MOUVEMENT BROWNIEN MATRICIEL ET FONCTIONS L

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FONCTIONS L

χ=caractère de Dirichlet, multiplicatif modulo d.

$$L_\chi(s) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$$

a un prolongement méromorphe à $\mathbf{C}$

Une équation fonctionnelle simple relie $\Gamma(\frac{s+e}{2})L_\chi(s)$ et $\Gamma(\frac{1-s+e}{2})L_{\chi^{-1}}(1-s)$.

TEMPS DE SORTIE

On va calculer la loi de temps de sortie du mouvement Brownien de différents ensembles.

1. Dimension 1

B=mouvement Brownien réel.

$B_0 = 0$

T= temps de sortie de $[-1, 1]$.

$$E[\exp -\lambda^2 T/2] = 1/\cosh(\lambda)$$

$$E[T^s] = \Gamma(s+1)2^{s+1}\left(\frac{2}{\pi}\right)^{2s+1} L_{\chi_4(2s+1)}$$

2. Dimension 2

B=mouvement Brownien complexe

T=temps de sortie d'un triangle équilatéral centré en 0.

$$E[\exp -\lambda^2 T/2] = \frac{\sinh(\lambda)}{\sinh 3\lambda}$$

$$E[T^s] = \Gamma(s+1)2^{s+1}\left(\frac{2}{\pi}\right)^{2s+1} L_{\chi_3(2s+1)}$$

3. Dimension 3. B=mouvement Brownien de dimension 3. T=temps de sortie d'une boule centrée en 0.

$$E[\exp -\lambda^2 T/2] = \frac{\lambda}{\sinh \lambda}$$

$$E[T^s] = 2\left(\frac{2^{1-2s}-1}{1-2s}\right)\left(\frac{2}{\pi}\right)^s \xi(2s)$$

Si $S = T_1 + T_2$ copies indépendantes de T

$$E[\exp -\lambda^2 S/2] = \left(\frac{\lambda}{\sinh \lambda}\right)^2$$

$$E[S^s] = 2\left(\frac{2}{\pi}\right)^s \xi(2s)$$

BROWNIAN MOTION ON HERMITIAN MATRICES

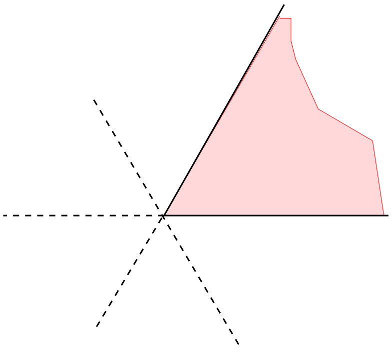
$$X_t = (X_{ij}(t))_{1 \leq i,j \leq n}$$

hermitian matrix with Brownian motion entries.
We assume $Tr(X(t)) = 0$.

$$\lambda_1(t) \leq \lambda_2(t) \leq \ldots \leq \lambda_n(t)$$

eigenvalues of $X(t)$.

The problem is to describe the process $(\lambda_1(t), \lambda_2(t), \ldots, \lambda_n(t))$ (Dyson's Brownian motion).



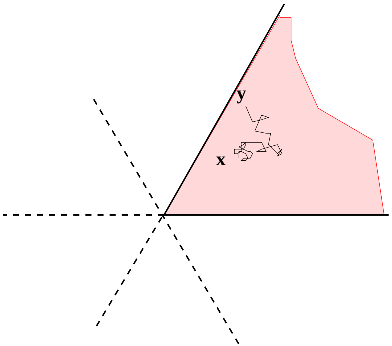
The cone

$$C: \qquad x_1 \leq x_2 \leq \ldots \leq x_n$$

in the hyperplane

$$H: \qquad x_1 + \ldots + x_n = 0$$

The symmetric group S_n acts by permutation of coordinates.



$p_t(x,y) = \frac{e^{-|x-y|^2/2t}}{(2\pi t)^{(n-1)/2}}$=gaussian kernel on H

$$p_t^0(x,y) = \sum_{\sigma \in S_n} \varepsilon(\sigma) p_t(x, \sigma(y))$$

is the transition probability kernel for Brownian motion killed at the boundary of C.

Theorem

There exists a unique (up to a positive constant) positive harmonic function on C

$$h(x) = \prod_{i>j}(x_i - x_j)$$

The motion of eigenvalues of X is the h process with probability transitions in C

$$q_t(x,y) = \frac{h(y)}{h(x)} p_t^0(x,y)$$

This is "Brownian motion conditionned to stay in the Weyl chamber C" It is a diffusion process with generator

$$\frac{1}{2}\Delta + \langle \nabla \log h, \nabla \cdot \rangle \, .$$

BROWNIAN MOTION ON $SL_n(\mathbf{C})$

$$dX_t = X_t dZ_t$$

where Z_t is a complex Brownian motion on $M_n(\mathbf{C})$. Assume $Tr(Z_t) = 0$, $det(X_t) = 1$.

$$e^{\lambda_1(t)} \leq e^{\lambda_2(t)} \leq \ldots \leq e^{\lambda_n(t)}$$

eigenvalues of $X_t X_t^*$
$=$singular values of X
$=A$-part in the KAK decomposition $=$radial part of Brownian motion on $SL_n(\mathbf{C})/SU(n)$.

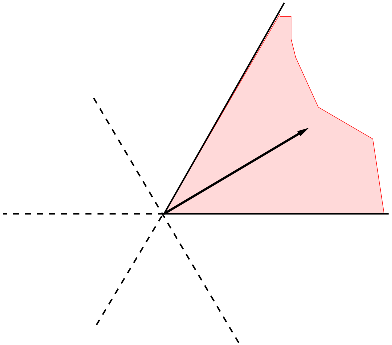
$$C: \qquad x_1 \leq x_2 \leq \ldots \leq x_n$$

$$H: \qquad x_1 + \ldots + x_n = 0$$

Consider a Brownian motion in H, with drift

$$\rho = (-n, -n+2, \ldots, n-2, n)$$

killed at the exit of C.



This process has a semigroup given by

$$p_t^{0,\rho}(x,y) = e^{\langle \rho, y-x\rangle - t\langle \rho,\rho\rangle/2} p_t^0(x,y) \ .$$

The function

$$h^\rho(y) = \prod_{i>j}(1 - e^{2(y_j - y_i)})$$

is a positive harmonic function for the semigroup $p_t^{0,\rho}$, in the cone C, and vanishes at the boundary of the cone.

Theorem

The motion of singular values of X_t is given by the Doob-transformed semigroup

$$q_t^\rho(x,y) = \frac{h^\rho(y)}{h^\rho(x)} p_t^{0,\rho}(x,y)$$

in the cone C.

Its infinitesimal generator is

$$\frac{1}{2}\Delta + \langle \rho, \cdot \rangle + \langle \nabla \log h^\rho, \nabla \cdot \rangle.$$

This is "Brownian motion with a drift conditionned to stay in the Weyl chamber C"

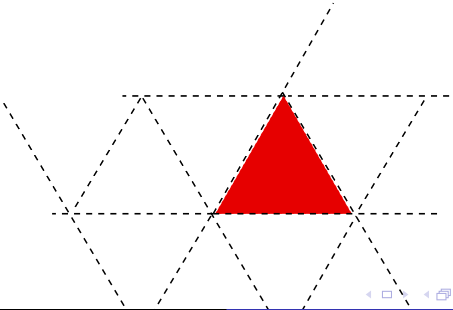
Brownian motion with values in $SU(n)$

$$dU_t = iU_t dH_t$$

where H_t is a hermitian (traceless) Brownian motion.
$e^{i\theta_1(t)}, \ldots, e^{i\theta_n(t)}$ = eigenvalues of U_t

$$\sum_i \theta_i = 0, \qquad \theta_1 \geq \theta_2 \geq \ldots \geq \theta_n \qquad \theta_1 - \theta_n \leq 2\pi$$

These conditions determine a simplex Δ_n in H, which is a fundamental domain for the action of the affine Weyl group on H.



$$p_t^0(\theta,\xi) = \sum_{w\in\tilde{W}} \epsilon(w) p_t(\theta, w(\xi)).$$

is the semigroup of Brownian moton in this simplex killed at the boundary. The function

$$h^u(\theta) = \prod_{j>k} (e^{i\theta_j} - e^{i\theta_k}).$$

is positive inside the simplex Δ_n, it vanishes on the boundary, and it is the eigenfunction corresponding to the Dirichlet eigenvalue with smallest module for the Laplacian on Δ_n.

The Doob-transformed semigroup

$$q_t^u(x,y) = \frac{h^u(y)}{h^u(x)} e^{-\lambda t} p_t^0(x,y)$$

is a Markov diffusion semigroup in Δ_n, with infinitesimal generator

$$\frac{1}{2}\Delta + \langle \nabla \log h^u, \nabla \cdot \rangle - \lambda.$$

The process of eigenvalues of a unitary Brownian motion is a diffusion with values in Δ_n with probability transition semigroup q_t^u.

Formally BM on $SU(n)$ is obtained by replacing ρ by $i\rho$ ("Wick rotation").

Pólya's paper

Pólya starts from Riemann's ξ function

$$\xi(s) = s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)$$

ζ = Riemann's zeta function.
with $s = 1/2 + iz$

$$\xi(z) = 2\int_0^\infty \Phi(u)\cos(zu)du$$

with

$$\Phi(u) = 2\pi e^{5u/2}\sum_{n=1}^{\infty}(2\pi e^{2u}n^2 - 3)n^2 e^{-\pi n^2 e^{2u}}$$

one has

$$\Phi(t) = \Phi(-t) \qquad \xi(z) = \xi(-z)$$

This lead Pólya to define a "falsified" ξ function

$$\xi^*(z) = 8\pi^2 \int_0^\infty e^{-\pi(e^{2u}+e^{-2u})} \cos(zu) du.$$

Theorem

The function ξ^* is entire, satisfies the functional equation

$$\xi^*(z) = \xi^*(-z)$$

its zeros are real and simple.

$N(r), N^*(r)$= number of zeros of $\xi(z), \xi^*(z)$ with real part in $[0, r]$, then

$$N(r) \sim \frac{r}{2\pi} \log r$$

$$N(r) - N^*(r) = O(\log r)$$

"This beautiful paper takes you within a hair's breadth of Riemann's hypothesis"

Marc Kac in Pólya's selected papers.

I will replace hair's breadth by a Wick rotation.

Spectral interpretation of the zeros of ξ^*

The zeros of ξ^* are the $\pm\gamma$ where the γ^2 are the eigenvalues of the Schrödinger operator

$$-\frac{d^2}{dx^2} + e^{2x}$$

on $[0, +\infty[$ with Dirichlet boundary condition at 0.

Gamma distributions

$$P(\gamma_{\omega,c} \in dt) = \frac{c^{-\omega}}{\Gamma(\omega)} t^{\omega-1} e^{-t/c} dt = \Gamma_{\omega,c}(dt) \qquad \omega, c > 0$$

Laplace transforms

$$E[e^{-\lambda\gamma_{\omega,c}}] = (1 + \lambda/c)^{-\omega}.$$

Generalized gamma convolutions are the distributions of linear combinations, with positive coefficients, of independent gamma variables, and their weak limits.
Characterized by the Thorin measure μ= a positive measure on $[0, +\infty[$.

$$E[e^{-\lambda X}] = \exp(-\int \log(1 + \lambda/c) d\mu(c)).$$

Consider the two symmetric spaces $SU(2)$ and $SL_2(\mathbf{C})/SU(2)$. One obtains the second by a "Wick rotation" of the first. Analyzing the Laplace operators on these spaces gives the spectral measures

$$\nu_1(dc) = 2\sum_{n=1}^{\infty} \delta_{n^2}(dc) \; ;$$

$$\nu_2(dc) = \frac{dc}{\pi\sqrt{c-1}} \qquad c > 1$$

=spectral measures associated with the infinitesimal generators of Brownian motions

$$\frac{d^2}{dx^2} \quad \text{on} \quad [0, \pi]$$

$$\frac{d^2}{dx^2} + \frac{d}{dx} \quad \text{on} \quad [0, +\infty[$$

Consider the Generalized Gamma Convolutions with Thorin measures ν_1 and ν_2.
Let X_1 and X_2 be random variables with these distributions.

Proposition

X_1 is distributed as a hitting time of a three dimensional Bessel process.
X_2 is distributed as a hitting time of Brownian motion with a drift.
Furthermore

$$E[X_1^s] = \xi(2s)$$

$$E[X_2^s] = \xi^*(s)$$