

Probability, homework 1, due September 9.

Exercise 1. Let $(\mathcal{G}_\alpha)_{\alpha \in A}$ be an arbitrary family of σ -fields defined on an abstract space Ω , with A possibly uncountable. Show that $\cap_{\alpha \in A} \mathcal{G}_\alpha$ is also a σ -field.

Exercise 2. Let $\emptyset \subsetneq A \subsetneq B \subsetneq \Omega$ (these are strict inclusions). What is the σ -field generated by $\{A, B\}$?

Exercise 3. Let $\mathcal{F}, \mathcal{G}$ be σ -fields for the same Ω . Is $\mathcal{F} \cup \mathcal{G}$ a σ -field?

Exercise 4. For $\Omega = \mathbb{N}$ and $n \geq 0$, let $\mathcal{F}_n = \sigma(\{\{0\}, \dots, \{n\}\})$. Show that $(\mathcal{F}_n)_{n \geq 0}$ is a non-decreasing sequence but that $\cup_{n \geq 0} \mathcal{F}_n$ is not a σ -field.

Exercise 5. Let Ω be an infinite set (countable or not). Let $\mathcal{A}$ be the set of subsets of Ω that are either finite or with finite complement in Ω . Prove that $\mathcal{A}$ is a field but not a σ -field.

Exercise 6. Can you build an infinite, countable σ -field?

Exercise 7. A monotone class is a collection $\mathcal{M}$ of sets closed under both monotone increasing and monotone decreasing (i.e. if $A_i \in \mathcal{M}$ and either $A_i \uparrow A$ or $A_i \downarrow A$ then $A \in \mathcal{M}$)

Prove that if $\mathcal{A} \subset \mathcal{M}$ with $\mathcal{A}$ a field and $\mathcal{M}$ a monotone class, then $\sigma(\mathcal{A}) \subset \mathcal{M}$.

Exercise 8. Let $\mathbb{P}$ be a probability measure on Ω , endowed with a σ -field $\mathcal{A}$.

- (i) What is the meaning of the following events, where all A_n 's are elements of $\mathcal{A}$?

$$\liminf_{n \rightarrow \infty} A_n = \bigcup_{n \geq 1} \bigcap_{k \geq n} A_k, \quad \limsup_{n \rightarrow \infty} A_n = \bigcap_{n \geq 1} \bigcup_{k \geq n} A_k.$$

- (ii) Prove that $\limsup_{n \rightarrow \infty} A_n$ and $\liminf_{n \rightarrow \infty} A_n$ are in $\mathcal{A}$.

- (iii) In the special case $\Omega = \mathbb{R}$, for any $p \geq 1$, let

$$A_{2p} = \left[-1, 2 + \frac{1}{2p}\right), \quad A_{2p+1} = \left(-2 - \frac{1}{2p+1}, 1\right].$$

What are $\liminf_{n \rightarrow \infty} A_n$ and $\limsup_{n \rightarrow \infty} A_n$?

- (iv) Prove that the following always holds:

$$\mathbb{P}\left(\liminf_{n \rightarrow \infty} A_n\right) \leq \liminf_{n \rightarrow \infty} \mathbb{P}(A_n), \quad \mathbb{P}\left(\limsup_{n \rightarrow \infty} A_n\right) \geq \limsup_{n \rightarrow \infty} \mathbb{P}(A_n).$$

Exercise 9. The symmetric difference of two events A and B , denoted $A \triangle B$, is the event that precisely one of them occurs: $A \triangle B = (A \cup B) \setminus (A \cap B)$.

- (i) Write a formula for $A \triangle B$ that only involves the operations of union, intersection and complement, but no set difference.

- (ii) Define $d(A, B) = \mathbb{P}(A \triangle B)$. Show that for any three events A, B, C ,
- $$d(A, B) + d(B, C) - d(A, C) = 2(\mathbb{P}(A \cap B^c \cap C) + \mathbb{P}(A^c \cap B \cap C^c)).$$
- (iii) Assume $A \subset B \subset C$. Prove that $d(A, C) = d(A, B) + d(B, C)$.

Exercise 10. Prove the Bonferroni inequalities: if $A_i \in \mathcal{A}$ is a sequence of events, then

- (i) $\mathbb{P}(\cup_{i=1}^n A_i) \geq \sum_{i=1}^n \mathbb{P}(A_i) - \sum_{i < j} \mathbb{P}(A_i \cap A_j),$
(ii) $\mathbb{P}(\cup_{i=1}^n A_i) \leq \sum_{i=1}^n \mathbb{P}(A_i) - \sum_{i < j} \mathbb{P}(A_i \cap A_j) + \sum_{i < j < k} \mathbb{P}(A_i \cap A_j \cap A_k).$