Probability, homework 3 due September 27.

Exercise 1. Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. Prove that if $A \cap B = \emptyset$ and A, B are independent, then $\mathbb{P}(A) = 0$ or $\mathbb{P}(B) = 0$.

Exercise 2. Let X be a nonnegative random variable with null expectation. Prove that it is 0 almost surely.

Exercise 3. Calculate $\mathbb{E}(X)$ for the following probability measures $\mathbb{P}^X$.

(i) $\mathbb{P}^X$ has Gaussian density $\frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/(2\sigma^2)}$, for some $\sigma > 0$ and $\mu \in \mathbb{R}$;
(ii) $\mathbb{P}^X$ has exponential dentity $\lambda e^{-\lambda x} \mathbb{1}_{x \geq 0}$ for some $\lambda > 0$;
(iii) $\mathbb{P}^X = p\delta_a + q\delta_b$ where $p + q = 1$, $p, q \geq 0$ and $a, b \in \mathbb{R}$;
(iv) $\mathbb{P}^X$ is the Poisson distribution: $\mathbb{P}^X(\{n\}) = e^{-\lambda} \frac{\lambda^n}{n!}$ for any integer $n \geq 0$, for some $\lambda > 0$.

Exercise 4. Let X be a standard Gaussian random variable. What is the density of $1/X^2$?

Exercise 5. Let X be uniformly distributed on $[0, 1]$ and $\lambda > 0$. Show that $-\lambda^{-1} \log X$ has the same distribution as an exponential random variable with parameter λ.

Exercise 6. A samouraï wants to create a triangle with a (rigid) spaghetti. With his saber, he cuts this spaghetti on two places, chosen uniformly and independently along this traditional pasta. What is the probability that he can create a triangle with sides these three pieces of spaghetti?

Exercise 7. Assume that $X_1, X_2, \dots$ are independent random variables uniformly distributed on $[0, 1]$. Let $Y^{(n)} = n \inf\{X_i, 1 \leq i \leq n\}$. Prove that it converges weakly to an exponential random variable, i.e. for any continuous bounded function $f : \mathbb{R}^+ \to \mathbb{R}$,

$$\mathbb{E}\left(f(Y^{(n)})\right) \underset{n\to\infty}{\longrightarrow} \int_{\mathbb{R}^+} f(u) e^{-u} \mathrm{d}u.$$

Exercise 8. Let $f : [0, 1] \to \mathbb{R}$ be continuous. For any $y \in \mathbb{R}$, let $N(y) \in \mathbb{R} \cup \{\infty\}$ be the number of solutions to $f(x) = y$. Prove that N is measurable.

Exercise 9. Let $f : \mathbb{R} \to \mathbb{R}$ be additive ($f(a + b) = f(a) + f(b)$ for all a, b) and measurable. Prove that it is linear.

What if f is not assumed to be measurable? What about $f : \mathbb{R}^d \to \mathbb{R}$?