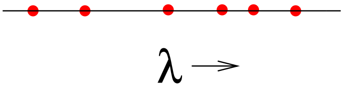
$$Z_N(t) = \int_{-\infty}^{t} \dots \int_{-\infty}^{t} \{\prod_i d\lambda_i\} \exp\left[-\frac{\beta}{2}\left\{\sum_{i=1}^{N} \lambda_i^2 - \sum_{j\neq k} \log|\lambda_j - \lambda_k|\right\}\right]$$

Left Large Deviation: Sketch of the Method

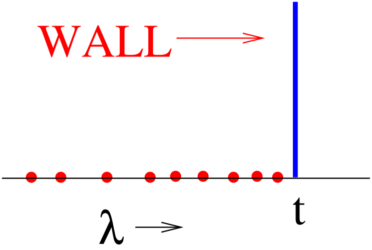
$$\text{Prob}[\lambda_{\max} \le t,\, N] = \frac{Z_N(t)}{Z_N(\infty)}$$

$$Z_N(t) = \int_{-\infty}^{t} \dots \int_{-\infty}^{t} \{\prod_i d\lambda_i\} \exp\left[-\frac{\beta}{2}\left\{\sum_{i=1}^{N} \lambda_i^2 - \sum_{j\neq k} \log|\lambda_j - \lambda_k|\right\}\right]$$

denominator



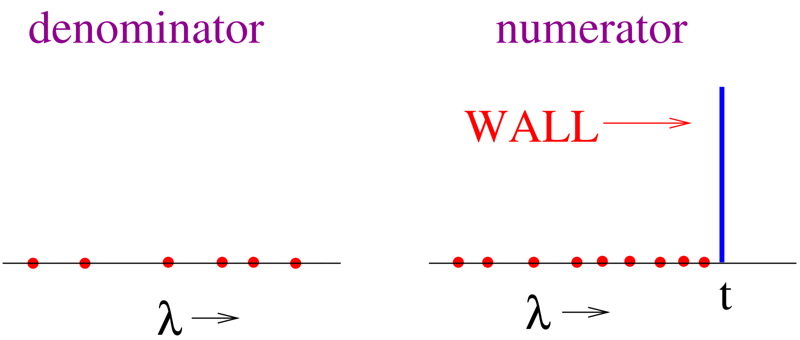
numerator



Left Large Deviation: Sketch of the Method

$$\mathrm{Prob}[\lambda_{\max} \leq t,\, N] = \frac{Z_N(t)}{Z_N(\infty)}$$

$$Z_N(t) = \int_{-\infty}^{t} \cdots \int_{-\infty}^{t} \{\prod_i d\lambda_i\} \exp\left[-\frac{\beta}{2}\left\{\sum_{i=1}^{N} \lambda_i^2 - \sum_{j\neq k} \log|\lambda_j - \lambda_k|\right\}\right]$$



Multidimensional Integral $\longrightarrow$ Functional Integral

- Scaled variables: $x_i = \lambda_i/\sqrt{N}$;   maximum x_i: $w = t/\sqrt{N}$

Multidimensional Integral $\longrightarrow$ Functional Integral

- Scaled variables: $x_i = \lambda_i/\sqrt{N}$; maximum x_i: $w = t/\sqrt{N}$
- $$\boxed{Z_N(t) \propto \int_{-\infty}^{t} \prod_i dx_i \exp\left[-\beta N^2 H\left(\{x_i\}\right)\right]}$$

$$H\left(\{x_i\}\right) = \frac{1}{2N}\sum_i x_i^2 - \frac{1}{2N^2}\sum_{j\neq k} \log|x_j - x_k|$$

Multidimensional Integral $\longrightarrow$ Functional Integral

- Scaled variables: $x_i = \lambda_i/\sqrt{N}$; maximum x_i: $w = t/\sqrt{N}$
- $$\boxed{Z_N(t) \propto \int_{-\infty}^{t} \prod_i dx_i \exp\left[-\beta N^2 H\left(\{x_i\}\right)\right]}$$

$$H\left(\{x_i\}\right) = \frac{1}{2N}\sum_i x_i^2 - \frac{1}{2N^2}\sum_{j\neq k} \log|x_j - x_k|$$

- Introduce counting function (scaled density): $f(x) = \frac{1}{N}\sum_i \delta(x - x_i)$

Multidimensional Integral $\longrightarrow$ Functional Integral

- Scaled variables: $x_i = \lambda_i/\sqrt{N}$; maximum x_i: $w = t/\sqrt{N}$
- $$Z_N(t) \propto \int_{-\infty}^{t} \prod_i dx_i \exp\left[-\beta N^2 H\left(\{x_i\}\right)\right]$$

$$H\left(\{x_i\}\right) = \frac{1}{2N}\sum_i x_i^2 - \frac{1}{2N^2}\sum_{j\neq k} \log|x_j - x_k|$$

- Introduce counting function (scaled density): $f(x) = \frac{1}{N}\sum_i \delta(x - x_i)$
- discrete sum $\longrightarrow$ continuous integral:

$$H\left[f(x)\right] = \int_{-\infty}^{w} x^2\, f(x)\, dx - \int_{\infty}^{w}\int_{-\infty}^{w} \ln|x - x'|\, f(x)\, f(x')\, dx\, dx'$$

Multidimensional Integral $\longrightarrow$ Functional Integral

- Scaled variables: $x_i = \lambda_i/\sqrt{N}$; maximum x_i: $w = t/\sqrt{N}$
- $$\boxed{Z_N(t) \propto \int_{-\infty}^{t} \prod_i dx_i \exp\left[-\beta N^2 H\left(\{x_i\}\right)\right]}$$

$$H\left(\{x_i\}\right) = \frac{1}{2N}\sum_i x_i^2 - \frac{1}{2N^2}\sum_{j\neq k} \log|x_j - x_k|$$

- Introduce counting function (scaled density): $f(x) = \frac{1}{N}\sum_i \delta(x - x_i)$
- discrete sum $\longrightarrow$ continuous integral:

$$\boxed{H\left[f(x)\right] = \int_{-\infty}^{w} x^2 f(x)\,dx - \int_{\infty}^{w}\int_{-\infty}^{w} \ln|x - x'|\, f(x)\, f(x')\,dx\,dx'}$$

$$Z_N(w) \propto \int \mathcal{D}f(x)\,\exp\left[-\beta\, N^2\left\{H\left[f(x)\right] + C\left(\int f(x)dx - 1\right)\right\} + O(N)\right]$$

Multidimensional Integral $\longrightarrow$ Functional Integral

- Scaled variables: $x_i = \lambda_i/\sqrt{N}$; maximum x_i: $w = t/\sqrt{N}$
- $$Z_N(t) \propto \int_{-\infty}^{t} \prod_i dx_i \exp\left[-\beta N^2 H(\{x_i\})\right]$$

$$H(\{x_i\}) = \frac{1}{2N}\sum_i x_i^2 - \frac{1}{2N^2}\sum_{j\neq k} \log|x_j - x_k|$$

- Introduce counting function (scaled density): $f(x) = \frac{1}{N}\sum_i \delta(x - x_i)$
- discrete sum $\longrightarrow$ continuous integral:

$$H[f(x)] = \int_{-\infty}^{w} x^2 f(x)\,dx - \int_{\infty}^{w}\int_{-\infty}^{w} \ln|x - x'|\, f(x)\, f(x')\,dx\,dx'$$

$$Z_N(w) \propto \int \mathcal{D}f(x)\, \exp\left[-\beta N^2 \left\{H[f(x)] + C\left(\int f(x)dx - 1\right)\right\} + O(N)\right]$$

- for large N, minimize the action $S[f(x)] = H[f(x)] + C\left[\int f(x)dx - 1\right]$ (saddle point): $\frac{\delta S}{\delta f} = 0 \rightarrow f_w(x)$

Multidimensional Integral $\longrightarrow$ Functional Integral

- Scaled variables: $x_i = \lambda_i/\sqrt{N}$; maximum x_i: $w = t/\sqrt{N}$
- $$\boxed{Z_N(t) \propto \int_{-\infty}^{t} \prod_i dx_i \exp\left[-\beta N^2 H(\{x_i\})\right]}$$

$$H(\{x_i\}) = \frac{1}{2N}\sum_i x_i^2 - \frac{1}{2N^2}\sum_{j\neq k} \log|x_j - x_k|$$

- Introduce counting function (scaled density): $f(x) = \frac{1}{N}\sum_i \delta(x - x_i)$
- discrete sum $\longrightarrow$ continuous integral:

$$\boxed{H[f(x)] = \int_{-\infty}^{w} x^2 f(x)\, dx - \int_{\infty}^{w}\int_{-\infty}^{w} \ln|x - x'|\, f(x)\, f(x')\, dx\, dx'}$$

$$Z_N(w) \propto \int \mathcal{D}f(x)\, \exp\left[-\beta N^2 \left\{H[f(x)] + C\left(\int f(x)dx - 1\right)\right\} + O(N)\right]$$

- for large N, minimize the action $S[f(x)] = H[f(x)] + C\left[\int f(x)dx - 1\right]$

(saddle point): $\frac{\delta S}{\delta f} = 0 \rightarrow f_w(x) \rightarrow$ $\boxed{Z_N(w) \sim \exp\left[-\beta N^2 S[f_w(x)]\right]}$

Saddle Point Solution

- saddle point $\frac{\delta S}{\delta f} = 0 \longrightarrow$ singular integral Eq. for $f_w(x)$

Saddle Point Solution

- saddle point $\frac{\delta S}{\delta f} = 0 \longrightarrow$ singular integral Eq. for $f_w(x)$

- $$\boxed{x = \mathcal{P} \int_{-\infty}^{w} \frac{f_w(y)\, dy}{x - y}}$$ $\longrightarrow$ Semi-Hilbert transform

$\longrightarrow$ Inverse electrostatic problem $\longrightarrow$ Given the potential x find the charge density $f_w(x)$

Saddle Point Solution

- saddle point $\frac{\delta S}{\delta f} = 0 \to$ singular integral Eq. for $f_w(x)$

- $$\boxed{x = \mathcal{P}\int_{-\infty}^{w} \frac{f_w(y)\,dy}{x-y}}$$ $\longrightarrow$ Semi-Hilbert transform

$\longrightarrow$ Inverse electrostatic problem $\longrightarrow$ Given the potential x find the charge density $f_w(x)$

- General method for solving such singular integral equations $\longrightarrow$ Tricomi (1957)

Exact Saddle Point Solution

- Exact solution (D. Dean and S.M., 2008):

$$f_w(x) = \frac{\sqrt{x + L_1(w)}}{2\pi\sqrt{w - x}} \, [w + L_1(w) - 2x]$$

where $-L_1(w) \leq x \leq w$ and $L_1(w) = [2\sqrt{w^2 + 6} - w]/3$

Exact Saddle Point Solution

- Exact solution (D. Dean and S.M., 2008):

$$f_w(x) = \frac{\sqrt{x + L_1(w)}}{2\pi\sqrt{w - x}} \, [w + L_1(w) - 2x]$$

where $-L_1(w) \leq x \leq w$ and $L_1(w) = [2\sqrt{w^2 + 6} - w]/3$

- When $w \to \infty$, $L_1(w) \to \sqrt{2}$ and $f_\infty(x) = \sqrt{2 - x^2}/\pi \to$ semicircle

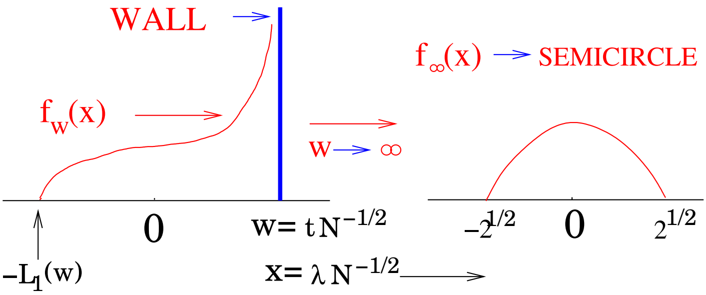
Exact Saddle Point Solution

- Exact solution (D. Dean and S.M., 2008):

$$f_w(x) = \frac{\sqrt{x + L_1(w)}}{2\pi\sqrt{w - x}}\,[w + L_1(w) - 2x]$$

where $-L_1(w) \le x \le w$ and $L_1(w) = [2\sqrt{w^2+6} - w]/3$

- When $w \to \infty$, $L_1(w) \to \sqrt{2}$ and $f_\infty(x) = \sqrt{2 - x^2}/\pi \to$ semicircle



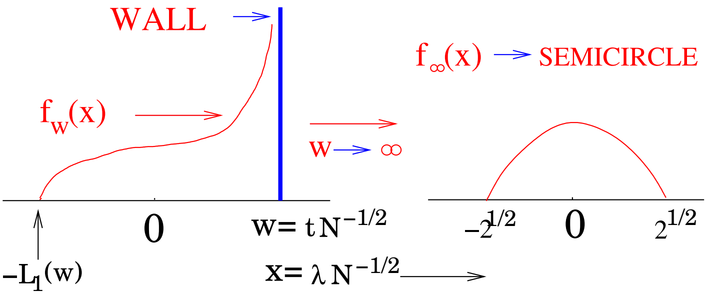
Exact Saddle Point Solution

- Exact solution (D. Dean and S.M., 2008):

$$f_w(x) = \frac{\sqrt{x+L_1(w)}}{2\pi\sqrt{w-x}}\,[w+L_1(w)-2x]$$

where $-L_1(w) \le x \le w$ and $L_1(w) = [2\sqrt{w^2+6}-w]/3$

- When $w \to \infty$, $L_1(w) \to \sqrt{2}$ and $f_\infty(x) = \sqrt{2-x^2}/\pi \to$ semicircle



Large Deviation Function

- $$\begin{aligned}\mathrm{Prob}[\lambda_{\max} \le t, N] &= \frac{Z_N(t)}{Z_N(\infty)} \\ &\sim \exp\left[-\beta N^2 \left\{S[f_{w=t/\sqrt{N}}(x)] - S[f_\infty(x)]\right\}\right] \\ &\sim \exp\left[-\beta N^2\, \Phi_-\left(\frac{\sqrt{2N}-t}{\sqrt{N}}\right)\right]\end{aligned}$$

Large Deviation Function

•

$$\begin{aligned}\mathrm{Prob}[\lambda_{\max} \le t, N] &= \frac{Z_N(t)}{Z_N(\infty)} \\ &\sim \exp\left[-\beta N^2 \left\{S[f_{w=t/\sqrt{N}}(x)] - S[f_\infty(x)]\right\}\right] \\ &\sim \exp\left[-\beta N^2 \, \Phi_-\left(\frac{\sqrt{2N}-t}{\sqrt{N}}\right)\right]\end{aligned}$$

• where $\Phi_-(y) \to$ left large deviation function:

$$\begin{aligned}\Phi_-(y = z+\sqrt{2}) &= \frac{1}{108}\left[36z^2 - z^4 + (15z + z^3)\sqrt{z^2+6}\right. \\ &+ \left. 27\left(\ln(18) - 2\ln(-z + \sqrt{6+z^2})\right)\right]\end{aligned}$$

Large Deviation Function

-

$$\mathrm{Prob}[\lambda_{\max} \le t, N] = \frac{Z_N(t)}{Z_N(\infty)}$$
$$\sim \exp\left[-\beta N^2 \left\{ S[f_{w=t/\sqrt{N}}(x)] - S[f_\infty(x)] \right\}\right]$$
$$\sim \exp\left[-\beta N^2\, \Phi_-\left(\frac{\sqrt{2N}-t}{\sqrt{N}}\right)\right]$$

- where $\Phi_-(y) \to$ left large deviation function:

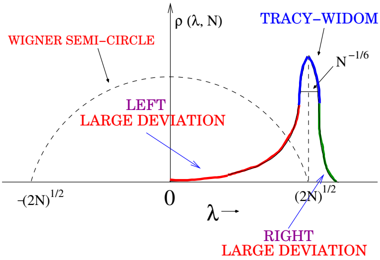
$$\Phi_-(y = z + \sqrt{2}) = \frac{1}{108}\Big[36z^2 - z^4 + (15z + z^3)\sqrt{z^2+6}$$
$$+ 27\left(\ln(18) - 2\ln(-z + \sqrt{6+z^2})\right)\Big]$$

- In particular, for $t = 0$, $P_N \sim \exp[-\beta\theta N^2]$

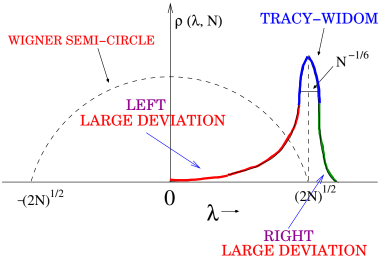
$$\boxed{\theta = \Phi_-(\sqrt{2}) = \frac{1}{4}\ln(3) = 0.274653\ldots}$$

(D.Dean & S.M., 2006,2008)

Matching the left tail of Tracy-Widom distribution:

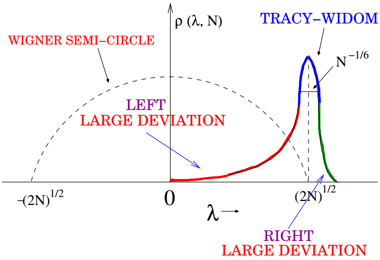


Matching the left tail of Tracy-Widom distribution:



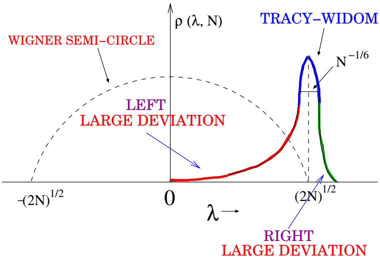
- $\mathrm{Prob}[\lambda_{\max} \leq t, N] \sim \exp\left[-\beta N^2\, \Phi_-\left(\frac{\sqrt{2N}-t}{\sqrt{N}}\right)\right]$; $\quad y = \frac{\sqrt{2N}-t}{\sqrt{N}}$

Matching the left tail of Tracy-Widom distribution:



- $\mathrm{Prob}[\lambda_{\max} \leq t, N] \sim \exp\left[-\beta N^2\, \Phi_-\left(\frac{\sqrt{2N}-t}{\sqrt{N}}\right)\right]$; $\quad y = \frac{\sqrt{2N}-t}{\sqrt{N}}$
- When $y << 1$, i.e., $\sqrt{2N} - t << \sqrt{N} \rightarrow$ left tail of Tracy-Widom

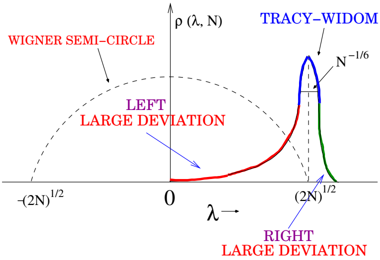
Matching the left tail of Tracy-Widom distribution:



- $\mathrm{Prob}[\lambda_{\max} \le t, N] \sim \exp\left[-\beta N^2 \, \Phi_-\left(\frac{\sqrt{2N}-t}{\sqrt{N}}\right)\right]$; $\quad y = \frac{\sqrt{2N}-t}{\sqrt{N}}$
- When $y << 1$, i.e., $\sqrt{2N} - t << \sqrt{N} \to$ left tail of Tracy-Widom
- As $y \to 0$, $\Phi_-(y) \to \frac{y^3}{6\sqrt{2}} \Rightarrow$

$$\mathrm{Prob}[\lambda_{\max} \le t, N] \approx \exp\left[-\frac{\beta}{24}\left|\sqrt{2}\, N^{1/6}\,(t - \sqrt{2N})\right|^3\right]$$

Matching the left tail of Tracy-Widom distribution:

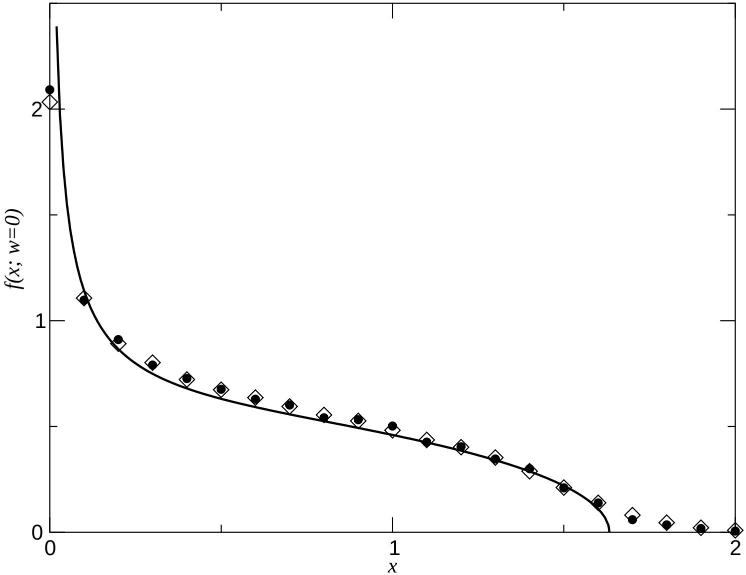


- $\mathrm{Prob}[\lambda_{\max} \leq t, N] \sim \exp\left[-\beta N^2\, \Phi_-\left(\frac{\sqrt{2N}-t}{\sqrt{N}}\right)\right]$; $\quad y = \frac{\sqrt{2N}-t}{\sqrt{N}}$
- When $y << 1$, i.e., $\sqrt{2N} - t << \sqrt{N} \rightarrow$ left tail of Tracy-Widom
- As $y \rightarrow 0$, $\Phi_-(y) \rightarrow \frac{y^3}{6\sqrt{2}} \Rightarrow$

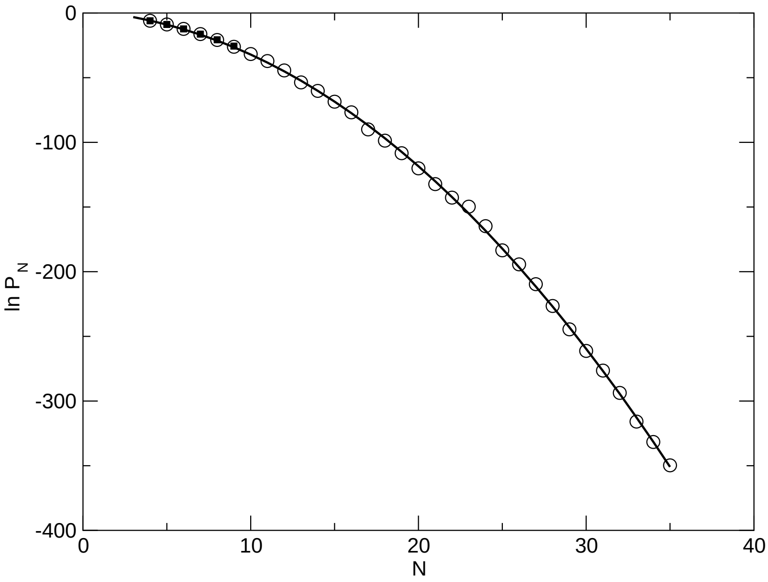
$$\mathrm{Prob}[\lambda_{\max} \leq t, N] \approx \exp\left[-\frac{\beta}{24}\left|\sqrt{2}\, N^{1/6}\, (t - \sqrt{2N})\right|^3\right]$$

- recovers the correct left tail of TW: $F_\beta(x) \sim \exp[-\frac{\beta}{24}\,|x|^3]$ as $x \rightarrow -\infty$

Numerical Results: Charge Density at $w = 0$



Numerical Result for P_N:



Right Large Deviation Function

- For large $\sim O(\sqrt{N})$ positive deviation: $t-\sqrt{2N} \sim O(\sqrt{N})$

$$\mathrm{Prob}[\lambda_{\max} = t, N] \sim \exp\left[-\beta\, N\, \Phi_{+}\left(\frac{t-\sqrt{2N}}{\sqrt{N}}\right)\right]$$

Right Large Deviation Function

- For large $\sim O(\sqrt{N})$ positive deviation: $t - \sqrt{2N} \sim O(\sqrt{N})$

$$\mathrm{Prob}[\lambda_{\max} = t, N] \sim \exp\left[-\beta\, N\, \Phi_{+}\left(\frac{t-\sqrt{2N}}{\sqrt{N}}\right)\right]$$

- $P(\lambda_{\max} = t, N) \propto \exp[-\beta \Delta E(t)/2]$ where

Right Large Deviation Function

- For large $\sim O(\sqrt{N})$ positive deviation: $t - \sqrt{2N} \sim O(\sqrt{N})$

$$\mathrm{Prob}[\lambda_{\max} = t, N] \sim \exp\left[-\beta\, N\, \Phi_{+}\left(\frac{t-\sqrt{2N}}{\sqrt{N}}\right)\right]$$

- $P(\lambda_{\max} = t, N) \propto \exp[-\beta \Delta E(t)/2]$ where

$$\boxed{\Delta E(t) = t^2 - 2\, N \int_{-\sqrt{2N}}^{\sqrt{2N}} \ln(t - \lambda)\, \rho_N(\lambda)\, d\lambda}$$

$\Longrightarrow$ energy cost in pulling a charge out of the Wigner sea

Right Large Deviation Function

- For large $\sim O(\sqrt{N})$ positive deviation: $t-\sqrt{2N} \sim O(\sqrt{N})$

$$\mathrm{Prob}[\lambda_{\max} = t, N] \sim \exp\left[-\beta\, N\, \Phi_+\left(\frac{t-\sqrt{2N}}{\sqrt{N}}\right)\right]$$

- $P(\lambda_{\max} = t, N) \propto \exp[-\beta \Delta E(t)/2]$ where

$$\boxed{\Delta E(t) = t^2 - 2\,N \int_{-\sqrt{2N}}^{\sqrt{2N}} \ln(t-\lambda)\,\rho_N(\lambda)\,d\lambda}$$

$\Longrightarrow$ energy cost in pulling a charge out of the Wigner sea

$$\rho(\lambda, N) \xrightarrow[N\to\infty]{} \sqrt{\frac{2}{N\pi^2}}\left[1-\frac{\lambda^2}{2N}\right]^{1/2} \rightarrow \text{av. den.}$$

Right Large Deviation Function

- For large $\sim O(\sqrt{N})$ positive deviation: $t-\sqrt{2N}\sim O(\sqrt{N})$

$$\mathrm{Prob}[\lambda_{\max}=t,N]\sim \exp\left[-\beta\, N\,\Phi_{+}\left(\frac{t-\sqrt{2N}}{\sqrt{N}}\right)\right]$$

- $P(\lambda_{\max}=t,N)\propto \exp[-\beta\Delta E(t)/2]$ where

$$\boxed{\Delta E(t)=t^2-2\,N\int_{-\sqrt{2N}}^{\sqrt{2N}}\ln(t-\lambda)\,\rho_N(\lambda)\,d\lambda}$$

$\Longrightarrow$ energy cost in pulling a charge out of the Wigner sea

$$\rho(\lambda,N)\xrightarrow[N\to\infty]{}\sqrt{\frac{2}{N\pi^2}}\left[1-\frac{\lambda^2}{2N}\right]^{1/2}\to \text{av. den.}$$

$\Longrightarrow$ exact $\Phi_{+}(y)$ (S.M. & Vergassola, 2009)

For $\beta=1$, $\Phi_{+}(y)\to$ different method by Ben-Arous et. al. (2001).

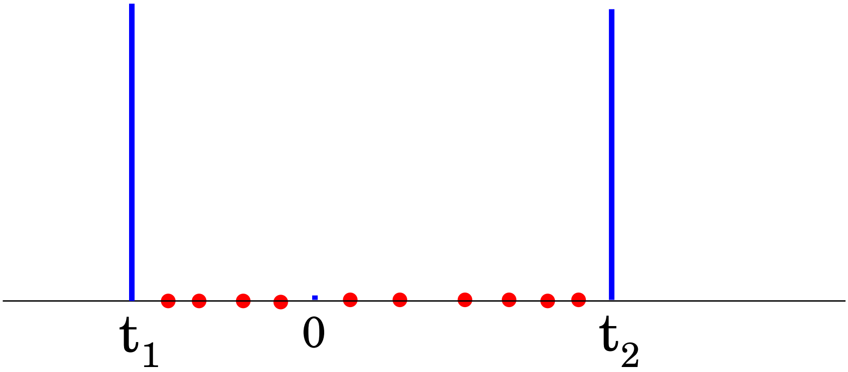
Joint distribution of $\lambda_{\min}$ and $\lambda_{\max}$:

- $\mathrm{Prob}[\lambda_{\min} \geq t_1, \lambda_{\max} \leq t_2] = \frac{Z_N(t_1,t_2)}{Z_N(-\infty,\infty)}$
- $Z_N(t_1, t_2) =$ Partition function with *two* walls

Joint distribution of $\lambda_{\min}$ and $\lambda_{\max}$:

- $\mathrm{Prob}[\lambda_{\min} \geq t_1, \lambda_{\max} \leq t_2] = \frac{Z_N(t_1,t_2)}{Z_N(-\infty,\infty)}$

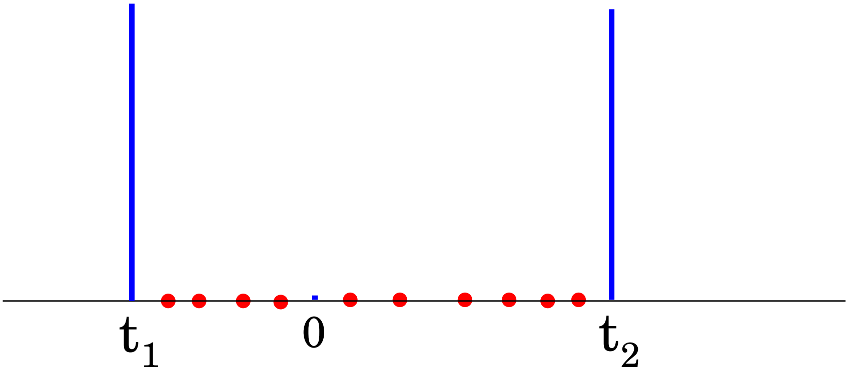
- $Z_N(t_1, t_2) =$ Partition function with *two* walls



Joint distribution of $\lambda_{\min}$ and $\lambda_{\max}$:

- $\mathrm{Prob}[\lambda_{\min} \geq t_1, \lambda_{\max} \leq t_2] = \frac{Z_N(t_1,t_2)}{Z_N(-\infty,\infty)}$

- $Z_N(t_1, t_2) =$ Partition function with *two* walls



Joint distribution of $\lambda_{\min}$ and $\lambda_{\max}$:

- When both $t_1 \sim O(\sqrt{N})$ and $t_2 \sim O(\sqrt{N})$

Joint distribution of $\lambda_{\min}$ and $\lambda_{\max}$:

- When both $t_1 \sim O(\sqrt{N})$ and $t_2 \sim O(\sqrt{N})$

$$\mathrm{Prob}[\lambda_{\min} \geq t_1, \lambda_{\max} \leq t_2] = \exp\left[-\beta\, N^2\, \Psi\left(\frac{t_1}{\sqrt{N}}, \frac{t_2}{\sqrt{N}}\right) + O(N)\right]$$

Joint distribution of $\lambda_{\min}$ and $\lambda_{\max}$:

- When both $t_1 \sim O(\sqrt{N})$ and $t_2 \sim O(\sqrt{N})$

$$\mathrm{Prob}[\lambda_{\min} \geq t_1, \lambda_{\max} \leq t_2] = \exp\left[-\beta\, N^2\, \Psi\left(\frac{t_1}{\sqrt{N}}, \frac{t_2}{\sqrt{N}}\right) + O(N)\right]$$

Joint distribution of $\lambda_{\min}$ and $\lambda_{\max}$:

- When both $t_1 \sim O(\sqrt{N})$ and $t_2 \sim O(\sqrt{N})$

$$\mathrm{Prob}[\lambda_{\min} \geq t_1, \lambda_{\max} \leq t_2] = \exp\left[-\beta\, N^2\, \Psi\left(\frac{t_1}{\sqrt{N}}, \frac{t_2}{\sqrt{N}}\right) + O(N)\right]$$

- $\Psi(z_1, z_2) = U(z_1, z_2 - z_1) - \frac{3+\ln(4)}{8}$

$$\begin{aligned} U(x,y) &= \frac{1}{32}\left[32\,\ln(2) - 16\,\ln(y) + 16\,x^2 + 6\,y^2\right. \\ &+ \left. 16\,y\,x - 2\,x^2\,y^2 - 2\,y^3\,x - \frac{9}{16}\,y^4\right] \end{aligned}$$

(D.Dean & S.M., 2008)

Largest Eigenvalue of Wishart Matrix

- Let $X = [x_{i,j}] \rightarrow (M \times N)$ rectangular matrix

Largest Eigenvalue of Wishart Matrix

- Let $X = [x_{i,j}] \rightarrow (M \times N)$ rectangular matrix
- $W = X^{\dagger} X \rightarrow (N \times N)$ square covariance matrix (Wishart)

Largest Eigenvalue of Wishart Matrix

- Let $X = [x_{i,j}] \to (M \times N)$ rectangular matrix
- $W = X^\dagger X \to (N \times N)$ square covariance matrix (Wishart)
- Entries of X Gaussian: $\mathrm{Pr}[X] \propto \exp\left[-\frac{\beta}{2}\mathrm{Tr}(X^\dagger X)\right]$

Largest Eigenvalue of Wishart Matrix

- Let $X = [x_{i,j}] \to (M \times N)$ rectangular matrix
- $W = X^\dagger X \to (N \times N)$ square covariance matrix (Wishart)
- Entries of X Gaussian: $\mathrm{Pr}[X] \propto \exp\left[-\frac{\beta}{2}\mathrm{Tr}(X^\dagger X)\right]$
- N real eigenvalues of W: $\lambda_1 \geq 0$, $\lambda_2 \geq 0$, $\ldots$, $\lambda_N \geq 0$

Largest Eigenvalue of Wishart Matrix

- Let $X = [x_{i,j}] \rightarrow (M \times N)$ rectangular matrix
- $W = X^{\dagger} X \rightarrow (N \times N)$ square covariance matrix (Wishart)
- Entries of X Gaussian: $\mathrm{Pr}[X] \propto \exp\left[-\frac{\beta}{2}\mathrm{Tr}(X^{\dagger}X)\right]$
- N real eigenvalues of W: $\lambda_1 \geq 0$, $\lambda_2 \geq 0$, $\ldots$, $\lambda_N \geq 0$
- Joint distribution of eigenvalues (James, 1964):

$$P(\{\lambda_i\}) \propto \exp\left[-\frac{\beta}{2}\sum_{i=1}^{N}\lambda_i\right]\prod_i \lambda_i^{\frac{\beta}{2}(1+M-N)-1}\prod_{j<k}|\lambda_j - \lambda_k|^{\beta}$$

Largest Eigenvalue of Wishart Matrix

- Let $X = [x_{i,j}] \to (M \times N)$ rectangular matrix
- $W = X^\dagger X \to (N \times N)$ square covariance matrix (Wishart)
- Entries of X Gaussian: $\mathrm{Pr}[X] \propto \exp\left[-\frac{\beta}{2}\mathrm{Tr}(X^\dagger X)\right]$
- N real eigenvalues of W: $\lambda_1 \geq 0$, $\lambda_2 \geq 0$, $\ldots$, $\lambda_N \geq 0$
- Joint distribution of eigenvalues (James, 1964):

$$P(\{\lambda_i\}) \propto \exp\left[-\frac{\beta}{2}\sum_{i=1}^{N}\lambda_i\right]\prod_i \lambda_i^{\frac{\beta}{2}(1+M-N)-1}\prod_{j<k}|\lambda_j-\lambda_k|^\beta$$

- Av. density of states: $\rho(\lambda, N) = \langle \frac{1}{N}\sum_{i=1}^{N}\delta(\lambda-\lambda_i)\rangle \xrightarrow[N\to\infty]{} \frac{1}{N}\, f_{\rm MP}(\frac{\lambda}{N})$

Largest Eigenvalue of Wishart Matrix

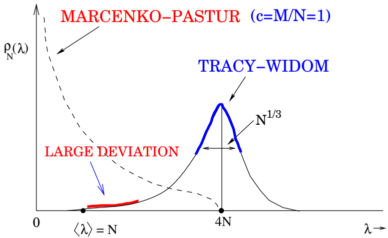
- Let $X = [x_{i,j}] \to (M \times N)$ rectangular matrix
- $W = X^\dagger X \to (N \times N)$ square covariance matrix (Wishart)
- Entries of X Gaussian: $\mathrm{Pr}[X] \propto \exp\left[-\frac{\beta}{2}\mathrm{Tr}(X^\dagger X)\right]$
- N real eigenvalues of W: $\lambda_1 \geq 0,\ \lambda_2 \geq 0,\ \ldots,\ \lambda_N \geq 0$
- Joint distribution of eigenvalues (James, 1964):

$$P(\{\lambda_i\}) \propto \exp\left[-\frac{\beta}{2}\sum_{i=1}^{N}\lambda_i\right] \prod_i \lambda_i^{\frac{\beta}{2}(1+M-N)-1} \prod_{j<k} |\lambda_j - \lambda_k|^\beta$$

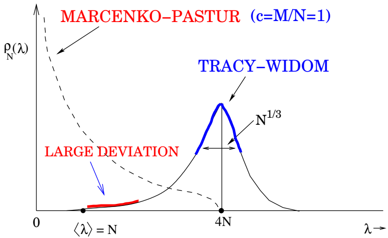
- Av. density of states: $\rho(\lambda, N) = \langle \frac{1}{N}\sum_{i=1}^{N}\delta(\lambda - \lambda_i)\rangle \xrightarrow[N\to\infty]{} \frac{1}{N} f_{\mathrm{MP}}(\frac{\lambda}{N})$
- Marcenko-Pastur law (1967): $f_{\mathrm{MP}}(x) = \frac{1}{2\pi x}\sqrt{(x_+ - x)(x - x_-)}$

$x_\pm = (1 \pm \frac{1}{\sqrt{c}})^2$ where $c = N/M \leq 1$

Largest Eigenvalue of the Wishart Matrix

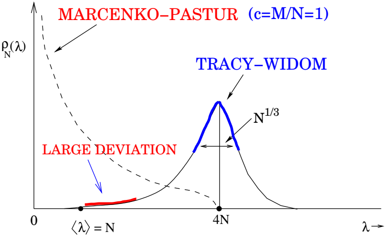


Largest Eigenvalue of the Wishart Matrix



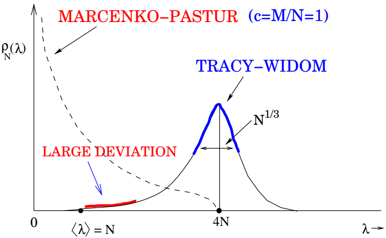
- average eigenvalue $\langle \lambda \rangle = N$ (e.g., for $M = N$)

Largest Eigenvalue of the Wishart Matrix



- average eigenvalue $\langle \lambda \rangle = N$ (e.g., for $M = N$)
- largest eigenvalue (e.g., for $M = N$): $\lambda_{\max} = 4N + 2^{4/3} N^{1/3} \Xi_{\mathrm{TW}}$ valid for $|4N - \lambda_{\max}| \sim O(N^{1/3})$ (Johansson 2000, Johnstone 2001).

Largest Eigenvalue of the Wishart Matrix



- average eigenvalue $\langle\lambda\rangle = N$ (e.g., for $M = N$)
- largest eigenvalue (e.g., for $M = N$): $\lambda_{\max} = 4N + 2^{4/3} N^{1/3} \Xi_{\mathrm{TW}}$ valid for $|4N - \lambda_{\max}| \sim O(N^{1/3})$ (Johansson 2000, Johnstone 2001).
- for $4N - t \sim O(N)$ (large negative fluctuation):

$$\mathrm{Prob}[\lambda_{\max} \le t, N] \sim \exp\left[-\beta\, N^2\, \Phi_W\left(\frac{4N - t}{N}\right)\right]$$

Large Deviation for Wishart Matrix:

- large deviation function $\Phi_W(y) \rightarrow$ computed explicitly for all $c = N/M \leq 1$

(Vivo, S.M., & Bohigas, 2007)

for the special case $c = 1$ Deift et. al. computed $\Phi_W(y)$ by a different method (Deift, Its & Krasovsky, 2006).

Large Deviation for Wishart Matrix:

- large deviation function $\Phi_W(y) \to$ computed explicitly for all $c = N/M \leq 1$

(Vivo, S.M., & Bohigas, 2007)

for the special case $c = 1$ Deift et. al. computed $\Phi_W(y)$ by a different method (Deift, Its & Krasovsky, 2006).

- In particular, $P_N = \mathrm{Prob}[\lambda_{\max} \leq \langle\lambda\rangle = N/c] \sim \exp[-\beta\theta(c)N^2]$

Large Deviation for Wishart Matrix:

- large deviation function $\Phi_W(y) \rightarrow$ computed explicitly for all $c = N/M \leq 1$

(Vivo, S.M., & Bohigas, 2007)

for the special case $c = 1$ Deift et. al. computed $\Phi_W(y)$ by a different method (Deift, Its & Krasovsky, 2006).

- In particular, $P_N = \mathrm{Prob}[\lambda_{\max} \leq \langle \lambda \rangle = N/c] \sim \exp[-\beta\theta(c)N^2]$

- For example, for the square matrix case $M = N$, i.e., $c = 1$

Large Deviation for Wishart Matrix:

- large deviation function $\Phi_W(y) \to$ computed explicitly for all $c = N/M \leq 1$

(Vivo, S.M., & Bohigas, 2007)

for the special case $c = 1$ Deift et. al. computed $\Phi_W(y)$ by a different method (Deift, Its & Krasovsky, 2006).

- In particular, $P_N = \mathrm{Prob}[\lambda_{\max} \leq \langle \lambda \rangle = N/c] \sim \exp[-\beta\theta(c)N^2]$

- For example, for the square matrix case $M = N$, i.e., $c = 1$

$$\boxed{\theta(1) = \log(2) - \frac{33}{64} = 0.177522\ldots}$$

(Vivo, S.M., & Bohigas, 2007)

Large Deviation for Wishart Matrix:

- large deviation function $\Phi_W(y) \to$ computed explicitly for all $c = N/M \leq 1$

(Vivo, S.M., & Bohigas, 2007)

for the special case $c = 1$ Deift et. al. computed $\Phi_W(y)$ by a different method (Deift, Its & Krasovsky, 2006).

- In particular, $P_N = \mathrm{Prob}[\lambda_{\max} \leq \langle\lambda\rangle = N/c] \sim \exp[-\beta\theta(c)N^2]$

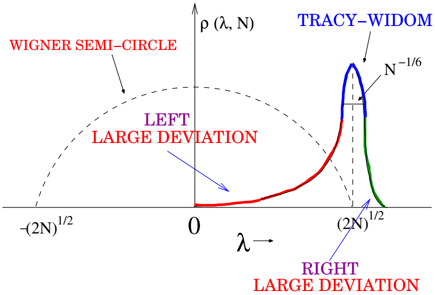
- For example, for the square matrix case $M = N$, i.e., $c = 1$

$$\boxed{\theta(1) = \log(2) - \frac{33}{64} = 0.177522\ldots}$$

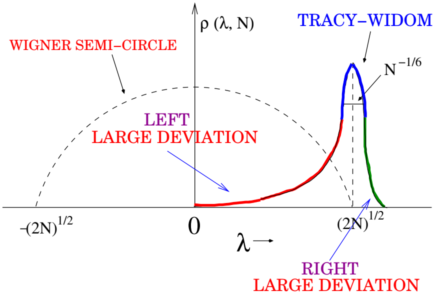
(Vivo, S.M., & Bohigas, 2007)

- Similarly, the right large deviation function (S.M. & Vergassola, 2009).

Summary and Conclusions

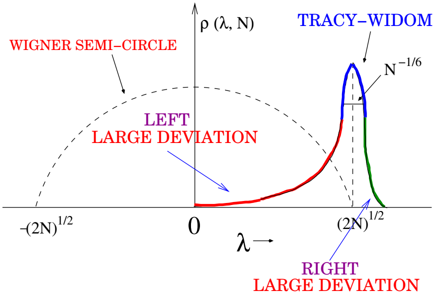


Summary and Conclusions



- Prob. distr. of $\lambda_{\max} \Longleftrightarrow$ Coulomb gas with a wall

Summary and Conclusions

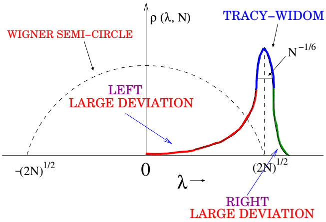


- Prob. distr. of $\lambda_{\max}$ $\Longleftrightarrow$ Coulomb gas with a wall
- for $(\sqrt{2N} - t) \sim N^{-1/6}$

$$\mathrm{Prob}[\lambda_{\max} \le t, N] \approx F_{\mathrm{TW}}\left[N^{1/6}\,(t - \sqrt{2N})\right]$$

Tracy-Widom Distribution

Summary and Conclusions: Large Deviations



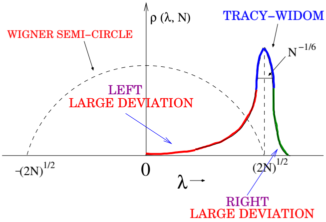
Large Deviations:

- for $(\sqrt{2N} - t) \sim \sqrt{N}$

$$\mathrm{Prob}[\lambda_{\max} = t, N] \approx \exp\left[-\beta\, N^2\, \Phi_-\left(\frac{\sqrt{2N} - t}{\sqrt{N}}\right)\right]$$

- for $(t - \sqrt{2N}) \sim \sqrt{N}$

$$\mathrm{Prob}[\lambda_{\max} = t, N] \sim \exp\left[-\beta\, N\, \Phi_+\left(\frac{t - \sqrt{2N}}{\sqrt{N}}\right)\right]$$



Large Deviations:

- for $(\sqrt{2N} - t) \sim \sqrt{N}$

$$\mathrm{Prob}[\lambda_{\max} = t, N] \approx \exp\left[-\beta\, N^2\, \Phi_{-}\left(\frac{\sqrt{2N} - t}{\sqrt{N}}\right)\right]$$

- for $(t - \sqrt{2N}) \sim \sqrt{N}$

$$\mathrm{Prob}[\lambda_{\max} = t, N] \sim \exp\left[-\beta\, N\, \Phi_{+}\left(\frac{t - \sqrt{2N}}{\sqrt{N}}\right)\right]$$

Exact functional forms of $\Phi_{-}(y)$ (Dean & S.M., 2006) and $\Phi_{+}(y)$ (S.M. & Vergassola, 2009).

Conclusions and Related Problems

- In particular, $P_N = \mathrm{Prob}[\lambda_1 \leq 0, \lambda_2 \leq 0, \ldots, \lambda_N \leq 0] \sim \exp[-\beta\theta N^2]$

$$\boxed{\theta = \frac{1}{4}\ln(3) = 0.274653..}$$ (Dean & S.M., 2006)

Conclusions and Related Problems

- In particular, $P_N = \mathrm{Prob}[\lambda_1 \leq 0, \lambda_2 \leq 0, \ldots, \lambda_N \leq 0] \sim \exp[-\beta\theta N^2]$

$$\boxed{\theta = \frac{1}{4}\ln(3) = 0.274653..}$$ (Dean & S.M., 2006)

- Joint distribution of $\lambda_{\min}$ and $\lambda_{\max}$ (Dean & S.M., 2008)

Conclusions and Related Problems

- In particular, $P_N = \mathrm{Prob}[\lambda_1 \leq 0, \lambda_2 \leq 0, \ldots, \lambda_N \leq 0] \sim \exp[-\beta\theta N^2]$

$$\boxed{\theta = \frac{1}{4}\ln(3) = 0.274653..}$$ (Dean & S.M., 2006)

- Joint distribution of $\lambda_{\min}$ and $\lambda_{\max}$ (Dean & S.M., 2008)

- Extension to Wishart matrices (Vivo, S.M. & Bohigas 2007, S.M. & Vergassola 2009), Non-intersecting Brownian Interfaces near a wall (C. Nadal & S.M., 2009)

Conclusions and Related Problems

• In particular, $P_N = \mathrm{Prob}[\lambda_1 \leq 0, \lambda_2 \leq 0, \ldots, \lambda_N \leq 0] \sim \exp[-\beta\theta N^2]$

$$\boxed{\theta = \frac{1}{4}\ln(3) = 0.274653..} \quad \text{(Dean \& S.M., 2006)}$$

• Joint distribution of $\lambda_{\min}$ and $\lambda_{\max}$ (Dean & S.M., 2008)

• Extension to Wishart matrices (Vivo, S.M. & Bohigas 2007, S.M. & Vergassola 2009), Non-intersecting Brownian Interfaces near a wall (C. Nadal & S.M., 2009)

• Technique ⇒ useful for other problems: Expected number of stationary points for Gaussian random fields (Bray & Dean 2007, Fyodorov et. al. 2007)

Conclusions and Related Problems

- In particular, $P_N = \mathrm{Prob}[\lambda_1 \leq 0, \lambda_2 \leq 0, \ldots, \lambda_N \leq 0] \sim \exp[-\beta\theta N^2]$

$$\boxed{\theta = \frac{1}{4}\ln(3) = 0.274653..}$$ (Dean & S.M., 2006)

- Joint distribution of $\lambda_{\min}$ and $\lambda_{\max}$ (Dean & S.M., 2008)

- Extension to Wishart matrices (Vivo, S.M. & Bohigas 2007, S.M. & Vergassola 2009), Non-intersecting Brownian Interfaces near a wall (C. Nadal & S.M., 2009)

- Technique $\Rightarrow$ useful for other problems: Expected number of stationary points for Gaussian random fields (Bray & Dean 2007, Fyodorov et. al. 2007)

- Large deviation of the conductance and shot noise in quantum dots (Vivo, S .M., & Bohigas 2008)

Conclusions and Related Problems

- In particular, $P_N = \mathrm{Prob}[\lambda_1 \le 0, \lambda_2 \le 0, \ldots, \lambda_N \le 0] \sim \exp[-\beta\theta N^2]$

$$\boxed{\theta = \frac{1}{4}\ln(3) = 0.274653..}$$ (Dean & S.M., 2006)

- Joint distribution of $\lambda_{\min}$ and $\lambda_{\max}$ (Dean & S.M., 2008)

- Extension to Wishart matrices (Vivo, S.M. & Bohigas 2007, S.M. & Vergassola 2009), Non-intersecting Brownian Interfaces near a wall (C. Nadal & S.M., 2009)

- Technique $\Rightarrow$ useful for other problems: Expected number of stationary points for Gaussian random fields (Bray & Dean 2007, Fyodorov et. al. 2007)

- Large deviation of the conductance and shot noise in quantum dots (Vivo, S .M., & Bohigas 2008)

- Exact smallest eigenvalue distribution of an entangled random pure state (S.M., Bohigas, & Lakshminarayan 2008)

Collaborators

- O. Bohigas (Orsay, FRANCE)
- D.S. Dean (Toulouse, FRANCE)
- A. Lakshminarayan (IIT Chennai, INDIA)
- C. Nadal (Ph.D student, Orsay, FRANCE)
- M. Vergassola (Institut Pasteur, Paris, FRANCE)
- P. Vivo (ICTP, Trieste, ITALY)

References

• D.S. Dean and S.N. Majumdar, " Large Deviations of Extreme Eigenvalues of Random Matrices", Phys. Rev. Lett., 97, 160201 (2006)

• D.S. Dean and S.N. Majumdar, "Extreme Value Statistics of Eigenvalues of Gaussian Random Matrices", Phys. Rev. E., 77, 041108 (2008)

• P. Vivo, S.N. Majumdar, and O. Bohigas, "Large Deviations of the Maximum Eigenvalue in Wishart Random Matrices", J. Phys. A. Math-Gen, 40, 4317 (2007)

• S.N. Majumdar, O. Bohigas and A. Lakshminarayan, "Exact Minimum Eigenvalue Distribution of an Entangled Random Pure State", J. Stat. Phys., 131, 33 (2008)

• P. Vivo, S.N. Majumdar, and O. Bohigas, "Distributions of Conductance and Shot Noise and associated Phase Transitions", Phys. Rev. Lett. 101, 216809 (2008)

• S.N. Majumdar and M. Vergassola, "Large Deviations of the Maximum Eigenvalue for Wishart and Gaussian Random Matrices", Phys. Rev. Lett. 102, 060601 (2009)

• S.N. Majumdar, Review on Tracy-Widom and related problems, Les Houches lecture notes (2006)